SOME PROBLEMS IN TESTING OF COINTEGRATION
Employment and production example

In the recent years more and more interest has been aroused by the investigations into the cointegration of variables and it is increasingly often treated as a typical stage of analysing time series. The popular methods of investigating the cointegration between variables is the Engle and Granger procedure being a two-step model estimation, where at the first stage the parameters of the long term equilibrium are estimated using the least squares method, and then stationarity of the residuals estimated at the first stage is examined. Engle and Granger procedure is a very simple tool used to examine cointegration, yet it has some drawbacks. Since the beginning of nineties the most popular method of testing cointegration has been the Johansen procedure (see S. Johansen [1988], pp. 231-54) based on the estimation of the vector autoregressive models (VAR) and precisely on its representation of error correction - VECM, that is:

$$\Delta x_t = \Psi_0 D_t + \Pi x_{t-1} + \sum_{i=1}^{k-1} \Pi_i \Delta x_{t-i} + \in_t \tag{1.1}$$

where:

Δx_t - vector of dimensions (n×1) with first differences of $I(1)$ variables included into VAR model: $x_t = [x_{1t} \quad x_{2t} \quad \ldots \quad x_{nt}]$;

Ψ_0 - vector of parameters of deterministic variables;

Π_i- matrix of parameters of lagged variables of vector Δx_t;

$\in_t$ - is (n×1) vector of random disturbances.

Π - is matrix of our special interest, because Johansen proved that in the multidimensional case, the rank of matrix Π can be used to investigate cointegration, which rank is equal to the number of independent cointegrating vectors. Particularly, three statements can be proved:

1. If rank Π is 0, then all elements of this matrix have to be zero and model (1.1) is a typical VAR model on first differences, for which no error correction representation exists. Therefore, linear combination of these variables, which is stationary, does not exist;
2. If matrix Π is of full rank then series of vector x_t are stationary;

3. If rank Π is one, then there is only one cointegrating vector and the term Πx_{t-1} is the factor of error correction. In the other cases, i.e. $1 < rank(\Pi) < n$ there are many cointegrating vectors.

According to the fact that the rank of matrix is equal to the number of its characteristic roots that differ from zero, we can use these roots to test cointegration. In practice, we can obtain only estimates of Π and the characteristic roots. So it should be provided suitable a test for the number of characteristic roots that are significantly different from zero[1].

But it should be stressed that result of testing matrix Π also depends on the position of deterministic regressors in vector D_t. Localisation of intercept and time trend in VECM have a meaning for process generating variables x_t. For example constant term in vector D_t in the equation (1.1) will generate a linear trend in the x_t process, and a linear term will generate a quadratic trend in the process x_t. We can avoid these effects by excluding these terms from the equation or including constant, trend or both into cointegrating vector. In every situation we will have different consequences.

The issue of the intercept can be solved in three ways. Namely, it is possible:

(a) Omit the intercept;

(b) Include the intercept as an unrestricted form[2], bearing in mind at the same time that this method will not make its representation appear in the cointegrating vector(s);

(c) Include the intercept in the cointegrating vector(s).

Apart from the techniques of making the intercept appear in the cointegrating vector[3], the selection of VAR model from among (a), (b) and (c) is important because of the assumptions concerning the variable generating processes. We will show it using the example of the model:

$$\begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} -0.5 & 1.25 \\ 0.2 & -0.5 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{yt} \\ e_{zt} \end{bmatrix} \qquad (1.1a)$$

which is a form of model (1.1), for k=1 and A_0=0.

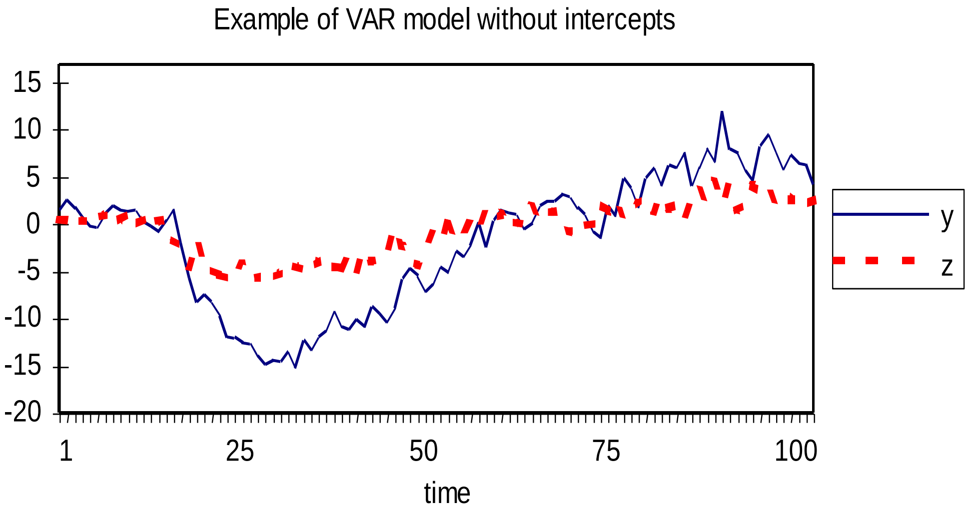
[1] I mentioned the most popular test statistic such a λ_{trace} or λ_{max} . Their description one can find in majority adequate handbooks, for instance W. Enders [1995], p. 391 or S. Johansen [1996], p. 97

[2]Here, an unrestricted term means that this term is including to the vector D_t, restricted term means that it is including in cointegrating vector, no term means that it is excluding from the whole VECM expression.

[3] One of these techniques is shown in i.e. E. Kusideł [1997], s. 9.

As the rank of matrix Π is 1, we find that there exists one cointegrating vector. In graph 1.1a we see the diagram of 100 realisations of variables y_t and z_t from model (1.1a):

Graph 1.1a



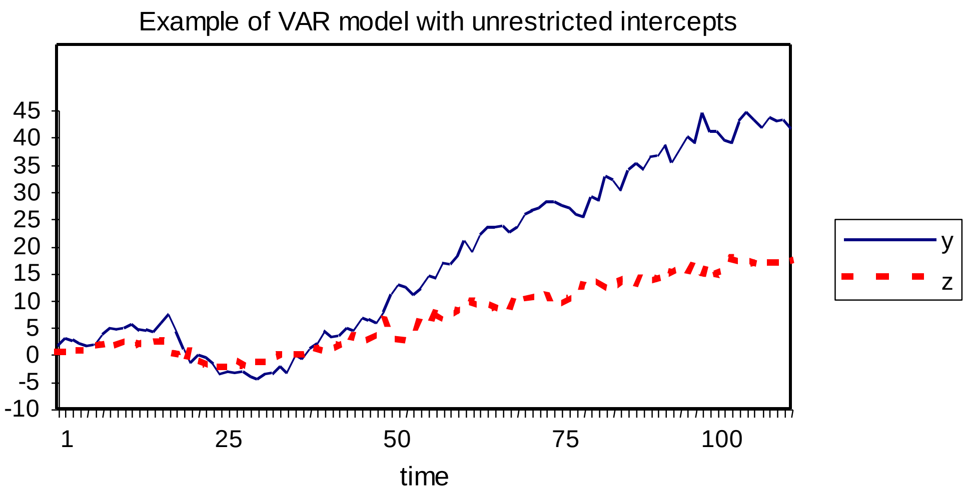
Series of data in graph 1.1a resemble the random walk process.

The course of the variables becomes a totally different if the intercept is added to the variable generating process, for instance:

(1.1b)
$$\begin{bmatrix}\Delta y_t\\ \Delta z_t\end{bmatrix}=\begin{bmatrix}0.5\\ 0.1\end{bmatrix}+\begin{bmatrix}-0.5 & 1.25\\ 0.2 & -0.5\end{bmatrix}\begin{bmatrix}y_{t-1}\\ z_{t-1}\end{bmatrix}+\begin{bmatrix}e_{yt}\\ e_{zt}\end{bmatrix}$$

Model (1.1b) is a classical case of model (1.1) with unrestricted intercept , for k=1. The course of variables for model (1.1b) is illustrated in graph 1.1b:

Graph 1.1b

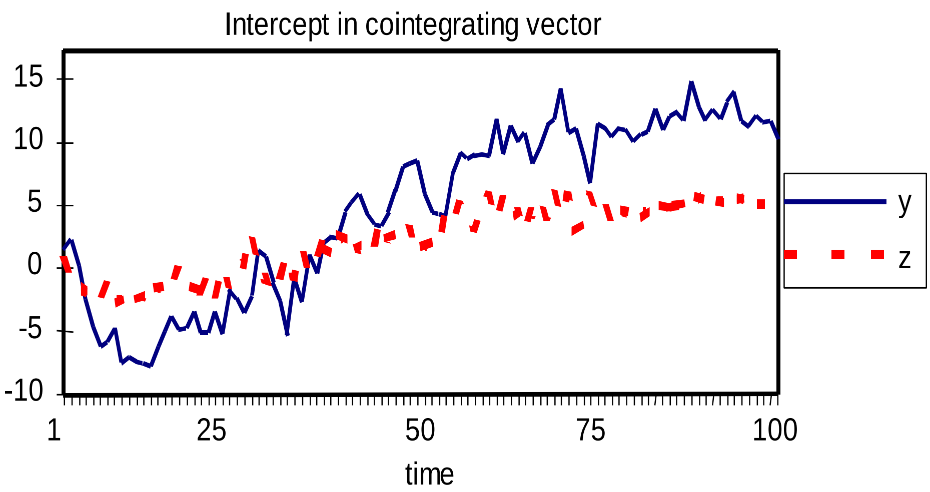


In this case variables are also shaped in a similar way, but the processes that generate them are characterised by a clear deterministic trend.

Finally, we can choose a model with the intercept included in the cointegrating vector (restricted intercept). In the case of one cointegrating vector (r=1) it is known that particular ranks of matrices Π may differ only by the scalar. Having divided the first row of matrix Π in model (1.1a) we have the value of c=-2.5. Therefore, if the intercepts will differ only by c=-2.5, we can write the formula of the model with the intercept included in the cointegrating vector. For example:

$$\text{(1.1c)} \qquad \begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} -0.5 & 1.25 & -0.25 \\ 0.2 & -0.5 & 0.1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \\ 1 \end{bmatrix} + \begin{bmatrix} e_{yt} \\ e_{zt} \end{bmatrix}$$

Graph 1.1c



Graph 1.1c showing the variables changes in the model (1.1c) is similar to graph 1.1a. Variables are shaped similarly and they resemble the random walk process (the emergence of upward trend is only apparent, as plotting the next, say 100, of realisations of variables shows cyclically declining and climbing curves).

The examples given were to show that it is important how the intercept is included into the VECM model as the process entails a variety of consequences and interpretations. Firstly, the parameter estimation of a model with the so-called unrestricted intercept (approach b) is identical with taking the assumption on the occurrence of the deterministic (linear) trend in the variable generating processes. If the intercept is skilfully included into the cointegrating vector then the same system of equations will not show either climbing or declining trends. Secondly, the probability of finding a stationary combination of n variables is higher in the case of a cointegrating vector with the intercept than in the case when it is absent therein. It is important when testing for the numbers of the cointegrating vectors, the number of which can be artificially overstated in the model with the intercept in the cointegrating vector. Because of the aforementioned reasons the form of the VAR model should be selected cautiously, especially when the analysis of detecting the cointegrating vectors is to further serve building the ECM and its economic interpretation.

The problem is more complicated if we allow for appearing a linear trend in model (1.1). Generally, we should take into consideration 5 forms of VECM:

1. Model with unrestricted constant and unrestricted trend. As it was mentioned above constant term in vector D_t of the equation (1.1) will generate a linear trend in the x_t process, and a linear term will generate a quadratic trend in the process x_t;
2. Model with unrestricted constant and restricted trend. It still allows the possibility of linear trend in all components of processes x_t, a trend which cannot be eliminated by the cointegrating relations. Thus a linear trend is allowed even in the cointegrating relations, each of which therefore represents a stationary process plus a linear trend or a trend stationary process;
3. Model with unrestricted constant and no trend is the same as approach (b) above. It still has a linear trend, but this can be eliminated by the cointegrating relations, and the process contains no trend stationary components. Thus the model allows for linear trend in each variable but not in the cointegrating relations;
4. Model with restricted constant and no trend- see approach (c)- which does not allow for linear trend in process;
5. Model with no constant and no trend, as in (a), which does not allow for linear trend in process and means that all stationary linear combinations will have mean zero.

In summary we have seen the crucial role of the deterministic part of the VECM. By suitably restricting the deterministic terms we can use equation (1.1) to generate processes with different trending behaviour. When selecting or restricting the model, it may be helpful to use statistics testing the restricted cointegrating vector(s), i.e. the set of hypotheses:

H_0: model with restricted intercept or linear trend (i.e. included into the cointegrating vector)

H_1: model with unrestricted intercept or linear trend

To apply the test both model's forms should be estimated. We denote characteristic roots of matrix Π with restricted intercept or linear trend**,** ordered in the declining order, as $\lambda_1, \lambda_2, ..., \lambda_n$, and roots of restricted matrix Π as $\lambda^*_1, \lambda^*_2, ..., \lambda^*_n$,. If the unrestricted form of the model has *r* of non-zero characteristic roots then statistic:

(1.1a) $$-T \sum_{i=r+1}^{n} [\ln(1-\lambda^*_i) - \ln(1-\lambda_i)]$$

has asymptotic distribution χ^2 with $(n-r)$ degree of freedom.

If the calculated value is lower than the critical value then we have no grounds for rejecting the zero hypothesis. Values bigger than critical, result in the rejection of the zero hypotheses in favour of the alternative one.

The reliability of the test can be verified using simulated data. To do this the random walk and drift random walk processes were generated. Then they were aggregated into cointegrated groups. The application of test (1.1a) in the case when at least one of the variables revealed a deterministic trend allowed to reject the zero hypotheses in 99% of cases on the significance level α=0.1 and in w 94 per cent of cases on the significance level α=0.05. In the cases when all variables were pure random walk there were no grounds for rejecting the zero hypothesis on the significance level α=0.05 in almost 100 per cent. A similar investigation was run for non-cointegrated variables (but integrated of the first order). Also in this case the test detected the real form of the model in almost 100 per cent of cases, i.e. in the case of presence of the deterministic trend, although the zero hypothesis could be rejected for one variable. The above research made with artificial (generated) data proves that statistic (1.1a) detects the „true” form of the VAR model very well. The application of an appropriate functional form of the VAR model is important for the investigation into the number of the cointegrating vectors, the more that the inclusion of the intercept into the cointegrating vector may artificially overstate their value.

We can also test another set of hypotheses:

H_0: model with no intercept or linear trend term (i.e. excluded from whole model)

H_1: model with restricted (included into the cointegrating vector) intercept or linear trend term

The tests statistics involve comparing the number of cointegrating vectors under the null and alternative hypotheses. Again, let $\lambda_1, \lambda_2, ..., \lambda_n$ and $\lambda^*_1, \lambda^*_2, ..., \lambda^*_n$, denote the ordered characteristic root of the matrix Π with unrestricted cointegrating vector(s) and roots of matrix Π with restricted cointegrating vector(s) (i.e. with no intercept or linear trend in whole model). If the model (matrix Π) from H_1 has r of non-zero characteristic roots then to test restrictions on cointegrating vector(s) we use test statistic:

$$T\sum_{i=1}^{r}[\ln(1-\lambda_i^*)-\ln(1-\lambda_i)] \qquad (1.1a)$$

Asymptotically, this statistic has χ^2 distribution with degrees of freedom equal to the number of restrictions placed on cointegrating relations. The restriction embedded in the null hypothesis is binding if the calculated value of the test statistic exceeds critical value.

As an empirical example of consequences of placed deterministic regressors in VECM equations we present the analysis which was made for the following variables:

spu - revenues from the sale of products and services in the public sector in the Łódź region

zpu - number of employees in the public sector in the Łódź region

The data have the form of time series with monthly frequency and cover the period January 1994 through to June 1998 (54 observations). They are derived from the Bulletin of the Łódź Statistical Office. Variables are expressed in logarithms, and *spu* in real terms (fixed prices of January 1994).

Relevant tests (*DF* , *ADF*, autocorrelation function) indicate that variables are $I(1)$ but linear time trend is also visible. We build for these variables VAR model and on its base we build VECM, to check whether non- stationary variables are cointegrated.

The VECM model for these variables has the form:

$$\Delta x_t = \Psi_0 D_t + \Pi x_{t-1} + \sum_{i=1}^{5-1}\Pi_i \Delta x_{t-i} + \in_t \qquad (1.2)$$

where:

$x_t^{\mathrm{T}} = [zpu_t \;\; spu_t]$, $D_t = [1 \; t \; u_{1097}]$[4]

[4] To select the order of a lag of the model three alternative statistical tests were applied: *Likelihood ratio test* (*LR* test), *Akaike Information Criterion* (*AIC*) and *Schwartz-Bayesian Criterion* (*SBS*). Dummy variable: u_{1097} takes the value of unit in October 1997 and zero in other periods.

We decided to test 4 forms of the VECM model:

1. VECM with unrestricted constant and unrestricted linear trend;
2. VECM with unrestricted constant and restricted linear trend;
3. VECM with unrestricted constant and no linear trend;
4. VECM with restricted constant and no linear trend.

Taking into consideration our above remarks, we expected that the proper form of the VECM is model with unrestricted constant and no trend. The explanation of this choice is the fact, that in level of variables is no quadratic trend (as in VECM1), we do not expect that cointegrating relation is trended (as in VECM2), but we know that linear time trend is also visible in the diagram of variables.

This choice we will try to confirm by suitable tests. Before we use them we should calculate characteristic roots of matrix Π from all above mentioned specification of VECM. To establish the number of non-zero characteristic roots we have used λ_{max} and λ_{trace} statistics. The results of testing the cointegrating vectors are shown in the table 1.2

Table 1.2. Results of testing cointegration relations in AIDM.

Model specification	Ordered characterised roots		Rank of matrix Π
VECM 1	λ_1=0,38572	λ_2=0,039295	r=1
VECM 2	λ_1=0,39738	λ_2=0,039885	r=1
VECM 3	λ_1=0,35777	λ_2=0,007018	r=1
VECM 4	λ_1=0,59738	λ_2=0,276420	r=2

Source: Authors' calculations.

From table 1.2 we can see that form of VECM has great importance for result of testing of cointegration. We can realise existing one or two stationary relationships. Notice that acceptation of one cointegrating vector means that variables x_t are nonstationary and only cointegrating relation is stationary, whereas acceptation of two vectors means that variables in x_t are stationary. This latest conclusion arouses from the basis of the Johansen assumption that if matrix Π is of full rank then series of vector x_t are stationary.

The above results show that presumption of one of the form of VECM model *ad* hoc is not reasonable because of the possibility incorrect conclusion. So it is very desirable to choose specific form of the model before testing the cointegration. We can select it using test statistic (1.1a) and (1.1b). To test the hypothesis of appearance of deterministic component in cointegrating vectors (VECM1 form) against their appearance in unrestricted form (VECM2 form) we use test statistic (1.1a):

$$LR = -T\sum_{i=r+1}^{n}[\ln(1-\lambda_i^*) - \ln(1-\lambda_i)];$$

where r is number of non-zero characteristic roots of unrestricted form of the model- VECM1- and $\lambda_1, \lambda_2, ..., \lambda_n$, $\lambda_1^*, \lambda_2^*, ..., \lambda_n^*$, are ordered in the declining order roots of respectively unrestricted and restricted matrix Π. Because unrestricted form of the model has one non-zero root, so statistic (1.1a) has a value of:

$$LR = -T\sum_{i=r+1}^{n}[\ln(1-\lambda_i^*) - \ln(1-\lambda_i)] = -49[\ln(1-0{,}039885) - \ln(0{,}039295)] =$$

=-49*(-0,00061)= 0,0301.

Since calculated value is lower than the critical value (χ^2 distribution with n-r=2-1=1 degree of freedom which for 5% significant level is 3,841) then we have no grounds for rejecting the zero hypothesis about model with trend in cointegrating vector.

To test form without linear trend (VECM3), e.g. zero restriction for parameter of cointegrating relation against VECM2, one should calculate test statistic (1.1b). Its value is:

$$LR = T\sum_{i=1}^{r}[\ln(1-\lambda_i^*) - \ln(1-\lambda_i)] = 49[\ln(1-0{,}35777) - \ln(0{,}39738)] = 49 * 0{,}0637 = 3{,}119$$

This value is also lower than the 5% critical value of χ^2 distribution with n-r=2-1=1 degree of freedom thus we have no grounds for rejecting the zero hypothesis about model with no linear trend.

Testing zero hypothesis that the proper form is VECM4 (with constant in cointegrating vector and no trend) against form VECM2 (with unrestricted constant and no trend) we should again calculate value of (1.1a):

$$LR = -T\sum_{i=r+1}^{n}[\ln(1-\lambda_i^*) - \ln(1-\lambda_i)] = \text{-}49[\ln(1\text{-}0{,}27642)\text{-}\ln(0{,}007018)] = \text{-}49*(\text{-}0{,}3165) =$$

$$=15{,}509$$

Obtained value is bigger than critical one. Zero hypothesis of VECM4 form (with constant in cointegrating vector) should be rejected.

Considering above results we can realise that the proper specification for (1.2) is model with unrestricted intercept and with no trend, that is:

$$\text{(1.2a)} \quad \Delta x_t = \Psi D_t + \Pi x_{t-1} + \sum_{i=1}^{4}\Pi_i \Delta x_{t-4} + \epsilon_t \ ;$$

where x_t =[spu_t zpu_t]’, D_t=[1 u_{1096}], t=1,2,...,T.

Summing up the above remarks it should be stressed that it is very important to correctly ascertain the form of the deterministic regressors because it has a meaning for the number (and form) of cointegrating relationships. To settle this terms in unrestricted or restricted form we can use either knowledge about process generating variables or suitable tests statistics, keeping in mind that results of this two "methods" should be adequate.

References

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