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Chapter 14
New properties of the families of convergent and divergent permutations - Part II

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14.1 Introduction

Notation and terminology used in this paper are entirely adopted from Part I of this study [1].

We begin with the following lemma which gives the combinatoric characterization of divergent permutations used throughout the current paper.

Lemma 14.1. *If $p \in \mathfrak{D}$ then, for any $k,n \in \mathbb{N}$, there exists an interval I such that*

(i) $I > k$ *and* $p(I) > k$,
(ii) *the set* $p(I)$ *is a union of at least* n **MSI**,
(iii) *the set* $p(I)$ *contains an interval* J *having the cardinality* $\geq n$.

Proof. Let $k,n \in \mathbb{N}$. Let us choose some $t \in \mathbb{N}$ such that

(1) $p(T) > k$,

(2) $p^{-1}\left([\min p(T), +\infty)\right) > k$,

where $T := [t, +\infty)$.

Since $p \in \mathfrak{D}$ then there exists an interval $U^* \subset \mathbb{N}$ such that the set $p(U^*)$ is a union of at least $(t-1+n(2n-1))$ **MSI**. Hence, the set $p(U)$ is a union of at least $n(2n-1)$ **MSI**, where $U := T \cap U^*$.

Suppose that the set $p(U)$ does not contain any interval having the cardinality $\geq n$. Then there exists an interval I satisfying the conditions:

(3) $\min U = \min I$,
(4) the set $p(I)$ includes an interval having the cardinality $\geq n$,
(5) for any proper subinterval J of the interval I, if $U \subset J$ then the set $p(J)$ does not contain any interval of the cardinality $\geq n$.

Notice that the set $p(I)$ includes precisely one interval having the cardinality $\geq n$. At worst, this interval could be constructed inserting a natural number between two intervals, both having the cardinality $(n-1)$. Hence, any interval contained in $p(I)$ has the cardinality $\leq 2n-1$. On the other hand, we have

$$\text{card}\, p(I) > \text{card}\, p(U) \geq n(2n-1).$$

This clearly implies that the set $p(I)$ is a union of at least n **MSI**. Moreover, we obtain

$$I \subset T \subset p^{-1}([\min p(T), +\infty)) > (\text{by (2)}) > k$$

and

$$p(I) \geq \min p(T) > (\text{by (1)}) > k,$$

i.e. $I > k$ and $p(I) > k$. So, I is the desired interval which terminates the proof. □

Remark 14.1. There exists a permutation $p \in \mathfrak{D}$ such that, for any interval I, the set $p(I)$ contains at most one interval J having the cardinality > 1 (see Example 1.3 in Part I).

Remark 14.2. More subtle, than the one given in Lemma 14.1, combinatoric characterizations of the divergent permutations are given in papers [2] and [3].

14.2 The description of the families $\mathfrak{A} \circ \mathfrak{B}$ for $\mathfrak{A}, \mathfrak{B} = \mathfrak{C}\mathfrak{C}$, $\mathfrak{C}\mathfrak{D}$, $\mathfrak{D}\mathfrak{C}$ or $\mathfrak{D}\mathfrak{D}$

In this section three main theorems of this study will be presented.

Theorem 14.1. *The product $\mathfrak{DC} \circ \mathfrak{DC}$ is equal to $\mathfrak{DC}$.*

Proof. Let $p, q \in \mathfrak{DC}$. Then

$$p^{-1}, q^{-1} \in \mathfrak{C} \quad \text{and} \quad (pq)^{-1} = q^{-1}p^{-1} \in \mathfrak{C}$$

because $\mathfrak{C}$ is a semigroup. Suppose that $pq \in \mathfrak{C}$. Then also

$$(pq)q^{-1} = p \in \mathfrak{C},$$

which is impossible. So $pq \in \mathfrak{DC}$ and the inclusion below holds true

$$\mathfrak{DC} \circ \mathfrak{DC} \subset \mathfrak{DC}.$$

Now let $p \in \mathfrak{DC}$. We show that there exist permutations $p_1, p_2 \in \mathfrak{DC}$ such that $p_2 p_1 = p$. Let us start with choosing two sequences I_k and J_k, $k \in \mathbb{N}$, of intervals of $\mathbb{N}$, satisfying, for every $k \in \mathbb{N}$, the following assumptions:

(1) $J_k < J_{k+1}$,
(2) $I_k < p^{-1}(J_k) < I_{k+1}$,
(3) the set $p(I_k)$ is an union of at least k **MSI**,
(4) there exists an interval $G_k \subset p^{-1}(J_k)$ such that the set $p(G_k)$ is an union of at least k **MSI**.

Next we define the permutation p_1. Let p_1 be an increasing map of the set $\bigcup_{k\in\mathbb{N}} p^{-1}(J_k)$ onto the set $\bigcup_{k\in\mathbb{N}} J_k$ and let $p_1(n) = p(n)$ outside the set $\bigcup_{k\in\mathbb{N}} p^{-1}(J_k)$. Then the permutation p_2 is given by $p_2 = pp_1^{-1}$.

By assumption (2), we have

$$p^{-1}(J_k) < p^{-1}(J_{k+1}),$$

for every $k \in \mathbb{N}$. Therefore, from assumptions (1) and (4) and from the definition of p_1, we see that the set $p_1(G_k)$ is an interval for every $k \in \mathbb{N}$. Furthermore, from assumption (2) and from the definition of p_1, we have

$$p_1(I_k) = p(I_k), \quad k \in \mathbb{N}.$$

From this and from assumptions (3) and (4) we conclude that any of the following sets:

$$p_1(I_k) \quad \text{and} \quad p_2 p_1(G_k) = p(G_k)$$

is a union of at least k **MSI**, for every $k \in \mathbb{N}$. This shows that the permutations p_1 and p_2 are both divergent.

Now, let $s \in \mathbb{N}$ be given so that the set $p^{-1}(I)$ is a union of at most s **MSI** for any interval I. Then, by the definition of p_1 and by the equality

$$p_1^{-1}(J_k) = p^{-1}(J_k), \quad k \in \mathbb{N},$$

the set $p_1^{-1}(I)$ is a union of at most s **MSI**, whenever I is a subinterval of J_k for some $k \in \mathbb{N}$. Moreover, in view of the definition of p_1 we have

$$p_1^{-1}(I) = p^{-1}(I),$$

for each interval I such that

$$\text{either } I \subset \left(\mathbb{N} \setminus \bigcup_{k\in\mathbb{N}} J_k\right) \text{ or } I = J_k \text{ for some } k \in \mathbb{N}.$$

As the result we have that the set $p_1^{-1}(I)$ is a union of at most $3s$ **MSI**, for every interval I. Thus $p_1^{-1} \in \mathfrak{C}$.

To prove that $p_2^{-1} \in \mathfrak{C}$ let us notice that

(i) $p_2^{-1}(n) = n$ for every $n \in \left(\mathbb{N} \setminus \bigcup_{k\in\mathbb{N}} J_k\right)$,

(ii) the set $p_2^{-1}(I) = p_1 p^{-1}(I)$ is a union of at most s **MSI** whenever I is a subinterval of J_k for some $k \in \mathbb{N}$,

(iii) $p_2^{-1}(J_k) = J_k$ for each $k \in \mathbb{N}$.

Hence we easy deduce that $p_2^{-1}(I)$ is a union of at most $(2s+1)$ **MSI** for every interval I. So $p_2^{-1} \in \mathfrak{C}$. The proof is completed. $\square$

Corollary 14.1. *We have* $\mathfrak{CD} \circ \mathfrak{CD} = \mathfrak{CD}$. *More precisely, from the above proof it follows that for every* $p \in \mathfrak{CD}$ *there exist permutations* $p_1, p_2 \in \mathfrak{CD}$ *such that* $p = p_1 p_2$, $c(p_2) \le 3\,c(p)$ *and* $c(p_1) \le 1 + 2\,c(p)$, *where, for every convergent permutation* $q \in \mathfrak{P}$, *we set*

$$c(q) := \sup\{c(q;I) \colon I \subset \mathbb{N} \text{ is an interval}\},$$

where $c(q;A) := \operatorname{card}(J)$*,* J *is the family of* **MSI** *defined by the relation* $q(A) = \bigcup J$ *for every* $A \subset \mathbb{N}$.

Theorem 14.2. *We have*

$$\mathfrak{DC} \circ \mathfrak{DD} = \mathfrak{DD} \circ \mathfrak{DC} = \mathfrak{D}$$

and

$$\mathfrak{CD} \circ \mathfrak{DD} = \mathfrak{DD} \circ \mathfrak{CD} = \mathfrak{CD} \cup \mathfrak{DD}.$$

Proof. First of all we note that if $p \in \mathfrak{DD}$ and $q \in \mathfrak{DC}$ then $pq, qp \in \mathfrak{D}$. Indeed, suppose that either $pq \in \mathfrak{C}$ or $qp \in \mathfrak{C}$. Then

$$p = (pq)q^{-1} = q^{-1}(qp) \in \mathfrak{C} \circ \mathfrak{C} = \mathfrak{C} \quad \text{i.e.} \quad p \in \mathfrak{C}.$$

This is a contradiction. So, both pq and qp are elements of $\mathfrak{D}$. In other words, the following conclusions hold:

$$\mathfrak{DC} \circ \mathfrak{DD} \subset \mathfrak{D} \text{ and } \mathfrak{D} \supset \mathfrak{DD} \circ \mathfrak{DC}$$

and

$$\mathfrak{CD} \circ \mathfrak{DD} \subset \mathfrak{CD} \cup \mathfrak{DD} \text{ and } \mathfrak{CD} \cup \mathfrak{DD} \supset \mathfrak{DD} \circ \mathfrak{CD}.$$

To prove the converse inclusions we consider four cases.

First, suppose that $p \in \mathfrak{DD}$. We shall show that $p = p_2 p_1$, for some permutations $p_1 \in \mathfrak{DC}$ and $p_2 \in \mathfrak{DD}$. Suppose that the intervals I_k and J_k, $k \in \mathbb{N}$, are chosen so that:

(1) $\min I_1 = 1$,
(2) $1 + \max I_k = \min J_k \quad \text{and} \quad 1 + \max J_k = \min I_{k+1}$,
(3) $\operatorname{card} J_k = 2k$,
(4) there exist intervals $E_k \subset I_k$ and $F_k \subset p(I_k)$ such that any of the two following sets:

$$p(E_k) \quad \text{and} \quad p^{-1}(F_k)$$

is a union of at least k **MSI**.

Let us put $p_1(n) = n$, for $n \in \bigcup_{k \in \mathbb{N}} I_k$, and

$$p_1(i + \min J_k) = \begin{cases} 2i + \min J_k & \text{for } i = 0, 1, \dots, k-1, \\ 2(i-k) + 1 + \min J_k & \text{for } i = k, k+1, \dots, 2k-1, \end{cases}$$

for $k \in \mathbb{N}$, and let $p_2 = p p_1^{-1}$.

From this definition it results easily that $p_1 \in \mathfrak{DC}$ and that $p_2 p_1 = p$. Moreover, from conditions (2) and (3) we get that any of the two following sets:

$$p_2(E_k) = p(E_k) \quad \text{and} \quad p_2^{-1}(F_k) = p^{-1}(F_k)$$

is a union of at least k **MSI**, for every $k \in \mathbb{N}$. Hence, we have $p_2 \in \mathfrak{DD}$ as it was claimed.

Let us set again that $p \in \mathfrak{DD}$. We will construct two permutations $p_1 \in \mathfrak{DD}$ and $p_2 \in \mathfrak{DC}$ such that $p_2 p_1 = p$. Assume that sequences I_k and J_k, $k \in$

$\mathbb{N}$, of intervals obey the conditions (1)-(3) from above and, additionally, the following one:

(5) for each $k \in \mathbb{N}$, there exist intervals

$$G_k \subset I_k \quad \text{and} \quad H_k \subset p^{-1}(I_k)$$

such that any of the two sets

$$p(H_k) \quad \text{and} \quad p^{-1}(G_k)$$

is a union of at least k **MSI**.

Let us set

$$p_2(i+\min J_k) = \begin{cases} 2i+\min J_k & \text{for } i=0,2,\dots,k-1, \\ 2(i-k)+1+\min J_k & \text{for } i=k,k+1,\dots,2k-1, \end{cases}$$

for $k \in \mathbb{N}$ and $p_2(n) = n$, for every $n \in \bigcup_{k\in\mathbb{N}} I_k$.

The permutation p_1 is given by $p_2 p_1 = p$. The verification that $p_1 \in \mathfrak{D}\mathfrak{D}$ and $p_2 \in \mathfrak{D}\mathfrak{C}$ may be peformed in a similar way as previously and will be omitted here.

Let us consider now the case $p \in \mathfrak{C}\mathfrak{D}$. We shall express p as the product $p_2 p_1$ of members $p_1 \in \mathfrak{C}\mathfrak{D}$ and $p_2 \in \mathfrak{D}\mathfrak{D}$. We start by choosing the intervals I_n, J_n and K_n, $n \in \mathbb{N}$, which form a partition of the set $\mathbb{N}$ and are such that

(6) $I_n < J_n < K_n < I_{n+1}$,

(7) $\min p^{-1}(I_n) < \min p^{-1}(J_n) < \min p^{-1}(K_n)$
and
$\max p^{-1}(I_n) < \max p^{-1}(J_n) < \max p^{-1}(K_n)$,

(8) $\max p^{-1}(J_n) < \min p^{-1}(I_{n+1})$
and
$\max p^{-1}(K_n) < \max p^{-1}(J_{n+1})$,

(9) $\text{card} J_n \geq 2n$,

(10) moreover, there exist the subintervals G_n of I_n and H_n of K_n such that any of the following sets:

$$p^{-1}(G_n) \quad \text{and} \quad p^{-1}(H_n)$$

is a union of at least n **MSI** and, additionally, the inclusion holds:

(11) $\left[\min p^{-1}(G_n), \max p^{-1}(G_n)\right] \subset p^{-1}(I_n)$,

for every $n \in \mathbb{N}$. Next, we define the permutations p_1 and p_2.

Let us assume that p_1 is an increasing map of the following sets:

$$p^{-1}\left(\{2i+\min J_n : i=0,1,...,n-1\}\right),$$
$$p^{-1}\left(J_n \setminus \{2i+\min I_n : i=0,1,...,n-1\}\right)$$

and

$$p^{-1}(I_n)$$

onto the intervals $[\min J_n, n-1+\min J_n]$, $[n+\min J_n, \max J_n]$ and I_n, respectively, for every $n \in \mathbb{N}$. Moreover, we set $p_1(m) = p(m)$ for each $m \in \bigcup_{n\in\mathbb{N}} p^{-1}(K_n)$. Since $p_1 \in \mathfrak{P}$, we may define the permutation p_2 by putting

$$p_2(n) = pp_1^{-1}(n), \quad n \in \mathbb{N}.$$

First we show that $p_1 \in \mathfrak{CD}$. Let L be an interval. In view of the conditions (7) and (8) we may write

$$L = L \cap p^{-1}\left(\mathfrak{I} \cup \mathfrak{J} \cup \mathfrak{K} \cup \mathfrak{L}\right),$$

where any of the following sets $\mathfrak{I}$ and $\mathfrak{J}$ is a union of at most three elements of the sequences $\{I_n\}$ and $\{J_n\}$, respectively. The set $\mathfrak{K}$ is a union of at most four elements of the sequence $\{K_n\}$, and $\mathfrak{L}$ is an interval of $\mathbb{N}$, which is a union of the successive elements of the sequence $\{I_n \cup J_n \cup K_n\}$ such that $p^{-1}(\mathfrak{L}) \subset L$.

Since $p_1 p^{-1}(U) = U$, for any interval $U = I_n$, J_n or K_n, $n \in \mathbb{N}$, then the set $p_1(L)$ may be expressed in the form

(12) $$p_1(L) = \mathfrak{L} \cup p_1\left(L \cap p^{-1}(\mathfrak{I})\right) \cup p_1\left(L \cap p^{-1}(\mathfrak{J})\right) \cup p_1\left(L \cap p^{-1}(\mathfrak{K})\right).$$

The following facts are the direct consequence of the definition of p_1. If U is an interval then the set $p_1\left(U \cap p^{-1}(I_n)\right)$ is a subinterval of I_n. The set $p_1\left(U \cap p^{-1}(J_n)\right)$ is a union of at most two subintervals of J_n and

$$p_1\left(U \cap p^{-1}(K_n)\right) = p(U) \cap K_n$$

for every $n \in \mathbb{N}$. Hence, the set $p_1\left(L \cap p^{-1}(\mathfrak{I})\right)$ is a union of at most 3 **MSI** and the set $p_1\left(L \cap p^{-1}(\mathfrak{J})\right)$ is a union of at most 6 **MSI**. On the other hand, if $m \in \mathbb{N}$ is chosen so that the set $p(U)$ is a union of at most m **MSI** for any interval U, then $p_1\left(L \cap p^{-1}(\mathfrak{K})\right)$ is a union of at most $4m$ **MSI**. Taking these observations together, by (12), we see that $p_1(L)$ is a union of at most $(4m+10)$ **MSI**. So, $p_1 \in \mathfrak{C}$. By (10), each set $p_1^{-1}(H_n) = p^{-1}(H_n), n \in \mathbb{N}$, is a union of at least n **MSI** and hence, p_1^{-1} belongs to $\mathfrak{D}$. Therefore $p_1 \in \mathfrak{CD}$ as it was claimed.

Now we have to show that $p_2 \in \mathfrak{D}\mathfrak{D}$. Take a look at the following equality:

$$\begin{aligned} &p_2\left([\min J_n, n-1+\min J_n]\right) \\ &= p_2\left(p_1 p^{-1}\left(\{2i+\min J_n\colon\ i=0,1,\ldots,n-1\}\right)\right) \\ &= \{2i+\min J_n\colon\ i=0,1,\ldots,n-1\} \text{ (by the definition of } p_1). \end{aligned}$$

We get that the set $p_2\left([\min J_n, n-1+\min J_n]\right)$ is a union of n **MSI**, for every $n \in \mathbb{N}$, and consequently $p_2 \in \mathfrak{D}$. By using the conditions (10), (11) and the definition of p_1 we receive easily that the set $p_2^{-1}(G_n) = p_1 p^{-1}(G_n)$ is a union of at least n **MSI**. This implies that $p_2^{-1} \in \mathfrak{D}$.

Let us set again $p \in \mathfrak{C}\mathfrak{D}$. Now, our goal will be to construct the permutations $p_1 \in \mathfrak{D}\mathfrak{D}$ and $p_2 \in \mathfrak{C}\mathfrak{D}$ satisfying $p_2 p_1 = p$. Before we define p_1 and p_2 we need some basic assumptions. Let I_n and J_n, $n \in \mathbb{N}$, be the increasing sequences of intervals such that the family $\{I_n\colon\ n \in \mathbb{N}\} \cup \{J_n\colon\ n \in \mathbb{N}\}$ forms the partition of $\mathbb{N}$. Furthermore, we assume that the following conditions hold:

(13) $I_n < J_n < I_{n+1}$,
(14) $\min p^{-1}(J_n) < p^{-1}(I_{n+1}) < \max p^{-1}(J_{n+1})$,
(15) $p^{-1}(J_n) < p^{-1}(J_{n+1})$,
(16) there is a subinterval Ω_n of I_n such that the set $p^{-1}(\Omega_n)$ is a union of at least n **MSI**,
(17) there exist four intervals:

$$E_n, G_n \subset p^{-1}(J_n) \quad \text{and} \quad F_n, H_n \subset J_n$$

such that

$$\begin{gathered} p^{-1}(F_n) < E_n < p^{-1}(H_n) < G_n, \\ \operatorname{card}(E_n) = \operatorname{card}(p^{-1}(F_n)) \quad \text{and} \quad \operatorname{card}(G_n) = \operatorname{card}(p^{-1}(H_n)) \end{gathered}$$

and, additionally, any of the two following sets:

$$p^{-1}(F_n) \quad \text{and} \quad p^{-1}(H_n)$$

is a union of at least n **MSI**,

for every $n \in \mathbb{N}$.

It follows from (17) that p_1 may be defined to be the increasing map of the following three sets:

$$E_n, \quad p^{-1}(H_n) \quad \text{and} \quad p^{-1}(J_n) \setminus \left(p^{-1}(F_n) \cup G_n\right)$$

onto the sets:

$$p^{-1}(F_n), \quad G_n \quad \text{and} \quad p^{-1}(J_n) \setminus \left(p^{-1}(H_n) \cup E_n\right),$$

respectively, for every $n \in \mathbb{N}$. Furthermore, we set

$$p_1(i) = i \quad \text{for any } i \in \bigcup_{n\in\mathbb{N}} p^{-1}(I_n).$$

Since $p_1 \in \mathfrak{P}$, then the permutation p_2 is well defined by the equation $p_2 p_1 = p$.

First, let us notice that, in view of the condition (17) and the definition of p_1, the permutation p_1 belongs to $\mathfrak{D}\mathfrak{D}$. Next, since

$$p_2(i) = p(i) \quad \text{for } i \in \bigcup_{n\in\mathbb{N}} p^{-1}(I_n)$$

we receive, from (16), that $p_2^{-1} \in \mathfrak{D}$. We need only to show that $p_2 \in \mathfrak{C}$. The proof of this fact is based on the following observations. If $\Delta \subseteq \mathbb{N}$ is an interval then we have

$$p_2\left(\Delta \cap p^{-1}(I_n)\right) = p\left(\Delta \cap p^{-1}(I_n)\right) = p(\Delta) \cap I_n$$

and if $\Gamma_n := \Delta \cap G_n$ then

$$\begin{aligned} p_2(\Gamma_n) &= p p_1^{-1}(\Gamma_n) = p\left(p^{-1}(H_n) \cap \left[\min p_1^{-1}(\Gamma_n), \max p_1^{-1}(\Gamma_n)\right]\right) \\ &\qquad \text{(by the definition of the restriction to } p^{-1}(H_n) \text{ of } p_1) \\ &= H_n \cap p\left(\left[\min p_1^{-1}(\Gamma_n), \max p_1^{-1}(\Gamma_n)\right]\right). \end{aligned}$$

Furthermore, if $\Phi_n := \Delta \cap p^{-1}(F_n)$ then, by the definition of the restriction to E_n of p_1, we get

$$p_2(\Phi_n) = p p_1^{-1}(\Phi_n) = p\left(\left[p_1^{-1}(\min \Phi_n), p_1^{-1}(\max \Phi_n)\right]\right).$$

Hence, if we choose $m \in \mathbb{N}$ in such a way that for every interval I the set $p(I)$ is a union of at most m **MSI** then any of the following three sets:

$$p_2\left(\Delta \cap p^{-1}(I_n)\right) \quad \text{or} \quad p_2(\Gamma_n) \quad \text{or} \quad p_2(\Phi_n)$$

is a union of at most m **MSI**, for every $n \in \mathbb{N}$.

Let again Δ be an interval of $\mathbb{N}$. Then we have

$$\begin{aligned} & p_2\left(\Delta \cap \left(p^{-1}(J_n) \setminus \left(p^{-1}(H_n) \cup E_n\right)\right)\right) \\ &= p p_1^{-1}\left(\Delta \cap \left(p^{-1}(J_n) \setminus \left(p^{-1}(H_n) \cup E_n\right)\right)\right) \\ &= p\left(\Delta^* \cap \left(p^{-1}(J_n) \setminus \left(p^{-1}(F_n) \cup G_n\right)\right)\right) \end{aligned}$$

(by the definition of the restriction to the set $p^{-1}(J_n) \setminus \left(p^{-1}(F_n) \cup G_n\right)$ of the permutation p_1, where Δ^* is some interval of $\mathbb{N}$)

$$= p\left(p^{-1}\left(p(\Delta^*) \cap J_n\right) \setminus \left(p^{-1}(F_n) \cup G_n\right)\right) = \left(p(\Delta^*) \cap J_n\right) \setminus \left(p(G_n) \cup F_n\right).$$

The following set:

$$\left(p(\Delta^*) \cap J_n\right) \setminus \left(p(G_n) \cup F_n\right)$$

is a union of at most $(2m+1)$ **MSI**, because the set $p(\Delta^*) \cap J_n$ is a union of at most m **MSI** and the set $p(G_n) \cup F_n$ is a union of at most $(m+1)$ **MSI** for every $n \in \mathbb{N}$. Therefore, the set $p_2\left(\Delta \cap p^{-1}(J_n)\right)$ is a union of at most $(4m+1)$ **MSI**.

According to the conditions (14) and (15), any bounded interval Δ may be written in the form

$$\Delta = \mathfrak{I} \cup \mathfrak{J} \cup \mathfrak{K},$$

where the set $\mathfrak{K}$ is a union of successive elements of the sequence

$$\{p^{-1}(I_n \cup J_n) \colon n \in \mathbb{N}\},$$

and the set $\mathfrak{I}$ is an intersection of Δ and at most four sets of the form $p^{-1}(I_n)$ satisfying the following relations:

$$p^{-1}(I_n) \cap \Delta \neq \emptyset \quad \text{and} \quad p^{-1}(I_n \cup J_n) \setminus \Delta \neq \emptyset.$$

The set $\mathfrak{J}$ is also an intersection of Δ and at most three elements of the sequence $p^{-1}(J_n), n \in \mathbb{N}$, such that

$$p^{-1}(J_n) \cap \Delta \neq \emptyset \quad \text{and} \quad p^{-1}(I_n \cup J_n) \setminus \Delta \neq \emptyset.$$

From the definition of p_1 and the above considerations it follows that $p_2(\mathfrak{K})$ is an interval and that the set $p_2(\mathfrak{I})$ is a union of at most $(4m)$ **MSI** and $p_2(\mathfrak{J})$ is a union of at most $3(4m+1)$ **MSI**. Hence, the set $p_2(\Delta)$ is a union of at most $(16m+4)$ **MSI**. Thus, $p_2 \in \mathfrak{C}$ as it was desired. □

Remark 14.3. Some parts of the above proof can be strengthen and, in consequence, the obtained conclusions can be stronger.

For example, if we replace the condition (3) with

(3') $\operatorname{card} J_k = kt, \quad k \in \mathbb{N},$

and we set

$$p_1(i + sk + \min J_k) = it + s + \min J_k,$$

for every $i = 0,1,...,k-1$, $s = 0,1,...,t-1$, $k \in \mathbb{N}$, then $c_\infty(p_1^{-1}) = t$, where for any $q \in \mathfrak{C}$ we define

$$c_\infty(q) := \lim_{n\to\infty} \max\{c(q;I) : I \subset \mathbb{N} \text{ is an interval such that } I \geq n\}.$$

Consequently we receive the following result:
For every $p \in \mathfrak{DD}$ and $t \in \mathbb{N}, t \geq 2$, there exist permutations $p_1 \in \mathfrak{DC}$, $p_2, p_3 \in \mathfrak{DD}$, such that $p = p_2 p_1 = p_1 p_3$ and $c_\infty(p_1^{-1}) = t$.

Theorem 14.3. *We have*

$$\mathfrak{U} \circ \mathfrak{CC} = \mathfrak{CC} \circ \mathfrak{U} = \mathfrak{U},$$

for any $\mathfrak{U} = \mathfrak{CC}, \mathfrak{CD}, \mathfrak{DC}$ *or* $\mathfrak{DD}$.

Proof. In view of the equality $\mathfrak{C} \circ \mathfrak{C} = \mathfrak{C}$ and the fact that the identity permutation on $\mathbb{N}$ belongs to $\mathfrak{CC}$, it is easy to check that

$$\mathfrak{CC} \circ \mathfrak{CC} = \mathfrak{CC} \quad \text{and} \quad \mathfrak{CD} \circ \mathfrak{CC} = \mathfrak{CC} \circ \mathfrak{CD} = \mathfrak{CD}.$$

Hence, we get

$$\mathfrak{DC} \circ \mathfrak{CC} = \mathfrak{CC} \circ \mathfrak{DC} = \mathfrak{DC} \quad \text{and} \quad (\mathfrak{CC} \circ \mathfrak{DD}) \cup (\mathfrak{DD} \circ \mathfrak{CC}) \subset \mathfrak{D}.$$

Now, if $(\mathfrak{CC} \circ \mathfrak{DD}) \cap \mathfrak{DC} \neq \emptyset$ then also $\mathfrak{DD} \cap (\mathfrak{CC} \circ \mathfrak{DC}) \neq \emptyset$, i.e. $\mathfrak{DD} \cap \mathfrak{DC} \neq \emptyset$, which is impossible. So, $\mathfrak{CC} \circ \mathfrak{DD} = \mathfrak{DD}$. Similarly, we show that $\mathfrak{DD} \circ \mathfrak{CC} = \mathfrak{DD}$. □

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