Wigner functions and coherent states for the quantum mechanics on a circle

OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **54** (2021) 275302 (23pp)
https://doi.org/10.1088/1751-8121/ac019d

Wigner functions and coherent states for the quantum mechanics on a circle

K Kowalski* **and K Ławniczak**

Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland

E-mail: kowalski@uni.lodz.pl

Received 1 March 2021, revised 3 May 2021
Accepted for publication 14 May 2021
Published 10 June 2021

Abstract

The Wigner functions for the coherent states of a particle on a circle are discussed. The nontrivial analytic forms of these functions are derived. The classicality of the circular coherent states existing in the literature as well as the new ones constructed by means of the Fourier transformation of the Gaussian is compared based on negativity of the Wigner function.

Keywords: quantum mechanics, coherent states, Wigner function

(Some figures may appear in colour only in the online journal)

1. Introduction

In spite of a long history of research into Wigner functions and their importance in quantum optics the theory of these functions in the case of the quantum mechanics on a manifold with non-trivial topology can hardly be called complete. In this work we calculate and compare the Wigner functions in the circular coherent states for the quantum mechanics on a ring. More precisely, the two specific coherent states defined earlier by us and other authors that we refer to as the 'Gaussian coherent states on the circle' and 'circular squeezed states', respectively as well as the new ones introduced herein called by us the 'Gaussian–Fourier coherent states'. The motivation for the investigation of the quantum rotational motion is, among others, its important role in atomic and molecular physics. On the other hand, there are numerous examples of applications of coherent states including quantum optics, atomic physics, condensed matter physics, quantum gravity and quantum information theory. Using the Wigner functions for the discussed coherent states we investigate the classicality of these states. As well known,

*Author to whom any correspondence should be addressed.

the most important property of coherent states is that they can be regarded from the physical point of view as the states closest to the classical ones and it is plausible to treat the most classical coherent states as the best ones. In the case of the standard coherent states for a particle on a line such closeness is described by minimization of the Heisenberg uncertainty relations determining up to a unitary transformation the coherent states. In opposition to the case of the quantum mechanics on a line there are no generally accepted uncertainty relations for a particle on a circle. Therefore, we have decided in this work to utilize the Wigner function as an indicator of classicality of the coherent states for the quantum mechanics on a circle. More precisely, our criterion is the negativity of the Wigner function used as a measure of their non-classicality. We recall that the Wigner function for the standard coherent states is nonnegative. This is an example of application of the Hudson theorem [1]. Nevertheless, this is not the case for the coherent states on a circle. Indeed, the only pure states of a quantum particle on a circle with non-negative Wigner function are the eigenstates of the angular momentum [2]. The paper is organized as follows. In section 2 we recall the basic facts about the quantum mechanics on a circle. Section 3 is devoted to the short review of properties of the Wigner function for a particle on a circle. Section 4 deals with the circular squeezed states and the corresponding Wigner function. In section 5 we discuss the coherent states called by us the Gaussian circular coherent states and the Wigner function for these states. The starting point of section 6 are the coherent states discussed in reference [3]. Because of the problems with definition of the Wigner function in these states we introduce the new ones that we refer to as the Gaussian–Fourier coherent states and analyze the corresponding Wigner function. Section 7 is devoted to comparison of classicality of the coherent states for the quantum mechanics on a circle choosing as a criterion the negativity of their corresponding Wigner function.

2. Quantum mechanics on a circle

We now collect the basic facts about the quantum mechanics on a circle. The algebra adequate for the study of a quantum particle on a circle is the $e(2)$ algebra.

$$[J, X_1] = \mathrm{i}X_2, \qquad [J, X_2] = -\mathrm{i}X_1, \quad [X_1, X_2] = 0, \tag{2.1}$$

where J is the Hermitian angular momentum operator, X_1 and X_2 are the Hermitian position observables on a circle and we set $\hbar = 1$. Indeed, the algebra (2.1) has the Casimir operator given in a unitary irreducible representation by

$$X_1^2 + X_2^2 = r^2. \tag{2.2}$$

The $e(2)$ algebra can be written by means of the unitary operator U representing the position of a particle on a unit circle as

$$[J, U] = U, \tag{2.3}$$

where $X_1 = r(U + U^\dagger)/2$ and $X_2 = r(U - U^\dagger)/2\mathrm{i}$.

Consider the eigenvalue equation

$$J|j\rangle = j|j\rangle. \tag{2.4}$$

The eigenvalue j is of the form $j = k + \lambda$, where k is integer and $\lambda \in [0, 1)$. Demanding the time-reversal invariance of the algebra (2.3) we find that $\lambda = 0$ or $\lambda = \frac{1}{2}$, so j is integer or half-integer, respectively [4]. We assume in the sequel that j is integer. Of course, the eigenvectors

$|j\rangle$ form an orthogonal and complete set, so

$$\langle j|k\rangle = \delta_{ij}, \tag{2.5a}$$

$$\sum_{j=-\infty}^{\infty} |j\rangle\langle j| = I. \tag{2.5b}$$

The operators U and $U^\dagger$ act on the vectors $|j\rangle$ as the ladder ones. We have

$$U|j\rangle = |j+1\rangle, \qquad U^\dagger|j\rangle = |j-1\rangle. \tag{2.6}$$

Consider now the eigenvalue equation

$$U|\varphi\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\varphi\rangle. \tag{2.7}$$

We point out that we have formally $U = \mathrm{e}^{\mathrm{i}\hat{\varphi}}$, where $\hat{\varphi}$ is the angle operator defined by

$$\hat{\varphi}|\varphi\rangle = \varphi|\varphi\rangle. \tag{2.8}$$

Nevertheless, it must be borne in mind that in opposition to the unitary operator U, the operator $\hat{\varphi}$ is problematic. The vectors $|\varphi\rangle$ satisfy the orthogonality conditions

$$\langle\varphi|\varphi'\rangle = 2\pi\delta(\varphi-\varphi'), \tag{2.9}$$

where $\varphi - \varphi' \in [-\pi, \pi)$. They form the complete set, namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi|\varphi\rangle\langle\varphi| = I. \tag{2.10}$$

The resolution of the identity (2.10) gives rise to the functional coordinate representation $L^2(S^1)$ for the quantum mechanics on a circle specified by the scalar product

$$\langle f|g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, f^*(\varphi)g(\varphi), \tag{2.11}$$

where $f(\varphi) = \langle\varphi|f\rangle$. The operators J and U act in the representation (2.11) as follows

$$Jf(\varphi) = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\varphi}f(\varphi), \qquad Uf(\varphi) = \mathrm{e}^{\mathrm{i}\,\varphi}f(\varphi). \tag{2.12}$$

The basis vectors $|j\rangle$ are represented in the Hilbert space with the scalar product (2.11) by the functions

$$\langle\varphi|j\rangle = \mathrm{e}^{\mathrm{i}j\varphi}. \tag{2.13}$$

Hence, using the completeness condition (2.5b) we find

$$f(\varphi) = \langle\varphi|f\rangle = \sum_{j=-\infty}^{\infty} f_j\,\mathrm{e}^{\mathrm{i}j\varphi}, \tag{2.14}$$

where $f_j = \langle j|f\rangle$. Of course (2.14) is the Fourier series expansion of the function $f(\varphi)$. Therefore the elements of the Hilbert space $L^2(S^1)$ specified by the scalar product (2.11) are in the

discussed case of the integer eigenvalues j of the angular momentum operator J, 2π-periodic functions. On the other hand, (2.5b), (2.9) and (2.13) taken together yield

$$\frac{1}{2\pi}\sum_{j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}j\varphi} = \delta(\varphi), \tag{2.15}$$

where $\varphi \in [-\pi, \pi)$

3. Wigner function for a particle on a circle

Our purpose in this section is to discuss the basic properties of the Wigner function for the quantum mechanics on a circle. The generally accepted definition of the Wigner function in the position representation W_{f} for the pair angle φ and orbital momentum l and a state $f(\varphi)$ is of the form

$$W_{\mathrm{f}}(l,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, f^*\left(\varphi - \frac{\theta}{2}\right) f\left(\varphi + \frac{\theta}{2}\right) \mathrm{e}^{-\mathrm{i}\theta l}. \tag{3.1}$$

Nevertheless, some authors interpret $W_{\mathrm{f}}(l,\varphi)$ as a function on the classical phase space that is the cylinder $S^1 \times \mathbb{R}$ [5], while others consider $W_{\mathrm{f}}(l,\varphi)$ as a function on a partially quantized space $S^1 \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and assume that l is discrete (see for example reference [6]). We point out that the formula (3.1) was derived by means of different methods such as the group-theoretical one based on the analysis of representations of the $E(2)$ group [5], the Weyl–Wigner–Moyal formalism [6] and as a special case of the general construction of the Wigner function for the n-dimensional sphere S^n utilizing the solutions of the Laplace–Beltrami equations for spheres [7]. We have the marginal position distribution reproduced by the Wigner function such that

$$\int_{-\infty}^{\infty} \mathrm{d}l\, W_{\mathrm{f}}(l,\varphi) = |f(\varphi)|^2 \tag{3.2}$$

in the case of continuous l, and for discrete l

$$\sum_{l=-\infty}^{\infty} W_{\mathrm{f}}(l,\varphi) = |f(\varphi)|^2 \tag{3.3}$$

following directly from (2.15) and (3.1). Thus, it turns out that the marginal position space probability density is given by the same formula whether l is continuous or discrete. Further, using the resolution of the identity (2.5b) and (2.13) we find the marginal for the momentum space probability density

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, W_{\mathrm{f}}(l,\varphi) = \sum_{j=-\infty}^{\infty} |f_j|^2 \mathrm{sinc}\,\pi(j-l), \tag{3.4}$$

where sinc is the *sinus cardinalis* function defined by

$$\mathrm{sinc}\, x = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases} \tag{3.5}$$

and we utilized the identity

$$\operatorname{sinc} \pi x = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}x\theta}. \tag{3.6}$$

Obviously, for integer l the formula (3.4) reduces to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, W_{\mathrm{f}}(l,\varphi) = |f_l|^2, \tag{3.7}$$

so the marginal has the same form as for the quantum mechanics on a real line. In the case of continuous l an interesting interpretation of the right-hand side of (3.4) was provided by Kastrup [5] as an example of the Whittaker cardinal function interpolating the different discrete values $|f_j|^2$. Now, an immediate consequence of (3.6) is the relation

$$\int_{-\infty}^{\infty} \operatorname{sinc} \pi(x-a) = 1. \tag{3.8}$$

On the other hand, (3.6) and (2.15) taken together yield

$$\sum_{j=-\infty}^{\infty} \operatorname{sinc} \pi(j-k) = 1. \tag{3.9}$$

From (3.2) it follows that we have the normalization condition

$$\int_{-\infty}^{\infty} \mathrm{d}l\, \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, W_{\mathrm{f}}(l,\varphi) = 1. \tag{3.10}$$

For integer l we get with the use of (3.3)

$$\sum_{l=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\varphi\, W_{\mathrm{f}}(l,\varphi) = 1. \tag{3.11}$$

We now discuss the Wigner function in the momentum representation

$$W(l,\varphi) = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f_j^* f_k\, \mathrm{e}^{\mathrm{i}(k-j)\varphi} \operatorname{sinc} \pi\left(\frac{j+k}{2} - l\right) \tag{3.12}$$

following directly from (3.1), (2.5b), (2.13) and (3.6). On introducing in (3.12) the new summation indices $k-j$ and $j+k$ we find after some calculation that we can write (3.12) in the form

$$\begin{aligned} W(l,\varphi) = &\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f_{k-j}^* f_{k+j}\, \mathrm{e}^{2\mathrm{i}j\varphi} \operatorname{sinc} \pi(k-l) \\ &+ \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f_{k-j}^* f_{k+j+1}\, \mathrm{e}^{\mathrm{i}(2j+1)\varphi} \operatorname{sinc} \pi\left(k-l+\frac{1}{2}\right). \end{aligned} \tag{3.13}$$

In the case of integer l (3.13) reduces to

$$W(l,\varphi) = \sum_{j=-\infty}^{\infty} f_{l-j}^* f_{l+j}\, \mathrm{e}^{2\mathrm{i}j\varphi} + \frac{1}{\pi}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} f_{l-j+k}^* f_{l+j+k+1} \frac{(-1)^k}{k+\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}(2j+1)\varphi}. \tag{3.14}$$

Notice that the first sum from the right-hand side of (3.14) has the structure similar to the

Wigner function for a particle on a line in momentum representation. Up to normalization constant the formula (3.14) was derived from the quantizer kernel in reference [6], where the general Wigner function for the quantum mechanics on a circle was considered in the states described by the density matrix.

We finally write down the inequality satisfied by the Wigner function (3.1)

$$|W_{\mathrm{f}}(l,\varphi)| \leqslant 2 \tag{3.15}$$

following directly from (3.1), the Schwarz inequality and the inequality satisfied by an arbitrary nonnegative 2π-periodic function

$$\int_{-\frac{\pi}{2}+\alpha}^{\frac{\pi}{2}+\alpha} \mathrm{d}\varphi\, g(\varphi) \leqslant \int_{-\pi}^{\pi} \mathrm{d}\varphi\, g(\varphi), \tag{3.16}$$

where α is an arbitrary constant. The inequality (3.16) is obvious in view of the identity

$$\int_{-\pi}^{\pi} \mathrm{d}\varphi\, g(\varphi) = \int_{-\pi+\alpha}^{\pi+\alpha} \mathrm{d}\varphi\, g(\varphi) \tag{3.17}$$

that holds for an arbitrary 2π-periodic function and arbitrary α.

4. Wigner function for the circular squeezed states

4.1. Circular squeezed states

We begin with a brief account of the circular squeezed states [8, 9]. Consider the following form of the $e(2)$ algebra obtained from (2.3) by the formal identification $U = \mathrm{e}^{\mathrm{i}\hat{\varphi}}$

$$[J, \cos\,\hat{\varphi}] = \mathrm{i}\,\sin\,\hat{\varphi}, \qquad [J, \sin\,\hat{\varphi}] = -\mathrm{i}\,\cos\,\hat{\varphi}, \qquad [\sin\,\hat{\varphi}, \cos\,\hat{\varphi}] = 0. \tag{4.1}$$

The algebra (4.1) implies the uncertainty relations of the form

$$\Delta J \Delta\,\cos\,\hat{\varphi} \geqslant \frac{1}{2}|\,\langle \sin\,\hat{\varphi}\rangle\,|, \tag{4.2a}$$

$$\Delta J \Delta\,\sin\,\hat{\varphi} \geqslant \frac{1}{2}|\,\langle \cos\,\hat{\varphi}\rangle\,|, \tag{4.2b}$$

$$\Delta\,\sin\,\hat{\varphi}\Delta\,\cos\,\hat{\varphi} \geqslant 0. \tag{4.2c}$$

The circular coherent states are defined as the states minimizing (4.2b) with $\hat{\varphi}$ replaced by $\hat{\varphi} - \alpha$. In the position representation $L^2(S^1)$ these states are given by

$$f^s_{\alpha,m}(\varphi) = \frac{1}{\sqrt{I_0(2s)}}\,\exp[s\,\cos(\varphi-\alpha) + \mathrm{i}m(\varphi-\alpha)], \tag{4.3}$$

where the packet is peaked at $\varphi = \alpha$, m is the counterpart of the classical momentum, and $I_\nu(x)$ designates the modified Bessel function of the first kind. The real parameter $s \geqslant 0$ representing the angular momentum spread [9] is given by

$$s = \frac{\Delta J}{\Delta\,\sin(\hat{\varphi}-\alpha)} = \frac{|\,\langle\cos(\hat{\varphi}-\alpha)\rangle\,|}{2[\Delta\,\sin(\hat{\varphi}-\alpha))]^2}. \tag{4.4}$$

Let $f^{s}_{m,\alpha}(\varphi) = \langle \varphi | m, \alpha \rangle_s$. The projection of the abstract coherent states $|\alpha, m\rangle_s$ onto the basis spanned by the eigenvectors $|j\rangle$ of the angular momentum operator J is

$$\langle j | m, \alpha \rangle_s = \frac{\mathrm{e}^{-\mathrm{i}\alpha j}}{\sqrt{I_0(2s)}} I_{m-j}(s), \tag{4.5}$$

where the use was made of (2.10) and the identity [10]

$$I_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{x\,\cos\,\theta}\, \mathrm{e}^{\mathrm{i}n\theta} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{x\,\cos(\theta-a)}\, \mathrm{e}^{\mathrm{i}n(\theta-a)}, \tag{4.6}$$

where n is integer. The circular squeezed states are not orthogonal. Using (4.5) and the identity [11]

$$\sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ka} J_k(w) J_{k+\nu}(z) = \left(\frac{z - w\,\mathrm{e}^{-\mathrm{i}a}}{z - w\,\mathrm{e}^{\mathrm{i}a}}\right)^{\frac{\nu}{2}} J_\nu(\sqrt{w^2 + z^2 - 2wz\,\cos\,a}), \tag{4.7}$$

where $|w\mathrm{e}^{\pm \mathrm{i}a}| < |z|$ for $\nu \neq 0, \pm 1, \pm 2, \ldots$, and the Bessel function of the first kind $J_k(z)$ with the integer order k is related to the modified Bessel function of the first kind $I_k(z)$ by $I_k(z) = \frac{1}{\mathrm{i}^k} J_k(z)$, we get

$$\begin{aligned} {}_s\langle m, \alpha | m', \alpha' \rangle_{s'} = {} & \frac{\mathrm{e}^{\mathrm{i}m(\alpha-\alpha')}}{\sqrt{I_0(2s) I_0(2s')}} \left(\frac{s' + s\,\mathrm{e}^{-\mathrm{i}(\alpha'-\alpha)}}{s' + s\,\mathrm{e}^{\mathrm{i}(\alpha'-\alpha)}}\right)^{\frac{m'-m}{2}} \\ & \times I_{m'-m}\left(\sqrt{s^2 + s'^2 + 2ss'\,\cos(\alpha' - \alpha)}\right). \end{aligned} \tag{4.8}$$

For $s' = s$ the formula (4.8) reduces to

$${}_s\langle m, \alpha | m', \alpha' \rangle_s = \frac{\mathrm{e}^{\frac{\mathrm{i}}{2}(\alpha-\alpha')(m+m')}}{I_0(2s)} I_{m'-m}\left(2s|\cos\frac{\alpha' - \alpha}{2}|\right). \tag{4.9}$$

The authors did not find the formulas (4.8) and (4.9) for the overlap of the circular squeezed states in the literature. Furthermore, another consequence of (4.5), (4.7) and the relation $I_k(z) = \frac{1}{\mathrm{i}^k} J_k(z)$ is the completeness (over completeness) of the circular coherent states. Namely, the resolution of the identity for these states is of the form

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha |m, \alpha\rangle_{ss}\langle m, \alpha| = I. \tag{4.10}$$

We now discuss the average values of the observables in the circular coherent states. An immediate consequence of (4.6), (4.3) and the first equation of (2.12) is

$${}_s\langle m, \alpha |\, J\, | m, \alpha \rangle_s = m, \tag{4.11}$$

so, as mentioned earlier m is the parameter that can be regarded as a counterpart of the classical orbital momentum. Nevertheless, by virtue of (4.3) m, in opposition to the classical angular momentum can take only discrete values in the considered case of the Hilbert space of 2π-periodic functions. On the other hand, taking into account (2.7), (2.10), (4.3) and (4.6) we arrive at the formula

$${}_s\langle m, \alpha |\, U\, | m, \alpha \rangle_s = \mathrm{e}^{\mathrm{i}\alpha}\, \frac{I_1(2s)}{I_0(2s)}. \tag{4.12}$$

On introducing the relative expectation value

$$ {}_s\langle\langle m,\alpha|\, U\,|m,\alpha\rangle\rangle_s = \frac{{}_s\langle m,\alpha|\, U\,|m,\alpha\rangle_s}{{}_s\langle 0,0|\, U\,|0,0\rangle_s} \tag{4.13} $$

we arrive at the relation

$$ {}_s\langle\langle m,\alpha|\, U\,|m,\alpha\rangle\rangle_s = \mathrm{e}^{\mathrm{i}\alpha}. \tag{4.14} $$

It follows from (4.11) and (4.14) that up to discreteness of m the states $|m,\alpha\rangle_s$ are parametrized by the points of the classical phase space. Nevertheless, in opposition to the standard coherent states for a particle on a line the states $|m,\alpha\rangle_s$ are labeled by the extra parameter s controlling the angular spread of the packet (4.3). On the other hand, it is not clear what is the counterpart of the Bose annihilation operator whose eigenvectors are the circular coherent states (4.3). We finally remark that the discussed circular squeezed states were applied in the study of the Rydberg wave packets [9].

4.2. Wigner function for the circular squeezed states

Our purpose now is to study the Wigner function for the circular squeezed states. From (3.1) and (4.3) we get

$$ W^s_{\alpha,m}(l,\varphi) = \frac{1}{\pi I_0(2s)}\int_0^{\pi} \mathrm{d}\theta\, \exp\left[2s\,\cos(\varphi-\alpha)\cos\,\frac{\theta}{2}\right]\cos[\theta(l-m)]. \tag{4.15} $$

The Wigner function (4.15) was introduced and investigated for the first time by Kastrup [5] who considered the general case of arbitrary l and m, and checked consistency of the formulas (3.4) and (4.5). We now derive the analytic expression of the Wigner function (4.15) for integer l and m in terms of the finite series of special functions. We begin by writing (4.15) in the form

$$ W^s_{\alpha,m}(l,\varphi) = \frac{2}{\pi I_0(2s)}\int_0^{\frac{\pi}{2}} \mathrm{d}\vartheta\, \exp[2s\,\cos(\varphi-\alpha)\cos\,\vartheta]\cos[2\vartheta(l-m)]. \tag{4.16} $$

On using the identity [12]

$$ \cos\,2nx = \sum_{k=0}^{n}(-1)^k 2^{2n-2k-1}\frac{2n}{k!}\frac{(2n-k-1)!}{(2n-2k)!}\cos^{2n-2k}x, \tag{4.17} $$

where $n \geqslant 1$, substituting in (4.16) $\cos\vartheta = x$ and utilizing the integral [12]

$$ \begin{aligned} &\int_0^u x^{2\nu-1}(u^2-x^2)^{\rho-1}\mathrm{e}^{\mu x}\,\mathrm{d}x \\ &\quad= \frac{1}{2}B(\nu,\rho)u^{2\nu+2\rho-2}{}_1F_2\left(\nu;\frac{1}{2},\nu+\rho;\frac{\mu^2u^2}{4}\right) \\ &\qquad+\frac{\mu}{2}B\left(\nu+\frac{1}{2},\rho\right)u^{2\nu+2\rho-1} \\ &\qquad\times{}_1F_2\left(\nu+\frac{1}{2};\frac{3}{2},\nu+\rho+\frac{1}{2};\frac{\mu^2u^2}{4}\right), \quad \mathrm{Re}\,\rho>0,\ \mathrm{Re}\,\nu>0, \end{aligned} \tag{4.18} $$

where $B(x,y)$ is the Euler beta function, ${}_1F_2(\alpha;\beta_1,\beta_2;z)$ is the generalized hypergeometric function and we set $u=1$, $\nu=|l-m|-k+\frac{1}{2}$, $\rho=\frac{1}{2}$ and $\mu=2s\cos(\varphi-\alpha)$, we get for $l\neq m$

$$\begin{aligned} W^s_{\alpha,m}(l,\varphi) = \frac{1}{\pi I_0(2s)}\sum_{k=0}^{|l-m|}(-1)^k 2^{2|l-m|-2k-1}\frac{2|l-m|}{k!}\frac{(2|l-m|-k-1)!}{(2|l-m|-2k)!} \\ \times\left[B\left(|l-m|-k+\frac{1}{2},\frac{1}{2}\right)\right. \\ \times {}_1F_2\left(|l-m|-k+\frac{1}{2};\frac{1}{2},|l-m|-k+1,s^2\cos^2(\varphi-\alpha)\right) \\ + 2s\,\cos(\varphi-\alpha)B\left(|l-m|-k+1,\frac{1}{2}\right) \\ \left.\times {}_1F_2\left(|l-m|-k+1;\frac{3}{2},|l-m|-k+\frac{3}{2},s^2\cos^2(\varphi-\alpha)\right)\right]. \end{aligned} \tag{4.19}$$

Taking into account (4.16) and the identity

$$\int_0^{\frac{\pi}{2}} \mathrm{e}^{a\,\cos\,\theta}\,\mathrm{d}\theta = \frac{\pi}{2}\left[I_0(a)+\boldsymbol{L}_0(a)\right], \tag{4.20}$$

where $\boldsymbol{L}_\nu(x)$ is the modified Struve function, following directly from the first equation of (4.6) and the relation [12]

$$\boldsymbol{L}_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^{\frac{\pi}{2}}\sinh(z\,\cos\,\varphi)(\sin\,\varphi)^{2\nu}\mathrm{d}\varphi, \qquad \mathrm{Re}\nu > -\frac{1}{2}, \tag{4.21}$$

we find for $l=m$

$$W^s_{\alpha,m}(l,\varphi) = \frac{1}{I_0(2s)}(I_0[2s\,\cos(\varphi-\alpha)]+\boldsymbol{L}_0[2s\,\cos(\varphi-\alpha)]), \tag{4.22}$$

As far as we are aware the formulas (4.19) and (4.22) are new.

The plot of the Wigner function (4.15) is shown in figure 1 (top left). In opposition to the standard coherent states for a particle on a line, the Wigner function (4.15) can take negative values. Such behavior is depicted in figure 1 (top right and bottom left).

We now recall that the Wigner function for the standard coherent states:

$$\phi_{\bar{x},\bar{p}}(x) = \pi^{-\frac{1}{4}}\,\mathrm{e}^{-\frac{1}{2}(x-\bar{x})^2+\mathrm{i}x\bar{p}} \tag{4.23}$$

such that

$$W_{\bar{x},\bar{p}}(x,p) = \frac{1}{\pi}\,\mathrm{e}^{-(x-\bar{x})^2}\,\mathrm{e}^{-(p-\bar{p})^2} \tag{4.24}$$

is peaked at $x=\bar{x}$ and $p=\bar{p}$, where $\bar{x}$ and $\bar{p}$ are the average position and momentum parametrizing the coherent state, respectively. It turns out that the same holds true for the discussed Wigner function in the circular coherent state (4.15) which has maximum at $l=m$ and $\varphi=\alpha$ (see figure 1, bottom right). We remark that the maximum value of the Wigner function for integer m can be immediately obtained from (4.22) by setting $\varphi=\alpha$, so

$$W^s_{\max} = 1+\frac{\boldsymbol{L}_0(2s)}{I_0(2s)}. \tag{4.25}$$

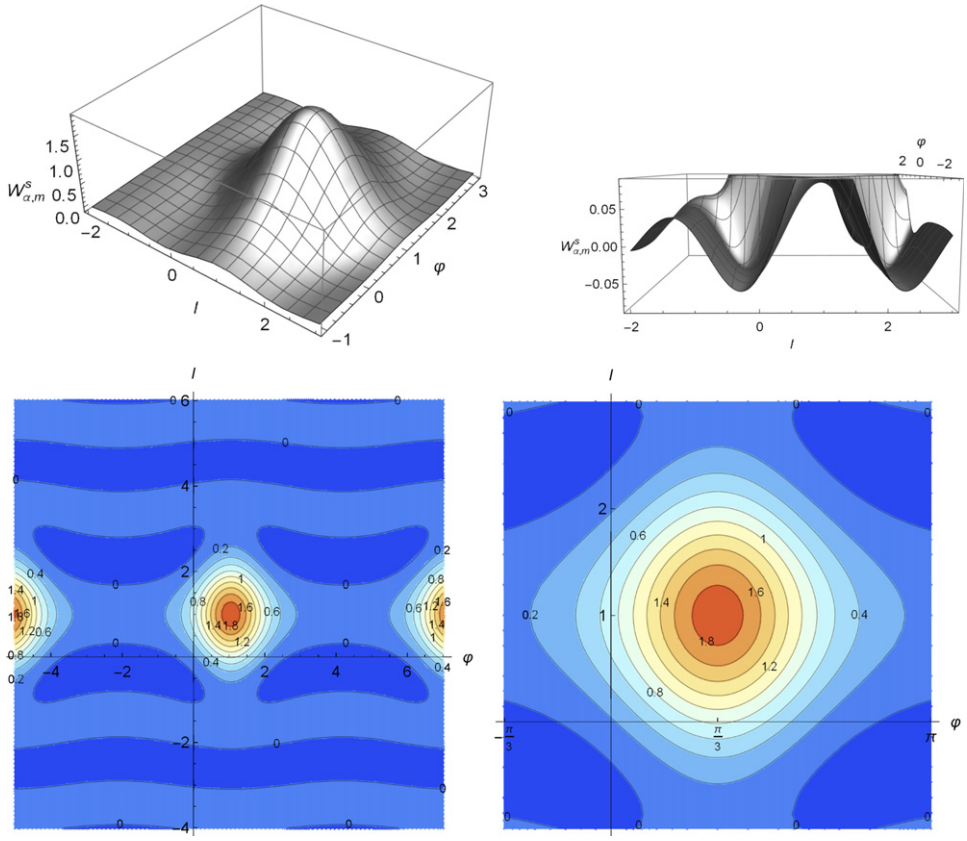

Figure 1. Top left: the plot of the Wigner function for the circular coherent states (4.11) with $s = 1.26$ (see (7.3) and (7.7)), $m = 1$ and $\alpha = \pi/3$ for the all panels. Top right: negativity of the Wigner function (4.15) in 3D presentation. Bottom left: the contour plot of the Wigner function (4.15). The regions of the negative Wigner function are bounded by contour levels 0. Bottom right: the contour plot of the Wigner function (4.15). The maximum of the Wigner function at $l = m$ and $\varphi = \alpha$, where $m = 1$ and $\alpha = \frac{\pi}{3}$ is easily seen.

Observe that the maximum value of the Wigner function is only the function of the squeezing parameter s. We point out that the maximum value of the Wigner function (4.24) for the standard coherent states also does not depend on parameters $\bar{x}$ and $\bar{p}$ labeling these states.

5. Wigner function for the Gaussian coherent states of a particle on a circle

5.1. Gaussian coherent states for the quantum mechanics on a circle

The Gaussian coherent states can be defined as the solution of the eigenvalue equation [4]

$$Z|z\rangle = z|z\rangle, \tag{5.1}$$

where $Z = \mathrm{e}^{-J+\frac{1}{2}}\, U$, and the complex number $z = \mathrm{e}^{-m+\mathrm{i}\alpha}$ parametrizes the circular cylinder which is the classical phase space for a particle on a circle, so m is the classical angular momentum and α is the classical angle. We shall use in the sequel the designation $|z\rangle \equiv |m, \alpha\rangle$. The

$L^2(S^1)$ representation of the (normalized) states $|m,\alpha\rangle$ is given by

$$f_{m,\alpha}(\varphi) = \frac{\theta_3\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}m)|\frac{\mathrm{i}}{2\pi}\right)}{\sqrt{\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}}, \tag{5.2}$$

where $f_{m,\alpha}(\varphi) = \langle\varphi|m,\alpha\rangle$ and $\theta_3(v|\tau)$ is the Jacobi theta function defined by

$$\theta_3(v|\tau) = \sum_{j=-\infty}^{\infty} q^{j^2}(\mathrm{e}^{\mathrm{i}\pi v})^{2j}, \tag{5.3}$$

where $q = \mathrm{e}^{\mathrm{i}\pi\tau}$ and $\mathrm{Im}\,\tau > 0$. It should be noted that $f_{m,\alpha}(\varphi)$ is a 2π-periodic function of the angle φ. Using the easily proven identity [13]

$$\theta_3(v+m\tau|\tau) = q^{-m^2}\,\mathrm{e}^{-2\mathrm{i}\pi m v}\theta_3(v|\tau), \tag{5.4}$$

where m is integer, we find that for discrete m the relation (5.2) can be written in the form

$$f_{m,\alpha}(\varphi) = \mathrm{e}^{\mathrm{i}(\varphi-\alpha)m}\frac{\theta_3(\frac{1}{2\pi}(\varphi-\alpha)|\frac{\mathrm{i}}{2\pi})}{\sqrt{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}}, \qquad m \in \boldsymbol{Z}. \tag{5.5}$$

Our motivation to use the (nonstandard) denomination 'Gaussian' for the coherent states defined by (5.1) was actuated by the alternative method of construction of these states based on the Zak transform [14]. We now present a simple version of this method in the case of integer parameter m labeling the coherent states. We first observe that in analogy with the standard coherent states for a particle on a line the vacuum vector $g_{0,0}(\varphi)$ should be annihilated by the operator ('annihilation operator') $\hat{\varphi} + \mathrm{i}J$ (the mathematically sound condition is $\mathrm{e}^{\mathrm{i}(\hat{\varphi}+\mathrm{i}J)}g_{0,0} = g_{0,0}$ leading to $Zg_{0,0}(\varphi) = g_{0,0}(\varphi)$ (see (5.1)). Hence using (2.8) and the first equation of (2.12) we find that up to normalization constant $g_{0,0}(\varphi) = \mathrm{e}^{-\frac{1}{2}\varphi^2}$. The tails of the Gaussian $g_{0,0}(\varphi)$ outside the range $[-\pi,\pi)$ can be wrapped around the circle according to

$$f_{0,0}(\varphi) = \sum_{j=-\infty}^{\infty} g_{0,0}(\varphi + 2j\pi). \tag{5.6}$$

On using the Poisson summation formula

$$\sum_{j=-\infty}^{\infty} f(x+2j\pi) = \frac{1}{\sqrt{2\pi}}\sum_{j=-\infty}^{\infty}\tilde{f}(j)\mathrm{e}^{\mathrm{i}jx}, \tag{5.7}$$

where $\tilde{f}(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}x$ is the Fourier transform of $f(x)$, and normalizing the obtained function $f_{0,0}(\varphi)$, we get

$$f_{0,0}(\varphi) = \frac{\theta_3(\frac{\varphi}{2\pi}|\frac{\mathrm{i}}{2\pi})}{\sqrt{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}}. \tag{5.8}$$

Now applying the Perelomov approach [15] we define the coherent states $f_{m,\alpha}(\varphi)$, where m is integer as

$$f_{m,\alpha}(\varphi) = \mathrm{e}^{\mathrm{i}m\hat{\varphi}-\mathrm{i}\alpha J}\,f_{0,0}(\varphi) \tag{5.9}$$

Utilizing the Baker–Hausdorff identity we obtain

$$\mathrm{e}^{\mathrm{i}m\hat{\varphi}-\mathrm{i}\alpha J} = \mathrm{e}^{-\frac{\mathrm{i}}{2}m\alpha}U^m\,\mathrm{e}^{-\mathrm{i}\alpha J} \tag{5.10}$$

Equations (2.12), (5.8), (5.9) and (5.10) taken together yield

$$f_{m,\alpha}(\varphi) = \mathrm{e}^{-\frac{\mathrm{i}}{2}m\alpha}\,\mathrm{e}^{\mathrm{i}m\varphi}\frac{\theta_3\left(\frac{1}{2\pi}(\varphi-\alpha)|\frac{\mathrm{i}}{2\pi}\right)}{\sqrt{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}}, \qquad m\in\mathbb{Z}. \tag{5.11}$$

Up to irrelevant phase factor $\mathrm{e}^{\frac{\mathrm{i}}{2}m\alpha}$ the coherent states (5.11) coincide with the coherent states (5.5). We point out that the states defined by (5.1) are the concrete realization of the very general abstract mathematical scheme of construction of Bargmann spaces introduced in [16, 17]. It should also be noted that the method for construction of the coherent states based on the eigenvalue equation (5.1) was generalized to the case of the n-dimensional sphere S^n by Hall [18]. The alternative constructions of coherent states for the sphere S^n based on the generalized Perelomov-type approach for the group $E(n+1)$ were introduced by De Bièvre [19] and Isham and Klauder [20].

We now collect some basic properties of the Gaussian circular coherent states. The Fourier coefficients of expansion of the normalized coherent state in the basis of eigenvectors of the angular momentum operator are given by

$$\langle j|m,\alpha\rangle = \mathrm{e}^{-\frac{1}{2}j^2}\frac{\mathrm{e}^{mj-\mathrm{i}j\alpha}}{\sqrt{\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}}. \tag{5.12}$$

Taking into account (2.10), (2.13) and (5.5) we find that for integer m the formula (5.12) reduces to

$$\langle j|m,\alpha\rangle = \frac{\mathrm{e}^{-\frac{1}{2}(j-m)^2}\,\mathrm{e}^{-\mathrm{i}j\alpha}}{\sqrt{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}}, \qquad m\in\boldsymbol{Z}. \tag{5.13}$$

Therefore, the probability distribution $|\langle j|m,\alpha\rangle|^2$ such that

$$|\langle j|m,\alpha\rangle|^2 = \frac{\mathrm{e}^{-(j-m)^2}}{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}, \qquad m\in\boldsymbol{Z} \tag{5.14}$$

is a discrete Gaussian one. We point out that $\theta_3(0|\frac{\mathrm{i}}{\pi})\approx\sqrt{\pi}$, where the relative error of approximation is of order 0.1 per mille. Furthermore, making use of (2.5b), (5.3) and (5.12) we get the overlap of the Gaussian coherent states (5.2). Namely, we have

$$\langle m,\alpha|m',\alpha'\rangle = \frac{\theta_3(\frac{\alpha-\alpha'}{2\pi}-\mathrm{i}\frac{m+m'}{2\pi}|\frac{\mathrm{i}}{\pi})}{\sqrt{\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)\theta_3\left(\frac{\mathrm{i}m'}{\pi}|\frac{\mathrm{i}}{\pi}\right)}}. \tag{5.15}$$

In the case with discrete m and m' the scalar product (5.15) can be written as

$$\langle m,\alpha|m',\alpha'\rangle = \mathrm{e}^{-\frac{1}{4}(m-m')^2}\,\mathrm{e}^{\frac{\mathrm{i}}{2}(m+m')(\alpha-\alpha')}\frac{\theta_3(\frac{\alpha-\alpha'}{2\pi}|\frac{\mathrm{i}}{\pi})}{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}, \qquad m,m'\in\boldsymbol{Z}. \tag{5.16}$$

The completeness relation satisfied by the coherent states (5.2) obtained with the help of (2.5a) and (5.12) is of the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha\int_{-\infty}^{\infty}\mathrm{d}\mu(m)|m,\alpha\rangle\langle m,\alpha| = I, \tag{5.17}$$

where $\mathrm{d}\mu(m) = \frac{1}{\sqrt{\pi}}\theta_3(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi})\mathrm{e}^{-m^2}\,\mathrm{d}m$. In the case with the discrete parameter m labeling the Gaussian circular coherent states the resolution of the identity is given by

$$\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha|m,\alpha\rangle\langle m,\alpha| = I \tag{5.18}$$

following directly from (2.5a) and (5.13).

We now discuss the parametrization of the Gaussian circular coherent states in a more detail. Using (2.5b) and (5.12) we find that the expectation value of the angular momentum J in the normalized coherent state is [4]

$$\begin{aligned}\langle m,\alpha|\hat{J}|m,\alpha\rangle &= m + 2\pi\,\sin(2m\pi)\\ &\quad\times\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\pi^2(2n-1)}}{(1+\mathrm{e}^{-\pi^2(2n-1)}\,\mathrm{e}^{2\mathrm{i}m\pi})(1+\mathrm{e}^{-\pi^2(2n-1)}\,\mathrm{e}^{-2\mathrm{i}m\pi})}.\end{aligned} \tag{5.19}$$

From (5.19) it follows that for integer m we have

$$\langle m,\alpha|\hat{J}|m,\alpha\rangle = m. \tag{5.20}$$

Otherwise

$$\langle m,\alpha|\hat{J}|m,\alpha\rangle \approx m. \tag{5.21}$$

where the approximation is very good—the maximal relative error is of order 0.1 per cent. Thus, it turns out that m can be really regarded as a classical orbital momentum. Proceeding analogously as with (5.21) and making use of (2.5b), (2.6) and (5.12) we get [4]

$$\begin{aligned}\langle m,\alpha|U|m,\alpha\rangle &= \mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\alpha}\frac{\theta_2\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}{\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}\\ &= \mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\alpha}\frac{1+2\sum_{n=1}^{\infty}\mathrm{e}^{-\pi^2n^2}\,\cos(2m+1)n\pi}{1+2\sum_{n=1}^{\infty}\mathrm{e}^{-\pi^2n^2}\,\cos\,2mn\pi},\end{aligned} \tag{5.22}$$

where

$$\theta_2(v|\tau) = \sum_{j=-\infty}^{\infty}q^{\left(j-\frac{1}{2}\right)^2}(\mathrm{e}^{\mathrm{i}\pi v})^{(2j-1)} \tag{5.23}$$

is the Jacobi theta function. An immediate consequence of (5.22) is

$$\langle m,\alpha|U|m,\alpha\rangle \approx \mathrm{e}^{-1/4}\,\mathrm{e}^{\mathrm{i}\alpha} \tag{5.24}$$

where the approximation is as good as in (5.21). Proceeding as with (4.12) and introducing the relative expectation value

$$\langle\langle m,\alpha|U|m,\alpha\rangle\rangle = \frac{\langle m,\alpha|U|m,\alpha\rangle}{\langle 0,0|U|0,0\rangle} \tag{5.25}$$

we obtain

$$\langle\langle m,\alpha|U|m,\alpha\rangle\rangle = \mathrm{e}^{\mathrm{i}\alpha}. \tag{5.26}$$

Therefore the parameter α can be regarded as a classical angle. We remark that the factors $\mathrm{e}^{-1/4}$ in (5.24) and $\frac{I_1(2s)}{I_0(2s)}$ in (4.12), respectively, are related to the fact that U is not diagonal in the coherent state basis—it is diagonal in the position representation spanned by the vectors $|\varphi\rangle$ (see (2.7)).

We finally mention that the Gaussian circular coherent states minimize the uncertainty relations of the form [21]

$$\Delta^2(J) + \Delta^2(\hat{\varphi}) \geqslant 1, \tag{5.27}$$

where the measure of the uncertainty of the angular momentum is defined by

$$\Delta^2(J) = \frac{1}{4}\ln\left(\langle \mathrm{e}^{-2J}\rangle\langle \mathrm{e}^{2J}\rangle\right) \tag{5.28}$$

and the measure of the uncertainty of the angle is given by

$$\Delta^2(\hat{\varphi}) = \frac{1}{4}\ln\frac{1}{|\langle U^2\rangle|^2}, \tag{5.29}$$

where $\langle A\rangle$ designates the average value of the observable A. In opposition to the standard coherent states for a particle on a line, the saturation of the uncertainty relations (5.27) does not uniquely determine up to a unitary transformation the Gaussian circular coherent states.

5.2. *Wigner function for the Gaussian circular coherent states*

We now investigate the Wigner function of the Gaussian coherent states for the quantum mechanics on a circle. Taking into account (3.1) and (5.2) we find that this function is given by

$$\begin{aligned} W_{m,\alpha}(l,\varphi) = \frac{1}{2\pi\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}\int_{-\pi}^{\pi}\theta_3\left(\frac{1}{2\pi}\left(\varphi-\alpha-\frac{\theta}{2}+\mathrm{i}m\right)\middle|\frac{\mathrm{i}}{2\pi}\right) \\ \times\,\theta_3\left(\frac{1}{2\pi}\left(\varphi-\alpha+\frac{\theta}{2}-\mathrm{i}m\right)\middle|\frac{\mathrm{i}}{2\pi}\right)\mathrm{e}^{-\mathrm{i}\theta l}\,\mathrm{d}\theta \end{aligned} \tag{5.30}$$

Using (3.6), (5.3) and (5.30) or (3.13) and (5.12) we get

$$\begin{aligned} W_{m,\alpha}(l,\varphi) = \frac{1}{\theta_3\left(\frac{\mathrm{i}m}{\pi}|\frac{\mathrm{i}}{\pi}\right)}\left(\theta_3\left(\frac{\varphi-\alpha}{\pi}|\frac{\mathrm{i}}{\pi}\right)\sum_{j=-\infty}^{\infty}\mathrm{e}^{-j^2}\,\mathrm{e}^{2mj}\,\mathrm{sinc}\,\pi(j-l)\right. \\ \left.+\,\theta_2\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\sum_{j=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{4}(2j+1)^2}\,\mathrm{e}^{m(2j+1)}\,\mathrm{sinc}\,\pi\left(j-l+\frac{1}{2}\right)\right). \end{aligned} \tag{5.31}$$

We have thus expressed the Wigner function for the Gaussian circular coherent states given by the integral (5.30) in terms of the Jacobi theta functions and quickly convergent series.

Integrating both sides of equations (5.30) and (5.31) over l and making use of (3.8), (5.3) and (5.23) we obtain the following relation for the Jacobi theta function

$$\left|\theta_3\left(\frac{1}{2\pi}(\varphi-\alpha-\mathrm{i}m)\middle|\frac{\mathrm{i}}{2\pi}\right)\right|^2 = \theta_3\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\theta_3\left(\frac{\mathrm{i}m}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)+\theta_2\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\theta_2\left(\frac{\mathrm{i}m}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right). \tag{5.32}$$

The formula (5.32) is a direct consequence of the Watson identity [13]

$$\theta_3\left(z\middle|\frac{\tau}{2}\right)\theta_3\left(w\middle|\frac{\tau}{2}\right)=\theta_3(z+w|\tau)\theta_3(z-w|\tau)+\theta_2\left(z+w|\tau\right)\theta_2(z-w|\tau). \tag{5.33}$$

We have thus verified the correctness of (5.31). The validity of (3.2) can be easily checked with the use of (5.2) and (5.30). The property (3.7) is a straightforward consequence of (5.31) and the relations (5.2), (5.3), (5.12) and (5.23).

Now, utilizing (5.4) we can write (5.31) for integer m as

$$\begin{aligned} W_{m,\alpha}(l,\varphi) \;=\; & \frac{1}{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}\left(\theta_3\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\sum_{j=-\infty}^{\infty}\mathrm{e}^{-j^2}\,\mathrm{sinc}\,\pi(j+m-l)\right.\\ & +\theta_2\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\sum_{j=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{4}(2j+1)^2}\\ & \left.\times\,\mathrm{sinc}\,\pi\left(j+m-l+\frac{1}{2}\right)\right), \quad m\in\mathbb{Z} \end{aligned} \tag{5.34}$$

Finally, for integer l the formula (5.34) reduces to

$$\begin{aligned} W_{m,\alpha}(l,\varphi) = & \frac{1}{\theta_3\left(0|\frac{\mathrm{i}}{\pi}\right)}\left(\theta_3\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\mathrm{e}^{-(l-m)^2}\right.\\ & \left.+\theta_2\left(\frac{\varphi-\alpha}{\pi}\middle|\frac{\mathrm{i}}{\pi}\right)\sum_{j=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{4}(2j+1)^2}\frac{(-1)^{j+m-l}}{j+m-l+\frac{1}{2}}\right), \quad m,l\in\mathbb{Z}. \end{aligned} \tag{5.35}$$

The expression for the Wigner function in the particular case of the integer m and l equivalent to (5.35) was originally obtained by Rigas *et al* [6]. More precisely, the formula obtained therein is an immediate consequence of (5.35) and the identity [22]

$$\theta_2(v|\tau)=\mathrm{e}^{\mathrm{i}\pi\left(\frac{\tau}{4}+v\right)}\theta_3\left(v+\frac{\tau}{2}\middle|\tau\right). \tag{5.36}$$

The Wigner function (5.30) is illustrated in figure 2 (top left). As with the Wigner function for the circular coherent states (4.15) the Wigner function in the Gaussian coherent states can be negative (see figure 2 top right and bottom left) and is peaked at $l=m$ and $\varphi=\alpha$ as shown in figure 2 (bottom right).

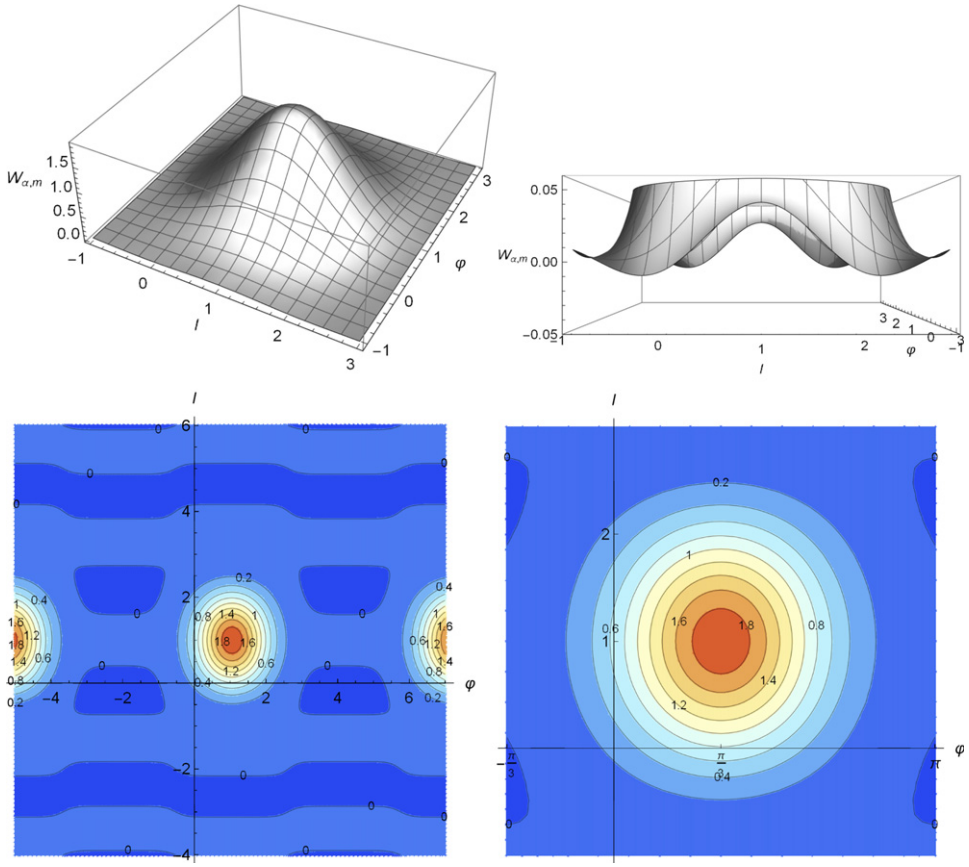

Figure 2. Top left: the plot of the surface given by the Wigner function for the Gaussian coherent states (5.30) where the parameters m and α are the same as in figure 1 for the all panels. Top right: the negative part of the Wigner function from the panel on the left. Bottom left: the contour plot of the Wigner function (5.30). The regions of the negative values of the Wigner function are easily seen. Bottom right: the contour plot of the Wigner function (5.30) illustrating the maximum of this function at $l = m$ and $\varphi = \alpha$.

6. Wigner function for the Gaussian–Fourier coherent states

6.1. Gaussian–Fourier coherent states

The third approach known from the literature on coherent states was introduced by Chadzitaskos, Luft and Tolar [3]. Roughly speaking it relies on restriction of the normalized Gaussian representing the vacuum state such that

$$f^T_{0,0}(\varphi) = A\,\mathrm{e}^{-\frac{1}{2}\varphi^2} \tag{6.1}$$

where $A = \sqrt{\frac{2\sqrt{\pi}}{\mathrm{erf}(\pi)}}$ and $\mathrm{erf}(x)$ is the error function, to the interval $\varphi \in [-\pi, \pi)$, and generating the family of coherent states by means of the action of the Weil operators on the vacuum state:

$$f^T_{m,\alpha}(\varphi) = \mathrm{e}^{\mathrm{i}m\hat{\varphi}}\,\mathrm{e}^{-\mathrm{i}\alpha J} f^T_{0,0}(\varphi) = U^m\,\mathrm{e}^{-\mathrm{i}\alpha J} f^T_{0,0}(\varphi), \tag{6.2}$$

where $m \in \mathbb{Z}$ and $\alpha \in [-\pi, \pi)$. In order to clip the defined coherent states to the range $[-\pi, \pi)$ they are defined in different regions as [3]

$$f^T_{m,\alpha}(\varphi) = \begin{cases} A\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\frac{1}{2}(\varphi-\alpha)^2} & \text{for } \varphi \in [-\pi, \pi+\alpha) \\ A\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\frac{1}{2}(\varphi-\alpha-2\pi)^2} & \text{for } \varphi \in [\pi+\alpha, \pi) \end{cases} \tag{6.3a}$$

where $\alpha \in [-\pi, 0)$, and

$$f^T_{m,\alpha}(\varphi) = \begin{cases} A\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\frac{1}{2}(\varphi-\alpha)^2} & \text{for } \varphi \in [-\pi+\alpha, \pi) \\ A\, \mathrm{e}^{\mathrm{i}m\varphi}\, \mathrm{e}^{-\frac{1}{2}(\varphi-\alpha+2\pi)^2} & \text{for } \varphi \in [-\pi, -\pi+\alpha) \end{cases} \tag{6.3b}$$

where $\alpha \in [0, \pi)$. A in (6.3) is the same normalization constant as in (6.1). In view of the method for construction of these states they would be called the clipped Gaussian coherent states. Of course, the functions (6.3) can be taken modulo 2π, nevertheless they are not explicitly periodic and it is not clear how to define the Wigner function for them. Therefore we introduce the proper coherent states based on the Fourier transform of the Gaussian $\mathrm{e}^{-\frac{1}{2}\varphi^2}$, hence the normalized vacuum state is

$$f^F_{0,0}(\varphi) = C \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}j^2}\, \mathrm{e}^{\mathrm{i}j\varphi} [\mathrm{erf}\left(\frac{\pi + \mathrm{i}j}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi - \mathrm{i}j}{\sqrt{2}}\right)], \tag{6.4}$$

where C is the normalization constant given by

$$C = \frac{1}{\sqrt{\sum_{j=-\infty}^{\infty} \mathrm{e}^{-j^2} \left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}j}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi-\mathrm{i}j}{\sqrt{2}}\right)\right]^2}} \tag{6.5}$$

and the use was made of the identity

$$\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}kx}\, \mathrm{e}^{-\frac{1}{2}x^2}\, \mathrm{d}x = \mathrm{e}^{-\frac{1}{2}k^2} \sqrt{\frac{\pi}{2}} \left[\mathrm{erf}\left(\frac{\pi + \mathrm{i}k}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi - \mathrm{i}k}{\sqrt{2}}\right)\right]. \tag{6.6}$$

We point out that by virtue of the Parseval theorem we have the relation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\varphi^2}\, \mathrm{d}\varphi = \frac{1}{8\pi} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-j^2} \left[\mathrm{erf}\left(\frac{\pi + \mathrm{i}j}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi - \mathrm{i}j}{\sqrt{2}}\right)\right]^2 \tag{6.7}$$

implying the nontrivial identity

$$\sum_{j=-\infty}^{\infty} \mathrm{e}^{-j^2} \left[\mathrm{erf}\left(\frac{\pi + \mathrm{i}j}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi - \mathrm{i}j}{\sqrt{2}}\right)\right]^2 = 4\sqrt{\pi}\mathrm{erf}(\pi). \tag{6.8}$$

The constant C calculated from (6.5) and (6.8) is $C = \frac{1}{2\sqrt{\sqrt{\pi}\mathrm{erf}(\pi)}} = 0.375\,564\,439$.

Proceeding as with (6.1) we define the coherent states for arbitrary α and integer m as

$$\begin{aligned} f^F_{m,\alpha}(\varphi) &= \mathrm{e}^{\mathrm{i}m\hat{\varphi}}\, \mathrm{e}^{-\mathrm{i}\alpha J} f^F_{0,0}(\varphi) = U^m\, \mathrm{e}^{-\mathrm{i}\alpha J} f^F_{0,0}(\varphi) \\ &= C\, \mathrm{e}^{\mathrm{i}m\varphi} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}j^2}\, \mathrm{e}^{\mathrm{i}j(\varphi-\alpha)} \left[\mathrm{erf}\left(\frac{\pi + \mathrm{i}j}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\pi - \mathrm{i}j}{\sqrt{2}}\right)\right]. \end{aligned} \tag{6.9}$$

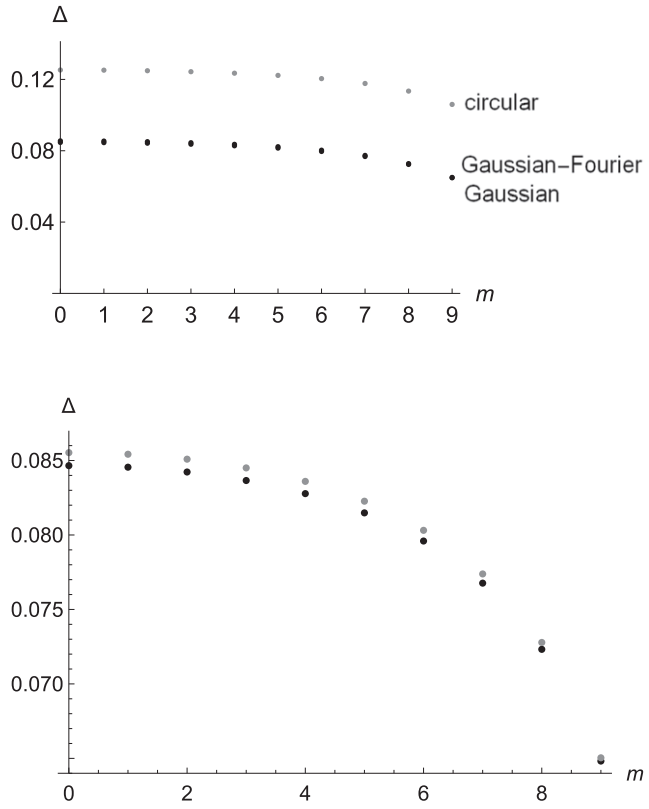

Figure 3. A comparison of the classicality of the investigated coherent states for the quantum mechanics on a circle, where the indicator of non-classicality is the quantity $\Delta(f)$ given by (7.9) and we set $\alpha = 0$ in the corresponding Wigner functions (4.15), (5.30) and (6.19) and $s = 1.26$ in (4.15). Because of the very small differences that cannot be seen in the top panel the non-classicality is compared separately in the bottom panel of the Wigner function (5.30) for the Gaussian coherent states (black disks) and the Wigner function (6.19) in the Gaussian–Fourier ones (gray disks). A look at both panels is enough to conclude that the most classical coherent states are the Gaussian ones.

Bearing in mind the procedure of construction of these states we have decided to call them the Gaussian–Fourier coherent states. We remark that the method of periodization of functions based on the utilization of the Fourier transform is an alternative of wrapping functions around the circle such as that applied in the case of the Gaussian coherent states (see (5.6)). As indicated by a referee such method is restricted to the case of a manifold with a non-trivial topology such as a circle. Indeed, for the quantum mechanics on a line the Fourier transform of a Gaussian is also a Gaussian so the discussed method cannot be applied for generation of coherent states. It should also be noted that the relations (5.9), (6.2) and (6.9) can be regarded as counterparts of the formula for generation of the standard coherent states by means of the unitary displacement operator applied to the ground state of the harmonic oscillator.

We now present some most important properties of the Gaussian–Fourier coherent states. Using (2.10) we find that the projection of the normalized coherent states $|m, \alpha\rangle_F$, where $f^F_{m,\alpha}(\varphi) = \langle\varphi|m, \alpha\rangle_F$, onto the eigenvectors $|j\rangle$ of the angular momentum is of the

form

$$\langle j|m,\alpha\rangle_F = C\,\mathrm{e}^{-\frac{1}{2}(j-m)^2}\,\mathrm{e}^{-\mathrm{i}(j-m)\alpha}\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(j-m)}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(j-m)}{\sqrt{2}}\right)\right]. \tag{6.10}$$

The coherent states (6.9) are not orthogonal. Using (2.5b) and (6.10) we obtain the following formula for the overlap of these states

$$\begin{aligned}{}_F\langle m,\alpha|m',\alpha'\rangle_F = C^2\mathrm{e}^{-\mathrm{i}m\alpha}\,\mathrm{e}^{\mathrm{i}m'\alpha'}\sum_{j=-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2}(j-m)^2}\,\mathrm{e}^{-\frac{1}{2}(j-m')^2}\\ \times\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(j-m)}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(j-m)}{\sqrt{2}}\right)\right]\\ \times\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(j-m')}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(j-m')}{\sqrt{2}}\right)\right]\end{aligned} \tag{6.11}$$

The Gaussian–Fourier coherent states form a complete (over complete) set. On making use of (2.5a) and (6.10) we arrive at the following resolution of the identity

$$\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\alpha|m,\alpha\rangle_{FF}\langle m,\alpha| = I. \tag{6.12}$$

We now discuss the reproduction of classical values by the Gaussian–Fourier coherent states. Utilizing (2.5b) and (6.10) we find

$${}_F\langle m,\alpha|\,J\,|m,\alpha\rangle_F = m. \tag{6.13}$$

Therefore the parameter m corresponds to the classical angular momentum. Nevertheless, it must be borne in mind that in opposition to the classical case m is discrete. Furthermore, taking into account (2.3) and the second equation of (6.9) we get

$${}_F\langle m,\alpha|\,U\,|m,\alpha\rangle_F = {}_F\langle 0,0|\,\mathrm{e}^{\mathrm{i}\alpha J}U\mathrm{e}^{-\mathrm{i}\alpha J}\,|0,0\rangle_F = \mathrm{e}^{\mathrm{i}\alpha}{}_F\langle 0,0|U|0,0\rangle_F, \tag{6.14}$$

where $|0,0\rangle_F$ is the abstract normalized vacuum vector such that $f^F_{0,0}(\varphi) = \langle\varphi|0,0\rangle_F$. Using(2.7), (2.10) and (6.4) we find

$$\begin{aligned}{}_F\langle 0,0|\,U\,|0,0\rangle_F = e^{-\frac{1}{4}}C^2\sum_{j=-\infty}^{\infty}e^{-\left(j+\frac{1}{2}\right)^2}\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}j}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}j}{\sqrt{2}}\right)\right]\\ \times\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(j+1)}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(j+1)}{\sqrt{2}}\right)\right].\end{aligned} \tag{6.15}$$

It follows that

$${}_F\langle 0,0|\,U\,|0,0\rangle_F \approx \mathrm{e}^{-\frac{1}{4}}, \tag{6.16}$$

where the relative error is 2 per mil, so the expression (6.14) for the expectation value of U has the same form as (5.24) in the case of the Gaussian coherent states. On introducing the relative average (see (5.25))

$${}_F\langle\langle m,\alpha|\,U\,|m,\alpha\rangle\rangle_F = \frac{{}_F\langle m,\alpha|\,U\,|m,\alpha\rangle_F}{{}_F\langle 0,0|\,U\,|0,0\rangle_F} \tag{6.17}$$

we get

$$ {}_F\langle\langle m,\alpha|\, U\, |m,\alpha\rangle\rangle_F = \mathrm{e}^{\mathrm{i}\alpha}, \tag{6.18}$$

so α can be identified with the classical angle. We finally point out that in opposition to the Gaussian circular coherent states satisfying (5.1) we do not know the eigenvalue equation defining the Gaussian–Fourier coherent states.

6.2. Wigner function for the Gaussian–Fourier coherent states

Our purpose now is to examine the Wigner function of the Gaussian–Fourier coherent states. From (3.1) and (6.9) it follows that this function is given by

$$\begin{aligned} W^F_{m,\alpha}(l,\varphi) = C^2 \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}j^2}\,\mathrm{e}^{-\frac{1}{2}k^2}\,\mathrm{e}^{\mathrm{i}(k-j)(\varphi-\alpha)} \times \left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}j}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}j}{\sqrt{2}}\right)\right] \\ \times \left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}k}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}k}{\sqrt{2}}\right)\right] \\ \times \operatorname{sinc}\pi\left[\frac{1}{2}(k+j)+m-l\right], \quad m\in\mathbb{Z} \end{aligned} \tag{6.19}$$

One can check that all the properties of the Wigner function presented in section 3 hold true in the case of the function (6.19). Unfortunately, we do not know any way to simplify the double sum in (6.19).

It follows from the numerical calculations that the behavior of the Wigner function (6.19) for the Gaussian–Fourier coherent states is very similar to that of the Wigner function (5.30) in the Gaussian coherent states presented in figure 2 and we have decided not to depict it herein. The difference of the two Wigner functions related to the degree of their negativity is demonstrated in the next section (see figure 3, right).

7. Comparison of classicality of the coherent states based on the Wigner function

Coherent states can be regarded as the states closest to the classical ones. In particular, they were introduced by Schrödinger in 1926 [23] as the states minimizing the Heisenberg uncertainty relations. In the discussed case of the quantum mechanics on a circle there are no generally accepted uncertainty relations. Instead, we have decided to use as an indicator of classicality of the investigated coherent states the negativity of the Wigner function [24]. In order to enable the comparison we should first fix the squeezing parameter s given by (4.4) parametrizing the circular coherent states. We assume for simplicity that the classical angle α labeling the coherent states is equal to zero. We also restrict to the case of integer classical momentum m marking the coherent states. Using (5.22) and the following identities for the Gaussian coherent states $|m,\alpha\rangle$

$$\begin{aligned} (\Delta J)^2 = \frac{1}{2} - \pi^2\,\cos\,2\pi m \sum_{n=1}^{\infty}\frac{1}{\cosh\,\pi^2(2n-1)+\cos\,2\pi m} \\ - \pi^2\sin^2 2\pi m \sum_{n=1}^{\infty}\frac{1}{(\cosh\,\pi^2(2n-1)+\cos\,2\pi m)^2}, \end{aligned} \tag{7.1}$$

where m is real, and

$$\langle m,\alpha|U^2|m,\alpha\rangle = \mathrm{e}^{-1}\,\mathrm{e}^{2\mathrm{i}\alpha} \tag{7.2}$$

we find that the counterpart of the squeezing parameter s for the Gaussian coherent states $|m,\alpha\rangle$ in the case of $\alpha = 0$ and integer m is

$$s_{\mathrm{G}} = \frac{\Delta J}{\Delta\,\sin\,\hat{\varphi}} = 1.256\,48 \tag{7.3}$$

Furthermore, utilizing (6.14) and the following relations that hold true for the Gaussian–Fourier coherent states

$$(\Delta J)^2 = \frac{1}{2\sqrt{\pi}\mathrm{erf}(\pi)}\sum_{n=1}^{\infty} n^2\,\mathrm{e}^{-n^2}\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}n}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}n}{\sqrt{2}}\right)\right]^2 \tag{7.4}$$

and

$${}_F\langle m,\alpha|\,U^2\,|m,\alpha\rangle_F = \mathrm{e}^{2\mathrm{i}\alpha}\,{}_F\langle 0,0|U^2|0,0\rangle_F, \tag{7.5}$$

where

$$\begin{aligned}{}_F\langle 0,0|\,U^2\,|0,0\rangle_F = e^{-1}\frac{1}{4\sqrt{\pi}\;\mathrm{erf}(\pi)}\Bigg\{&\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}}{\sqrt{2}}\right)\right]^2\\ &+2\sum_{n=1}^{\infty}\mathrm{e}^{-n^2}\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(n+1)}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(n+1)}{\sqrt{2}}\right)\right]\\ &\times\left[\mathrm{erf}\left(\frac{\pi+\mathrm{i}(n-1)}{\sqrt{2}}\right)+\mathrm{erf}\left(\frac{\pi-\mathrm{i}(n-1)}{\sqrt{2}}\right)\right]\Bigg\},\end{aligned} \tag{7.6}$$

we arrive at the following counterpart of the squeezing parameter in the case of the Gaussian–Fourier coherent states with $\alpha = 0$

$$s_{\mathrm{F}} = 1.257\,89 \tag{7.7}$$

We remark that (7.5) is a very good approximation of (7.2). Namely, we have ${}_F\langle m,\alpha|U^2|m,\alpha\rangle_F = 0.999\,985\cdot\mathrm{e}^{-1}\,\mathrm{e}^{2\mathrm{i}\alpha}$. In view of (7.3) and (7.7) we can set $s = 1.26$ for the circular coherent states to enable their comparison with the remaining coherent states.

Now let $W_{\mathrm{f}}(l,\varphi)$ be the Wigner function of the state $f(\varphi)$ of a particle on a circle. Following [24] we can consider the double volume $\delta(f)$ of the integrated negative part of the Wigner function $W_{\mathrm{f}}(l,\varphi)$ as an indicator of non-classicality of $f(\varphi)$, so

$$\delta(f) = \int_{-\infty}^{\infty}\mathrm{d}l\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi(|W_{\mathrm{f}}(l,\varphi)|-W_{\mathrm{f}}(l,\varphi)) \tag{7.8a}$$

$$= \int_{-\infty}^{\infty}\mathrm{d}l\,\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi|W_{\mathrm{f}}(l,\varphi)|-1, \tag{7.8b}$$

where (7.8b) is an immediate consequence of (7.8a) and (3.10). Bearing in mind the discreteness of the orbital momentum variable in the Wigner function preferred by some authors and simplicity of numerical calculations we use instead of $\delta(f)$ the following quantity

$$\Delta(f) = \sum_{l=-\infty}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\varphi|W_{\mathrm{f}}(l,\varphi)|-1, \tag{7.9}$$

Clearly, $\Delta(f)$ is the total double area bounded by negative part of the Wigner function $W_{\mathrm{f}}(l,\varphi)$ of the one variable φ with fixed l and the axis $\varphi = 0$, summed over all integer l.

A comparison of classicality of the discussed coherent states for a particle on a circle based on the application of $\Delta(f)$ defined by (7.9) as an indicator of negativity of the Wigner function is shown in figure 3. As easily seen the most classical and thus the best are the Gaussian coherent states.

8. Conclusion

In this work the Wigner functions are studied in the coherent states for the quantum mechanics on a circle. It seems that the introduced nontrivial analytic expressions of the Wigner functions (4.19), (4.22) and (5.31) would be of importance in practical applications. We only recall the utilization of the circular squeezed states in the investigation of the Rydberg wave packets. It should also be noted that the method for construction of the Gaussian–Fourier coherent states described herein relying on application of the Fourier transform would be a new effective tool in the theory of coherent states. We point out the very small differences between the Gaussian–Fourier states and the best as shown in this paper, Gaussian coherent states. An interesting property of the all discussed Wigner functions is the maximum at points of the classical phase space labeling the coherent states. As a matter of fact such behavior at particular point of the Wigner function for the Gaussian coherent states was reported for example in reference [6], nevertheless it seems that the peak of the Wigner functions is their general attribute regardless of the type of coherent states marked with point of a phase space. Moreover, it seems that the presence of such peak would be regarded as a correctness criterion for both Wigner functions and coherent states. We remark that the probability density for the coordinates and momenta in the standard coherent states and coherent states for a particle on a circle (see (4.3) and (5.2)) and sphere [25] are peaked at the position and momentum orangular momentum, respectively referring to the classical one parametrizing the coherent state. The maximum of the Wigner function at points labeling the coherent states can be viewed as a generalization of the behavior of the probability densities to the case of the phase space. We finally point out that the comparison of classicality of coherent states based on negativity of the corresponding Wigner function would be a useful tool for selecting the best coherent states. For instance it would be interesting to compare the classicality of the coherent states for a quantum particle on a sphere introduced in reference [26] with that discussed in reference [27].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

K Kowalski https://orcid.org/0000-0003-2160-0526
K Ławniczak https://orcid.org/0000-0002-9530-8846

References

[1] Hudson R L 1974 When is the Wigner quasi-probability density non-negative? *Rep. Math. Phys.* **6** 249–52

[2] Rigas I, Sánchez-Soto L L, Klimov A B, Řehaček J and Hradil Z 2010 Non-negative Wigner functions for orbital angular momentum states *Phys. Rev.* A **81** 012101
[3] Chadzitaskos G, Luft P and Tolar J 2012 Quantizations on the circle and coherent states *J. Phys. A: Math. Theor.* **45** 244027
[4] Kowalski K, Rembielinski J and Papaloucas L C 1996 Coherent states for a quantum particle on a circle *J. Phys. A: Math. Gen.* **29** 4149–67
[5] Kastrup H A 2016 Wigner function for the pair angle and orbital momentum *Phys. Rev.* A **94** 062113
[6] Rigas I, Sánchez-Soto L L, Klimov A B, Řehaček J and Hradil Z 2011 Orbital angular momentum in phase space *Ann. Phys., NY* **326** 426–39
[7] Alonso M A, Pogosyan G S and Wolf K B 2003 Wigner functions for curved spaces. II. On spheres *J. Math. Phys.* **44** 1472–89
[8] Carruthers P and Nieto M M 1968 Phase and angle variables in quantum mechanics *Rev. Mod. Phys.* **40** 411–40
[9] Bluhm R, Kostelecký V A and Tudose B 1995 Elliptical squeezed states and Rydberg wave packets *Phys. Rev.* A **52** 2234–44
[10] Arfken G B, Weber H J and Harris F E 2013 *Mathematical Methods for Physicists* (Amsterdam: Elsevier)
[11] Prudnikov A P, Brychkov Yu A and Marichev O I 2002 *Integrals and Series* (*Elementary Functions* vol 1) (Moscow: Fizmatlig)
[12] Gradshteyn I S and Ryzhik I M 2007 *Tables of Integrals, Series, and Products* (Amsterdam: Elsevier)
[13] Olver F W J (ed) 2010 *NIST Handbook of Mathematical Functions* (Cambridge: Cambridge University Press)
[14] Gonzáles J A and del Olmo M A 1998 Coherent states on the circle *J. Phys. A: Math. Gen.* **31** 8841–57
[15] Perelomov A M 1986 *Generalized Coherent States and Their Applications* (Berlin: Springer)
[16] Hall B C 1994 The Segal–Bargmann 'coherent state' transform for compact Lie groups *J. Funct. Anal.* **122** 103–51
[17] Stenzel M B 1999 The Segal–Bargmann transform on a symmetric space of compact type *J. Funct. Anal.* **165** 44–58
[18] Hall B C and Mitchell J J 2002 Coherent states on spheres *J. Math. Phys.* **43** 1211–36
[19] De Bièvre S 1989 Coherent states over symplectic homogeneous spaces *J. Math. Phys.* **30** 1401–7
[20] Isham C J and Klauder J R 1991 Coherent states for n-dimensional Euclidean groups $E(n)$ and their application *J. Math. Phys.* **32** 607–20
[21] Kowalski K and Rembieliński J 2002 On the uncertainty relations and squeezed states for the quantum mechanics on a circle *J. Phys. A: Math. Gen.* **35** 1405–14
[22] Korn G A and Korn T M 2000 *Mathematical Handbook for Scientist and Engineers* (New York: Dover)
[23] Schrödinger E 1926 Der stetige Übergang von der Mikro-zur Makromechanik *Naturwissenschaften* **14** 664–6
[24] Kenfack A and Zyczkowski K 2004 Negativity of the Wigner function as an indicator of non-classicality *J. Opt. B: Quantum Semiclass. Opt.* **6** 396–404
[25] Kowalski K and Rembieliński J 2001 The Bargmann representation for the quantum mechanics on a sphere *J. Math. Phys.* **42** 4138–47
[26] Kowalski K and Rembielinski J 2000 Quantum mechanics on a sphere and coherent states *J. Phys. A: Math. Gen.* **33** 6035–48
[27] Díaz-Ortiz E I and Villegas-Blas C 2012 On a Bargmann transform and coherent states for the n-sphere *J. Math. Phys.* **53** 062103