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Chapter 10
On the Darboux property of derivative multifunction

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10.1 Introduction

The concept of differentiability for multifunction has been considered by many authors from different point of view ([1], [2], [3], [5], [6], [10], [11], [12], [13]). We need to differentiate multifunctions as much as we need differentiate single-valued maps, for extending Darboux theorem on intermediate value property of derivative to multifunctions for instans, and for many other reasons.

How should we go about it? It is possible to define derivatives as adequate limits of differential quotients ([10], [11]). Starting from such a definition of derivative and using a theory of some new derivatives of single-valued functions given by Garg in [8] we will show that derivative multifunction has the Darboux property.

10.2 Preliminaries

We will use standard notations. In particular, the sets of positive integers and real numbers will be denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. $\mathbb{R}^n$ will denote the n-dimensional Euclidean space.

Let X and Y be two nonempty sets. By a multifunction from X to Y we mean a map which assigns to every point of X a nonempty subsets of Y; if F is a multifunction from X to Y, we denote it by $F\colon X \rightsquigarrow Y$.

The image of a set $A \subset X$ under multifunction $F\colon X \rightsquigarrow Y$ is defined by

$$F(A) = \bigcup\{F(x)\colon x \in A\}.$$

If $F\colon X \rightsquigarrow Y$ is amultifunction, then for a set $B \subset Y$ two inverse images of B under F are defined as follows:

$$F^+(B) = \{x \in X\colon F(x) \subset B\},$$

$$F^-(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}.$$

One sees immediately that

$$F^-(B) = X \setminus F^+(Y \setminus B) \quad \text{and} \quad F^+(B) = X \setminus F^-(Y \setminus B).$$

Let $(X, \mathscr{T}(X))$ be a topological spaces. We will use the notations $\mathrm{Int}A, \mathrm{Cl}A$ and $\mathrm{Fr}A$ for the interior, closure and boundary of $A \subset X$, respectively.

Let us still establish that $(Y, \mathscr{T}(Y))$ is also a topological space.

A multifunction $F\colon X \rightsquigarrow Y$ is called *upper* (resp. *lower*) *semicontinuous* at a point $x \in X$ if

(1) $\forall G \in \mathscr{T}(Y)(F(x) \subset G \Rightarrow x \in \mathrm{Int}F^+(G))$
(resp. $\forall G \in \mathscr{T}(Y)(F(x) \cap G \neq \emptyset \Rightarrow x \in \mathrm{Int}F^-(G))$).

F is called *continuous* at x if it is simultaneously upper and lower semicontinuous at x; F is continuous if it is continuous at each point $x \in X$.

Now suppose that (Y,d) is a metric space. Let $y \in Y$ and $A \subset Y$. We use $B(y,r)$ to denote an open ball in Y and $B(A,r) = \bigcup\{B(y,r)\colon y \in A\}$. In this case we have a set of more adjectives.

A multifunction $F\colon X \rightsquigarrow Y$ is called *h-upper* (resp. *h-lower*) *semicontinuous* at a point $x_0 \in X$ if the following condition holds:

(2) for each $\varepsilon > 0$ there exists a neighbourhood $U(x_0)$ of x_0 such that $F(x) \subset B(F(x_0), \varepsilon)$ (resp. $F(x_0) \subset B(F(x), \varepsilon)$) for each $x \in U(x_0)$.

F is *h-continuous* (or Hausdorff continuous) at x_0 if it is simultaneously h-upper and h-lower semicontinuous at x_0; F is *h-continuous* if it is h-continious at any point $x \in X$.

It is known (see [9]) that

(3) If F is upper (resp. h-lower) semicontinuous at $x \in X$, then F is h-upper (resp. lower) semicontinuous at x. If moreover $F(x)$ is compact for each $x \in X$, then conditions (1) and (2) are equivalent.

Let $\mathscr{P}_0(Y)$ denote the family of all nonempty subsets of Y. We denote the following families of sets:

$$\mathscr{C}(Y) = \{A \in \mathscr{P}_0(Y) \colon A \text{ is closed}\}$$

$$\mathscr{C}_b(Y) = \{A \in \mathscr{P}_0(Y) \colon A \text{ is closed and bounded}\}.$$

For $A, B \in \mathscr{C}_b(Y)$ let $d_H(A,B)$ denotes the Hausdorff distance of the sets A and B. Then the set $\mathscr{C}_b(Y)$ with Hausdorff distance becomes a metric space. Let us note that

(4) If $F \colon X \rightsquigarrow Y$ has closed and bounded values, then F is h-continuous if and only if F is continuous (with respect to d_H) as a function from X to $\mathscr{C}_b(Y)$.

Let $(Y, \|\cdot\|)$ be a real normed linear space. The symbol $\mathscr{C}_{ob}(Y)$ will be used to denote the collection of all nonempty, closed, bounded and convex subsets of Y.

If $A \subset Y$, $B \subset Y$, $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, one defines

$$A + B = \{a + b \colon a \in A, b \in B\}, \; \lambda A = \{\lambda a \colon a \in A\};$$

$$A - B = A + (-1)B \quad \text{and} \quad \frac{A}{\alpha} = \frac{1}{\alpha} A.$$

We will write $A + x$, if $B = \{x\}$.

(5) The following properties hold (see [14]):
(i) If $\alpha, \beta \in \mathbb{R}$ and $A, B \subset Y$ are convex, then $\alpha(\beta A) = (\alpha\beta)A$ and $\alpha(A + B) = \alpha A + \alpha B$.
(ii) If $A \subset B$ and $\alpha \geq 0$, then $\alpha A \subset \alpha B$.
(iii) If A is convex, $\alpha \geq 0$ and $\beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.
(iv) If $A \subset Y$ and $B \subset Y$ are closed and convex and $C \subset Y$ is bounded, then $A + C = B + C$ implies $A = B$.
(v) If $(Y, \|\cdot\|)$ is reflexive and $A, B \in \mathscr{C}_{ob}(Y)$, then $A + B \in \mathscr{C}_{ob}(Y)$.

10.3 The $\mathscr{D}$ and $\mathscr{D}_*$ properties of multifunctions

As we know a real function f defined on an interval $I \subset \mathbb{R}$ has the Darboux property if for each pair of distinct points $x_1, x_2 \in I$ and each y between $f(x_1)$ and $f(x_2)$ there is a point x_3 between x_1 and x_2 such that $y = f(x_3)$.

It is well known that f has the Darboux property if and only if $f(C)$ is a connected set for each connected set $C \subset I$.

It turns out that if we extend given above properties into multifunctions they are not already equivalent.

Let $I \subset \mathbb{R}$ be an interval. For each $a, b \in \mathbb{R}$ we will use $a \wedge b$ and $a \vee b$ to denote the minimum and maximum, respectively, of a and b.

In [7] the following definition of the Darboux property was given.

Definition 10.1. A multifunction $F\colon I \rightsquigarrow \mathbb{R}$ will be said to have the Darboux property (or $\mathscr{D}$ property) if for every connected set $C \subset I$, the image $F(C)$ is connected in $\mathbb{R}$.

In [4] the following definition was introdused.

Definition 10.2. A multifunction $F\colon I \rightsquigarrow \mathbb{R}$ will be said to have the intermediate value property (or $\mathscr{D}_*$ property) if for each pair of distinct points $x_1, x_2 \in I$ and each $y_1 \in F(x_1)$ there exists $y_2 \in F(x_2)$ such that $(y_1 \wedge y_2, y_1 \vee y_2) \subset F((x_1 \wedge x_2, x_1 \vee x_2))$.

Let us note that each of the properties $\mathscr{D}$ and $\mathscr{D}_*$ is equivalent to the usual Darboux property in the case when $F(x) = \{f(x)\}$, where $f\colon I \to \mathbb{R}$ is a function.

The following examples show that they are not equivalent in general.

Example 10.1. Let $F_1\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ be a multifunction defined by

$$F_1(x) = \begin{cases} [0,2], \text{ if } x = 0, \\ [0,1], \text{ if } x \neq 0. \end{cases}$$

Then F_1has the $\mathscr{D}$ property , but not the $\mathscr{D}_*$ property.

Example 10.2. Let $F_2(x) = [0,1] \cup [2,3]$ for each $x \in \mathbb{R}$. Then F_2 has the $\mathscr{D}_*$ property and does not have the $\mathscr{D}$ property .

However they showed the following theorem.

Theorem 10.1. *([4], Theorem 1]) Let $F\colon I \rightsquigarrow \mathbb{R}$ be a multifunction with connected values. If F has the $\mathscr{D}_*$ property, then it has the $\mathscr{D}$ property.*

Let us note that F_2 is continuous. Therefore, a continuous multifunction (with closed values) does not necessarily have the $\mathscr{D}$ property unlike the case of the $\mathscr{D}_*$ property. The following theorem was proved.

Theorem 10.2. *([4], Theorem 2]) If a multifunction $F\colon I \rightsquigarrow \mathbb{R}$ with closed values is continuous, then it has the $\mathscr{D}_*$ property.*

Remark 10.1. The assumption that a multifunction has closed values is important. In order to illustrate this, let us consider a multifunction $F\colon \mathbb{R} \rightsquigarrow \mathbb{R}$ defined by

$$F(x) = \begin{cases} \{y\colon y = \frac{1}{k}, k \in \mathbb{Z} \setminus \{0\}\}, & \text{if } x \in (0,1), \\ \{y\colon y = 0 \text{ or } y = \frac{1}{k}, k \in \mathbb{Z} \setminus \{0\}\}, & \text{if } x \notin (0,1), \end{cases}$$

where $\mathbb{Z}$ is the set of integers. Then F is continuous but does not have the $\mathscr{D}_*$ property .

10.4 Derivative multifunction

Let $(Y, \|\cdot\|)$ be a reflexive real normed linear space with the metric d determined by the norm in Y; θ will denote the neutral element of Y.

We define a difference $A \ominus B$ of the sets $A, B \in \mathscr{C}_{ob}(Y)$ as follows:

Definition 10.3. We will say the difference $A \ominus B$ is defined if there exists a set $C \in \mathscr{C}_{ob}(Y)$ such that either $A = B + C$ or $B = A - C$, and we define $A \ominus B$ to be the set C.

Using property (5) (iv) it is easy to show, that the difference $A \ominus B$ is uniquely determined.

Example 10.3. Let $A = \alpha P$ and $B = \beta P$, where $P \in \mathscr{C}_{ob}(Y), \alpha \geq 0$ and $\beta \geq 0$. Let us put $C = (\alpha - \beta)P$. Then, by (5) (iii), $B + C = A$ or $A - C = B$ depending on whether $\alpha \geq \beta$ or $\alpha < \beta$. Therefore $\alpha P \ominus \beta P$ exists and is equal to $(\alpha - \beta)P$.

Example 10.4. Consider the following sets:

$$A = \left\{(x,y) \in \mathbb{R}^2\colon 0 \leq x \leq 1, \quad 0 \leq y \leq 1 - x\right\},$$

$$B = \left\{(x,y) \in \mathbb{R}^2\colon 0 \leq x \leq 1, \quad 0 \leq y \leq \tfrac{1}{2}(1 - x)\right\}.$$

Then $A \ominus B$ does not exist. Indeed, suppose that there exists $C \in \mathscr{C}_{ob}(\mathbb{R}^2)$ such that $A = B + C$. Since $(0,1) \in A$, there exist $(a,b) \in B$ and $(c,d) \in C$ such that

$(0,1) = (a+c, b+d)$, where $a \geq 0$. Then $c = -a$ and $d = 1-b$. On the other hand $(0,0) \in B$. Therefore $(0,0)+(c,d) = (-a, 1-b) \in A$ and $-a \geq 0$. Hence $a = 0$. Since $(c,d) = (0, 1-b) \in C$ and $(1,0) \in B$, we have $(1,0)+(0,1-b) \in A$ and $b = 1$. Therefore we have $(a,b) = (0,1) \notin B$, which is a contradiction. Now let us suppose that there exists $C \in \mathscr{C}_{ob}(\mathbb{R}^2)$ such that $B = A - C$. Let $z \in C$. We observe that for every $x \in A$, $x - z \in A - C = B$. Hence we have $A - z \subset B$, i.e., some translate of A is contained in B, which is of course not possible.

Let $A, B \in \mathscr{C}_{bc}(Y)$. We write $B \subset_t A$, if for each $a \in \mathrm{Fr}A$ there is $y \in Y$ such that $a \in B + y \subset A$.

Theorem 10.3. *Suppose $A \in \mathscr{C}_{ob}(Y)$ and $B \in \mathscr{C}_{ob}(Y)$. Then*
(a) *$A \ominus B$ exists and is equal to a set $C \in \mathscr{C}_{ob}(Y)$ such that $A = B + C$ if and only if $B \subset_t A$.*
(b) *$A \ominus B$ exists and is equal to a set $C \in \mathscr{C}_{ob}(Y)$ such that $B = A - C$ if and only if $A \subset_t B$.*

Proof. To prove (a), suppose the existence of $C \in \mathscr{C}_{ob}(Y)$ such that $A = B + C$. If $a \in A$ (in particular $a \in \mathrm{Fr}(A)$), then $a \in B + C$. Therefore exist $b \in B$ and $c \in C$ such that $a = b + c$. If $z \in B$, then $z + c \in B + C = A$. Consequently $B + c \subset A$. Moreover $a = b + c \in B + c$. This proves that for a there is $y \in Y$ with $a \in B + y \subset A$.

Now let us suppose that for each $a \in \mathrm{Fr}A$ there exists $y \in Y$ such that $a \in B + y \subset A$. Assume that $C = \{x \colon B + x \subset A\}$. Then C is closed and bounded. We will show that C is convex. Let $c, c' \in C$. Then $B + c \subset A$ and $B + c' \subset A$. Let $\lambda \in [0,1]$. From (5) (ii) and (iii) we obtain

(6) $$(1-\lambda)(B+c) + \lambda(B+c') \subset A.$$

Furthermore

(7) $$(1-\lambda)(B+c) + \lambda(B+c') = B + (1-\lambda)c + \lambda c'.$$

We conclude from (6) and (7) that $B + (1-\lambda)c + \lambda c' \subset A$, hence that $z = (1-\lambda)c + \lambda c' \in C$, and finally that C is convex. Since $B + C \subset A$, we need to prove that $A \subset B + C$. Let $x \in A$. Since A is convex there exist $a, a' \in \mathrm{Fr}A$ and $\lambda \in [0,1]$ such that $x = (1-\lambda)a + \lambda a'$. Then by hypothesis there exist $y, y' \in Y$ such that $a \in B + y \subset A$ and $a' \in B + y' \subset A$. Thus there exist $b, b' \in B$ such that $a = b + y$ and $a' = b' + y'$ and $x = (1-\lambda)a + \lambda a' = b'' + (1-\lambda)y + \lambda y'$, where $b'' = (1-\lambda)b + \lambda b'$. Thus $x \in B + (1-\lambda)y + \lambda y'$. Since $y, y' \in C$ and C is convex, $u = (1-\lambda)y + \lambda y' \in C$. Therefore $x \in B + C$, which finishes the proof of (a).

To prove (b) we apply similar arguments, with $\{x\colon B+x \subset A\}$ replaced by $\{x\colon A-x \subset B\}$ in the second part of the proof. □

It is easy to see that

(8) If $B \in \mathscr{C}_{ob}(Y)$ and $y \in Y$ then $(B+y) \ominus B = \{y\}$. In particular $A \ominus A = \{\theta\}$.

(9) If $A \ominus B$ exists, then $d_H(A,B) = \|A \ominus B\|$, where $\|C\| = d_H(C,\{\theta\})$ for $C \subset Y$.

(10) If $Y = \mathbb{R}$ and $A, B \in \mathscr{C}_{ob}(\mathbb{R})$, then $A \ominus B$ exists and

$$A \ominus B = [(a-b) \wedge (x-y), (a-b) \vee (x-y)],$$

where $A = [a,x]$ and $B = [b,y]$.

Now we can present a definition of derivative of a multifunction.

Definition 10.4. A multifunction $F\colon I \rightsquigarrow Y$ with $F(x) \in \mathscr{C}_{ob}(Y)$ is said to be differentiable at a point $x_0 \in I$ if there exists a set $DF(x_0) \in \mathscr{C}_{ob}(Y)$ such that the limit (with respect to the Hausdorff metric)

$$\lim_{x \to x_0} \frac{F(x) \ominus F(x_0)}{x - x_0}$$

exists and is equal to $DF(x_0)$.

The set $DF(x_0)$ will be called the derivative of F at x_0. F will be called differentiable if it is differentiable at every point $x \in I$.

Of course, implicit in the definition of $DF(x_0)$ is the existence of the differences $F(x) \ominus F(x_0)$.

Example 10.5. A multifunction $F\colon [0,1] \rightsquigarrow \mathbb{R}^2$ defined by the formula

$$F(\alpha) = \{(x,y) \in \mathbb{R}^2\colon 0 \leq x \leq 1, 0 \leq y \leq \alpha - \alpha x\}$$

is not differentiable, since the required differences do not exist (see Example 10.4).

Theorem 10.4. *If a multifunction $F\colon I \rightsquigarrow Y$ with closed, bounded and convex values is differentiable at $x_0 \in I$, then it is h-continuous at this point.*

Proof. Suppose F is differentiable at x_0. Then we can assume that the differences $F(x) \ominus F(x_0)$ exist for x in some neighbourhood of x_0. Let $x \neq x_0$. By the differentiability of F at x_0, there exists a set $DF(x_0) \in \mathscr{C}_{ob}(Y)$ such that

$$\lim_{x\to x_0} d_H\left(\frac{F(x)\ominus F(x_0)}{x-x_0}, DF(x_0)\right) = 0. \tag{11}$$

Then (see (9))

$$\begin{aligned}(12)\quad d_H(F(x),F(x_0)) = \|F(x)\ominus F(x_0)\| = \left\|\tfrac{F(x)\ominus F(x_0)}{x-x_0}\right\| |x-x_0| \le \\ \le \left(d_H\left(\tfrac{F(x)\ominus F(x_0)}{x-x_0}, DF(x_0)\right) + \|DF(x_0)\|\right)|x-x_0|.\end{aligned}$$

Since the set $DF(x_0)$ is bounded, (11) and (12) shows that $d_H(F(x),F(x_0))$ converges to zero as x tends to x_0. Hence, F is h-continuous at x_0, by (4). $\square$

Definition 10.5. A multifunction $G\colon I \rightsquigarrow Y$ will be called a derivative if there exists a differentiable multifunction $F\colon I \rightsquigarrow Y$ with $G(x) = DF(x)$ for each $x \in I$.

Example 10.6. Let S be the closed unit ball in Y, and consider a multifunction $F\colon (0,2\pi) \rightsquigarrow \mathbb{R}^2$ defined by

$$F(x) = (2+\sin x)S.$$

Then F is differentiable and $DF(x) = (\cos x)S$ for each $x \in (0,2\pi)$ and the multifunction $G\colon (0,2\pi) \rightsquigarrow Y$ given by $G(x) = (\cos x)S$ is a derivative.

Now we will deal with the case when $Y = \mathbb{R}$. Let $F\colon I \rightsquigarrow Y$ be a multivfunction with compact and convex values. Then

$$F(x) = [i(x), s(x)], \tag{13}$$

where $i(x) = \inf F(x)$ and $s(x) = \sup F(x)$ for $x \in I$.

It should be noted that in this case $F(x)\ominus F(x_0)$ exists for $x \in I$ and

$$\frac{F(x)\ominus F(x_0)}{x-x_0} = \begin{cases} \left[\frac{i(x)-i(x_0)}{x-x_0}, \frac{s(x)-s(x_0)}{x-x_0}\right], & \text{if } \delta F(x) \ge \delta F(x_0), x > x_0, \\ \left[\frac{s(x)-s(x_0)}{x-x_0}, \frac{i(x)-i(x_0)}{x-x_0}\right], & \text{if } \delta F(x) \ge \delta F(x_0), x < x_0, \\ \left[\frac{s(x)-s(x_0)}{x-x_0}, \frac{i(x)-i(x_0)}{x-x_0}\right], & \text{if } \delta F(x) \le \delta F(x_0), x > x_0, \\ \left[\frac{i(x)-i(x_0)}{x-x_0}, \frac{s(x)-s(x_0)}{x-x_0}\right], & \text{if } \delta F(x) \le \delta F(x_0), x < x_0, \end{cases} \tag{14}$$

where δA denotes diameter of A.

It can be verified without difficulty that

Theorem 10.5. *If the functions $i\colon I \to \mathbb{R}$ and $s\colon I \to \mathbb{R}$ are differentiable at $x_0 \in I$, then multifunction F given by (13) is differentiable at x_0 and*

$$DF(x_0) = \begin{cases} [i'(x_0), s'(x_0)], & \text{if } i'(x_0) \le s'(x_0), \\ [s'(x_0), i'(x_0)], & \text{if } i'(x_0) > s'(x_0). \end{cases}$$

However, in general, differentiability of F does not imply differentiability of the functions i or s as the following example shows:

$$F(x) = \begin{cases} [0, x], & \text{if } x \ge 0, \\ [x, 0], & \text{if } x < 0. \end{cases}$$

But in the case F is differentiable at x_0, both functions i and s are either simultaneously differentiable at x_0 or simultaneously nondifferentiable at x_0.

As a consequence of Theorem 10.4 we have the following property.

(15) If a multifunction $F\colon I \rightsquigarrow \mathbb{R}$ given by (13) is differentiable at a point $x_0 \in I$, then F is h-continuous at x_0 and consequently the functions i and s are continuous at x_0.

Let us suppose that the multifunction F given by (13) is differentiable at a point $x_0 \in I$. According to Definition 10.4, there is a set $DF(x_0) \in \mathscr{C}_{ob}(\mathbb{R})$ such that

(16) $\lim\limits_{x \to x_0} \frac{F(x) \ominus F(x_0)}{x - x_0} = DF(x_0).$

This condition can be reinterpreted in terms of Dini derivatives of the functions i and s.

10.5 A new notion of derivative of functions

Garg in [8] has presented a unified theory of Dini derivatives and a theory of some new derivatives of functions. After the discovery of Weierstrass, it became well known that there are continuous functions that are not derivable at any point. The same holds in terms of various generalized derivatives that are known, e.g. the Dini, approximate and symmetric derivatives. Garg showed that in terms of new derivatives every continuous function is derivable at a set of points which has cardinality continuum in every interval, and the properties of f can in turn be investigated in terms of the values of its new derivative. Many of the known results in differentiation theory, like the mean value theorems and the Darboux property of derivative, are found to hold in terms of new derivatives without any derivability hypothesis.

Let $f\colon I \to \mathbb{R}$ be a function and $x \in \text{Int}I$. We will use $f'_-(x)$ and $f'_+(x)$ to denote the left-side and right-side derivatives of f at x, $D_- f(x)$, $D^- f(x)$, $D_+ f(x)$

and $D^+f(x)$ to denote the left and right lower and upper Dini derivatives of f at x. Further, given $x,y \in I$, $x \neq y$, we will use $Qf(x,y)$ to denote the following difference quotient of f on $[x,y]$ or $[y,x]$:

$$Qf(x,y) = \frac{f(x) - f(y)}{x - y}.$$

Following Garg we are accepting the following definitions.

Definition 10.6. A number $c \in \overline{\mathbb{R}} = [-\infty,\infty]$ is called a lower (resp. upper) gradient of f at $x \in \mathrm{Int}I$ if $D^-f(x) \leq c \leq D_+f(x)$ ($D^+f(x) \leq c \leq D_-f(x)$).

For example, for the norm function $f(x) = |x|$ each element of the interval $[-1,1]$ is a lower gradient of f at 0.

The lower and upper derivatives of f are also defined in terms of its Dini derivatives.

Definition 10.7. A function $f\colon I \to \mathbb{R}$ is lower (resp. upper) derivable at a point $x \in \mathrm{Int}I$, if $D^-f(x) \leq D_+f(x)$ (resp. $D^+f(x) \leq D_-f(x)$), and then the interval $[D^-f(x), D_+f(x)]$ (resp. $[D^+f(x), D_-f(x)]$) is called the lower (resp. upper) derivative of f at x and denoted by $Lf'(x)$ (resp. Uf'(x)). So $Lf'(x) = [D^-f(x), D_+f(x)]$ (resp. $Uf'(x) = [D^+f(x), D_-f(x)]$. We call further f semi-derivable at x, if it is either lower or upper derivable at x, and then its lower or upper derivative at x is called the semi-derivative of f at x and denoted by $Sf'(x)$.

When the lower, upper or semi-derivative of f at x is a singleton, f is said to be uniquely lower, upper or semi-derivable , respectively, at x, and then $Lf'(x)$, $Uf'(x)$ or $Sf'(x)$ are also used to denote its unique element.

Further, when f has a finite lower, upper or semi-gradient at x, f is called lower, upper or semi-differentiable, respectively, at x; and when this gradient is further unique, f is called uniquely lower, upper or semi-differentiable, respectively, at x. Also, when f has a finite ordinary derivative at x, f is called simply differentiable at x.

As the lower, upper and semi-derivatives are set-valued, and they are not defined in terms of a limit, the nature of results on these derivatives are quite different from the usual results. They include, however, most of the results on the ordinary derivative. We quote now these properties which will be essential for the proof of the main theorem our paper .

Let f be a function on $\mathbb{R}$ and $I = [x_1, x_2] \subset \mathbb{R}$. Moreover $\Delta_S(f)$ will denote the set of all points in I where f is semi-derivable.

Theorem 10.6. *[[8], Theorem 8.1.2] If f is continuous relative to I, then there is a point $x \in (x_1, x_2)$ such that f is semi-derivable at x and $Qf(x_1, x_2) \in Sf'(x)$.*

Theorem 10.7. *[[8], Theorem 10.4.1] The semi-derivatives of every continuous relative to I function f posses the Darboux property in the following sense: for each connected set $C \subset \mathbb{R}$ the set $\bigcup\{Sf'(x) \colon x \in C \cap \Delta_S(f)\}$ is connected in the set $\overline{\mathbb{R}}$.*

Finally we define some set valued medians of f which are associated with lower and upper derivatives of f.

Definition 10.8. If $D_+f(x) \le D^-f(x)$, then the interval $[D_+f(x), D^-f(x)]$ will be called the lower median of f at x; and when $D_-f(x) \le D^+f(x)$, the interval $[D_-f(x), D^+f(x)]$ will be called the upper median of f at x. We will use $\underline{M}f(x)$ and $\overline{M}f(x)$ to denote the lower and upper median, respectively, of f at the point x.

Now let us return to the multifunction F given by (13).

Let us suppose that $x_0 \in \mathrm{Int}I$, F is differentiable at x_0 and $DF(x_0) = [a,b]$, where $a,b \in \mathbb{R}$ and $a \le b$. Then (14) and (16) force a and b to be the only limit points of $Qi(x,x_0)$ and $Qs(x,x_0)$.

If $a = b$, then the four Dini derivatives of i and s at x_0 are equal, and hence the functions i and s are differentiable at x_0 with $i'(x_0) = s'(x_0)$.

If $a < b$ and the functions i and s are not differentiable at x_0, then they have a semi-derivative at x_0 or a lower or upper median at x_0. We consider this in four basically different cases.

Case **(i)**: There exists $h > 0$ such that $\delta F(x) \ge \delta F(x_0)$ for each point $x \in (x_0, x_0+h)$ and $\delta F(x) \le \delta F(x_0)$ for each $x \in (x_0-h, x_0)$. Note that in this case (16) holds if and only if $D_+i(x_0) = D^+i(x_0) = a$, $D_+s(x_0) = D^+s(x_0) = b$, and $D_-i(x_0) = D^-i(x_0) = a$, $D_-s(x_0) = D^-s(x_0) = b$. Thus the functions i and s are differentiable at the point x_0 and $DF(x_0) = [i'(x_0), s'(x_0)]$. Of course, $Li'(x_0) = Ui'(x_0) = a$ and $Us'(x_0) = Ls'(x_0) = b$.

Case **(ii)**: There exists $h > 0$ such that $\delta F(x) \ge \delta F(x_0)$ for each $x \in (x_0, x_0 + h)$ and $\delta F(x) \ge \delta F(x_0)$ for each $x \in (x_0-h, x_0)$. In this case (16) holds if and only if $D_+i(x_0) = D^+i(x_0) = a$, $D_+s(x_0) = D^+s(x_0) = b$, and $D_-s(x_0) = D^-s(x_0) = a$, $D_-i(x_0) = D^-i(x_0) = b$. Thus the function i is upper derivable at x_0, the function s is lower derivable at x_0, and $Ui'(x_0) = [a,b] = Ls'(x_0)$, and $DF(x_0) = [i'_+(x_0), s'_+(x_0)] = [s'_-(x_0), i'_-(x_0)]$.

Case **(iii)**: There exists $h > 0$ such that $\delta F(x) \ge \delta F(x_0)$ for each $x \in (x_0, x_0 + h)$ but for each $h > 0$ there exists $x \in (x_0-h, x_0)$ such that $\delta F(x) \ge \delta F(x_0)$ and there exists $x' \in (x_0-h, x_0)$ such that $\delta F(x') < \delta F(x_0)$. In this case (16) holds if and only if $D_+i(x_0) = D^+i(x_0) = a$ and $D_+s(x_0) = D^+s(x_0) = b$, $D_-s(x_0) = a$ and $D^-s(x_0) = b$, $D_-i(x_0) = a$ and $D^-i(x_0) = b$. Thus $Ui'(x_0) = a$, $Ls'(x_0) = b$, and $DF(x_0) = \underline{M}i(x_0) = \overline{M}s(x_0) = [i'_+(x_0), s'_+(x_0)]$.

Case **(iv)**: For each $h > 0$ there exists $x \in (x_0, x_0 + h)$ such that $\delta F(x) \geq \delta F(x_0)$ and there exists $x' \in (x_0, x_0 + h)$ such that $\delta F(x') < \delta F(x_0)$, and for each $h > 0$ there exists $x \in (x_0 - h, x_0)$ such that $\delta F(x) \geq \delta F(x_0)$ and there exists $x' \in (x_0 - h, x_0)$ such that $\delta F(x') < \delta F(x_0)$. In this case (16) holds iff $D_+ i(x_0) = a$ and $D^+ s(x_0) = b$, $D_+ s(x_0) = a$ and $D^+ i(x_0) = b$, $D_- s(x_0) = a$ and $D^- i(x_0) = b$, $D_- i(x_0) = a$ and $D^- s(x_0) = b$. Thus neither the function i nor the function s is semi-derivable at x_0, and $\underline{M} i(x_0) = \overline{M} i(x_0) = \underline{M} s(x_0) = \overline{M} s(x_0) = DF(x_0)$.

10.6 The $\mathscr{D}_*$ property of derivative multifunction

Now we restrict our attention to the well-known result on ordinary derivative of functions, namely the intermediate value property of derivative. We will extend this result to the multivalued case.

Theorem 10.8. *Suppose $F\colon I \rightsquigarrow \mathbb{R}$ is a multifunction with compact and convex values. If F is a derivative, then F has the intermediate value property.*

Proof. Assume the contrary. Then

(17) there exist two distinct points $x_1, x_2 \in I$, say $x_1 < x_2$, and a point $y_1 \in F(x_1)$ such that for any $y \in F(x_2)$ there exists a number α with $\alpha \in (y_1 \wedge y, y_1 \vee y) \setminus F((x_1, x_2))$.

Obviously $y_1 \notin F(x_2)$. Let $y_2 = \inf F(x_2)$. We have either $y_1 < y_2$ or $y_1 > y_2$. Let us suppose that $y_1 < y_2$ and

(18) $\alpha \in (y_1, y_2) \setminus F((x_1, x_2))$.

On the other hand, by hypothesis, there is a differentiable multifunction $\Phi\colon I \rightsquigarrow \mathbb{R}$ such that $F(x) = D\Phi(x)$ for each $x \in I$. It follows from Theorem 10.4 that Φ is h-continuous. Assume $\Phi(x) = [i(x), s(x)]$ (see (13)). Then the functions i and s are continuous (see (15)).

Let

$$K = \bigcup \{ S i'(x) \colon x \in (x_1, x_2) \cap \Delta_S(i) \}$$

and

$$L = \bigcup \{ S s'(x) \colon x \in (x_1, x_2) \cap \Delta_S(s) \},$$

where $\Delta_S(i)$ and $\Delta_S(s)$ denote the sets of points at which the functions i and s, respectively, are semi-derivable. By Theorem 10.7, both sets K and L are connected.

Let us notice that

(19) If $x \in [x_1, x_2]$ and $z \in \{D^+i(x), D_+i(x), D^-i(x), D_-i(x)\}$, then z is a limit point of K.

(20) If $x \in [x_1, x_2]$ and $z \in \{D^+s(x), D_+s(x), D^-s(x), D_-s(x)\}$, then z is a limit point of L.

Indeed, without loss of generality we can assume that $z = D^+i(x)$. Thus there is a sequence $(x_n)_{n\in\mathbb{N}}$ which converges to x from the right such that

(21) $\lim_{n\to\infty} Qi(x, x_n) = z.$

We conclude from Theorem 10.6 that for each $n \in \mathbb{N}$ exists $y_n \in (x, x_n)$ such that the function i is semiderivable at y_n and

(22) $Qi(x, x_n) \in Si'(y_n) \subset K.$

By (21) and (22) we have (19). Similarly we can schow (20). Suppose that $F(x_1) = [p, q]$ and $F(x_2) = [y_2, r]$. Then according to (18) we have

(23) $p \leq y_1 < \alpha \leq y_2 \leq r.$

Let us suppose that $p \in \{D_+i(x_1), D^+i(x_1)\}$. One of the points y_2 or r belongs to the set $\{D_-i(x_2), D^-i(x_2)\}$. Suppose y_2. Then according to (19) p and y_2 are the limit points of K. The set K is connected. Therefore $(p, y_2) \subset K$ and, by (23), $\alpha \in K$. Similarly if $p \in \{D_+s(x_1), D^+s(x_1)\}$, then $\alpha \in L$. Therefore

(24) $\alpha \in K \cup L.$

Let us note that $K \cup L \subset F((x_1, x_2))$. So, by (24) $\alpha \in F((x_1, x_2))$. But this contradicts (18).

We obtain a similar conclusion when $y_1 > y_2$. This completes the proof of Theorem 10.8. □

Observe that, by Theorem 10.1 and Theorem 10.8, we have the following Corollary.

Corollary 10.1. *If $F : I \rightsquigarrow \mathbb{R}$ is a derivative multifunction with compact and connected values, then F has the Darboux property.*

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