ACTA UNIVERSITATIS LODZIENSIS FOLIA OECONOMICA 225, 2009

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THE METHOD OF RISK MEASUREMENT IN CASE OF STOCHASTIC DEFINITION OF NET PRESENT VALUE

Abstract

The Net Present Value (NPV) rule is a base of modern finance theory. The classical definition of NPV is based on unrealistic assumptions: it treats the discount rate as a deterministic and constant function. The above fact may lead to some situations when the classical NPV may mark the wrong direction of investing. Moreover, the deterministic formula makes the measurement of risk impossible – we can not estimate the probability of obtaining any values of NPV. In this paper we consider a stochastic, general definition of Net Present Value. We propose the method of measurement of risk in case of using the stochastic definition. The risk is identified with probability, that stochastic process NPV(t), starting from a negative and real point B(0) (initial investment), will not leave the interval $(-\infty, 0)$ before end of the project (T). Presented considerations lead to Pontriagin's differential equation and its solution is the mentioned probability.

Key words: net present value, risk measurement, stochastis processes.

1. Introduction

Investment decisions should be based on the economic calculation, which requires qualification of methods of measurement the profitability. This is one of the major subjects studied in both theory of finance and financial mathematics. Most academics and professionals agree that the Net Present Value (*NPV*) rule is the most reliable criterion in ranking investment projects (W y p y c h a, 1999; G a j d k a, W a l i ń s k a, 1998). Although this method has some faults (see O e h m k e, 2000; M a g n i, 2002; B i a ł e k, 2005a) it is still very popular and recommended by banks and UNIDO. *NPV* rule discounts all cash flows connected with realization of our project. For the investment project specified by

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cash flows: $\{P_0, P_1, P_2..., P_T\}$, where real P_i – denotes the outflow or inflow of the money connected with *i*-th year of realization ($i \in \{1, 2, ..., T\}$), we define Net Present Value as:

$$NPV = \sum_{t=0}^{T} \frac{P_t}{(1+r)^t}$$
(1)

where:

T- is the (possibly infinite) life of the project,

r – is a considered discount rate.

Our project is acceptable only if NPV > 0. But this method takes into account no change of a discount rate during the time interval [0, T]. Besides NPV treats each cash flow as if it appears at the end of the year. The right, the most general definition of measure of the efficiency should treat both discount rate and cash flow as random variables depending on time.

In the contemporary economy we have general, stochastic definitions of Net Present Value. Omitting all technical assumptions (see B i a ł e k, 2005b), in the case of continuous time we can define Net Present Value using the Ito integral as below:

$$\overline{NPV}(T) = B(0) + \int_{0}^{T} \frac{\alpha(t)}{\alpha(t)} dt + \int_{0}^{T} \frac{\beta(t)}{\alpha(t)} dW(t)$$
(2)

where presented stochastic processes mean:

B(t) – all cash flows (connected with realization of our investment project) which appeared till moment t,

B(0) – the initial investment,

a(t) – the process of accumulation of money,

W(t) – the standard Brownian motion,

 $\alpha(t), \beta(t)$ – are progressively measurable processes on [0, T],

and the stochastic process B(t) is modeled as follows (the so called *outflow-inflow process*, see K o o (1998):

$$dB(t) = \alpha(t)dt + \beta(t)dW(t)$$
(3)

The formula (2) means that the new definition is a random variable – $\overline{NPV}: \Omega \rightarrow R$, on some probability space (Ω, F, P) . The \overline{NPV} takes into account changing values of discount rate and treats each cash flow as a random variable. It allows to calculate the probability of events like "the project is profitable" or "the project is unprofitable". And finally, we can measure the risk

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of the project. Let us notice that under some technical assumptions the *NPV* definition can be presented as a solution of the below stochastic differential equation:

$$dNPV(t) = A(t)dt + B(t)dW(t)$$
(4)

where

$$NPV(0) = B(0), \ A(t) = \frac{\alpha(t)}{a(t)}, \ B(t) = \frac{\beta(t)}{a(t)}$$
(5)

2. The pontriagin's equation

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Let us consider the process described by the following stochastic differential equation:

$$dx(t) = \hat{a}(t, x(t))dt + b(t, x(t))dW(t)$$
(6)

where

 $\hat{a}(t, x(t))$ – coefficient of drift, $\hat{b}(t, x(t))$ – coefficient of diffusion

and $\hat{a}(\cdot,\cdot)$, $\hat{b}(\cdot,\cdot)$ satisfy some technical assumptions – see Jakubowski *et al.* (2003). We are going to find an equation describing the probability that a stochastic process x(t), starting from some real point x(0), will not leave the set D before time t.

Let us denote: $x(0) = x_0$ and let us assume that $x_0 \in \text{int } D$. Let us denote by $P_D(t, x_0)$ – the probability, that a stochastic process x(t), starting from a real point x_0 , will **leave** the set D before time t. Hence, the probability of staying inside D during the time interval [0, t] equals: $Q_D(t, x_0) = \frac{1}{2} - P_D(t, x_0)$.

But the same probability can be calculated as $\int_D p(t, s, x_0) ds$, where $p(t, s, x_0)$ is a density of probability of finding the process x(t) in point s after time t but under the condition, that the process has not left the set D so far. Hence we get:

$$\int_{D} p(t, s, x_0) ds + P_D(t, x_0) = 1$$
(7)

The probability $P_D(t + \tau, x_0)$ is a sum of two, separable events as follows:

$$P_D(t+\tau, x_0) = P_D(\tau, x_0) + \int_D p(t, s, x_0) P_D(t, s) ds$$
(8)

Using the Taylor formula for $P_D(\tau, x)$ – near the point x_0 – we get

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$$\frac{P_D(t+\tau,x_0) - P_D(\tau,x_0)}{\tau} = \frac{P_D(\tau,x_0)}{\tau} (1 - P_D(t,x_0)) + \frac{\partial P_D(t,x_0)}{\partial x_0} \frac{1}{\tau} \int_D (s-x_0) p(\tau,s,x_0) ds + \frac{1}{2} \frac{\partial^2 P_D(t,x_0)}{\partial x_0^2} \frac{1}{\tau} \int_D (s-x_0)^2 p(\tau,s,x_0) ds + \frac{1}{3!} \frac{\partial^3 P_D(t,x_0 + \lambda(x'-x_0))}{\partial x_0^3} \frac{1}{\tau} \int_D (s-x_0)^3 p(\tau,s,x_0) ds$$
(9)

where $\lambda \in (0, 1)$, $x' \in \text{int } D$.

We hale also (see Rolski, Schmidli, Schmidt, Teugels 1999):

$$\lim_{\tau \to o^+} \frac{P_D(\tau, x_0)}{\tau} = 0 \qquad \forall x_0 \in D$$
(10)

$$\lim_{\tau \to 0^+} \frac{1}{\tau} \int_D (s - x_0)^n p(\tau, s, x_0) ds = K_n(x_0)$$
(11)

where:

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$$K_1(x_0) = \hat{a}(0, x_0)$$

$$K_2(x_0) = \hat{b}^2(0, x_0)$$

 $K_n(x_0) = 0, \text{ dla } n \ge 3$

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Under the limit $\tau \to 0^+$ we get

$$\frac{\partial P_D(t, x_0)}{\partial t} = \hat{a}(0, x_0) \frac{\partial P_D(t, x_0)}{\partial x_0} + \frac{1}{2} \hat{b}^2(0, x_0) \frac{\partial^2 P_D(t, x_0)}{\partial x_0^2}$$
(12)

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Similarly, in the case of $Q_D(t, x_0)$, we get

$$\frac{\partial Q_D(t, x_0)}{\partial t} = \hat{a}(0, x_0) \frac{\partial Q_D(t, x_0)}{\partial x_0} + \frac{1}{2} \hat{b}^2(0, x_0) \frac{\partial^2 Q_D(t, x_0)}{\partial x_0^2}$$
(13)

The formula (12) and (13) is known in the literature as a first Pontriagin's differential equation. To solve it we must consider additionally some frontier conditions. Let us consider the equation described in (12). Let us notice that

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$$P_D(0, x_0) = 0 \text{ if } x_0 \in \text{int } D,$$
 (14)

so the process x(t), starting from x_0 , has no chance to leave the set D during the infinitely short time. Additionally, we have

$$P_D(t,s) = 1 \quad \forall s \in FrD \tag{15}$$

where *FrD* means the frontier of set *D*.

3. The definition of risk

Let us consider the equation (4) as a special case of (6). The right Pontriagin's differential equation is as follows:

$$\frac{\partial P_D(t, x_0)}{\partial t} = A(0) \frac{\partial P_D(t, x_0)}{\partial x_0} + \frac{1}{2} B^2(0) \frac{\partial^2 P_D(t, x_0)}{\partial x_0^2}$$
(16)

We assume that a(0) = 1 and it implicates

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$$A(0) = \frac{\alpha(0)}{a(0)} = \alpha(0), \ B(0) = \frac{\beta(0)}{a(0)} = \beta(0)$$
(17)

We consider $D = (-\infty, 0)$ because in the case of process NPV(t) we want to know the chance of leaving it. Let us notice that $FrD = \{0\}$ and it implicates the following frontier conditions: $P_D(t, 0) = 1$ and $P_D(0, x_0) = 0$ for $x_0 \in (-\infty, 0)$. To solve an equation:

$$\frac{\partial P_D(t, x_0)}{\partial t} = \alpha(0) \frac{\partial P_D(t, x_0)}{\partial x_0} + \frac{1}{2} \beta^2(0) \frac{\partial^2 P_D(t, x_0)}{\partial x_0^2}$$
(18)

we use the Laplace transform (see Sneddon (1972)) with regard on variable t, defined as follows

$$L_{s}P_{D}(t,x_{0}) = \hat{P}_{D}(s,x_{0}) = \int_{0}^{\infty} \exp(-st)P_{D}(t,x_{0})dt$$
(19)

Using the transform for both sides of equation (18) we get:

$$s\hat{P}_D(s,x_0) - P_D(0,x_0) = \alpha(0)\frac{d\hat{P}_D(s,x_0)}{dx_0} + \frac{1}{2}\beta^2(0)\frac{d^2\hat{P}_D(s,x_0)}{dx_0^2}$$
(20)

The frontier condition $P_D(0, x_0) = 0$ leads to

$$\alpha(0)\frac{d\hat{P}_D(s,x_0)}{dx_0} + \frac{1}{2}\beta^2(0)\frac{d^2\hat{P}_D(s,x_0)}{dx_0^2} - s\hat{P}_D(s,x_0) = 0$$
(21)

The characteristic elements of equation (21) equal:

$$\omega_{\rm f} = -\frac{\sqrt{\alpha^2(0) + 2s\beta^2(0)} + \alpha(0)}{\beta^2(0)} < 0$$
(22)

$$\omega_2 = \frac{\sqrt{\alpha^2(0) + 2s\beta^2(0)} - \alpha(0)}{\beta^2(0)} > 0 \quad \text{(for } \alpha(0)\beta(0) \neq 0\text{)}$$
(23)

And the common solution of (21) is

$$\hat{P}_D(s, x_0) = M \exp(\omega_1 x_0) + N \exp(\omega_2 x_0)$$
(24)

Let us notice that $\lim_{x_0\to\infty} \exp(\omega_1 x_0) = \infty$ and it implicates M = 0. From (24) we get

$$\dot{P}_D(s, x_0) = N \exp(\omega_2 x_0) \tag{25}$$

Let us notice that on the one hand we have

$$L_s P_D(t,0) = L_s 1 = \frac{1}{s}$$
(26)

and on the other hand we have

$$L_{s}P_{D}(t,0) = \hat{P}(s,0) = N \exp(0) = N$$
(27)

From (26) and (27) we get

$$N = \frac{1}{s} \tag{28}$$

and from (25) we get finally

$$\hat{P}_{D}(s, x_{0}) = \frac{1}{s} \exp(\omega_{2} x_{0}) = \frac{1}{s} \exp(\frac{\sqrt{\alpha^{2}(0) + 2s\beta^{2}(0)} - \alpha(0)}{\beta^{2}(0)} x_{0}) =$$
$$= \exp(\frac{-\alpha(0)}{\beta^{2}(0)} x_{0}) \frac{1}{s} \exp(\frac{\sqrt{\alpha^{2}(0) + 2s\beta^{2}(0)}}{\beta^{2}(0)} x_{0})$$
(29)

To return to the variable t and $P_D(t, x_0)$ we must use the inverse Laplace transform

$$P_D(t, x_0) = L_t^{-1} \hat{P}_D(s, x_0)$$
(30)

After some technical operations (see for example Sobczyk (1996), Sneddon (1972)) we get

$$P_D(t, x_0) = 2\exp(-\frac{\alpha(0)}{\beta^2(0)}x_0)\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{x_0^2}{\beta^2(0)t}}}^{\infty}\exp(-\frac{\alpha^2(0)x_0^2}{2\beta^4(0)z^2} - \frac{z^2}{2})dz$$
(31)

And the risk can be defined as follows

$$R(t, x_0) = Q_D(t, x_0) =$$

$$1 - 2\exp(-\frac{\alpha(0)}{\beta^2(0)}x_0)\frac{1}{\sqrt{2\pi}}\int_{\sqrt{\frac{x_0^2}{\beta^2(0)t}}}^{\infty} \exp(-\frac{\alpha^2(0)x_0^2}{2\beta^4(0)z^2} - \frac{z^2}{2})dz \qquad (32)$$

Let us notice that in the case of negligible drift ($\alpha \approx 0$) we obtain from (32)

$$R(t, x_{0}) \approx 1 - 2 \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{x_{0}^{2}}{\beta^{2}(0)t}}}^{\infty} \exp(-\frac{z^{2}}{2}) dz = 1 - 2(1 - \Phi(\sqrt{\frac{x_{0}^{2}}{\beta^{2}(0)t}})) =$$
$$= 2\Phi(\sqrt{\frac{x_{0}^{2}}{\beta^{2}(0)t}}) - 1$$
(33)

where $\Phi(\cdot)$ – Gaussian distribution with mean 0 and standard deviation 1.

Certainly, the bigger the initial investment the higher the risk. Asymptotically we have

$$\lim_{x_0 \to -\infty} R(t, x_0) = 1$$
(34)

4. Examples

Let us consider the investment project whose function of cash flows B(t) is described by arithmetic Brownian motion:

$$dB(t) = \mu dt + \delta dW(t), \text{ where } \mu \in R, \delta \in R_+, B(0) = x_0$$
(35)

We are going to consider three cases depending on parameters in (35).

Example 1. We assume that the initial investment equals $x_0 = -2$ monetary units (for example 10 000 PLN). Let us consider the process described in (35) for $\mu = 3$ and $\delta = 0.5$. We assume that the time horizontal equals T = 2 time units (for example years). The following graph presents the example of realization B(t):



Graph 1. The realization of process B(t) for $\mu = 3$ and $\delta = 0.5$ and B(0) = -2

After calculations, the risk – defined in (32) – equals: R(2,-2) = 0.00246. We can see that the chance that the process NPV(t) will not leave the interval $(-\infty, 0)$ before time T = 2 is negligible. The risk is very small and the presented graph can verify it.

Example 2. We assume that the initial investment equals $x_0 = -6$ monetary units. We consider the process described in (35) for $\mu = 3$ and $\delta = 0.5$ (see Example 1). As in the previous case we assume that the time horizontal equals T = 2 time units. The following graph presents the example of realization B(t):



Graph 2. The realization of process B(t) for $\mu = 3$ and $\delta = 0.5$ and B(0) = -6

After calculations, the risk – defined in (32) – equals: R(2, -6) = 0.483024. We can see that the chance that the process NPV(t) will not leave the interval $(-\infty, 0)$ before time T = 2 is not negligible in this case. The risk is appreciable and the presented graph can verify it.

Example 3. We assume that the initial investment equals $x_0 = -8$ monetary units. We consider the process described in (35) for $\mu = 3$ and $\delta = 0.5$ (as in the previous cases). We assume that the time horizontal equals T = 2 time units. The following graph presents the example of realization B(t):

After calculations, the risk – defined in (32) – equals: R(2, -8) = 0.99998. We can see that the chance that the process NPV(t) will not leave the interval $(-\infty, 0)$ before time T = 2 is huge. The risk equals almost 1 (maximum value) and the presented graph can verify it.



Graph 3. The realization of process B(t) for $\mu = 3$ and $\delta = 0.5$ and B(0) = -8

5. Conclusions

The formula (32) can be used for measuring the risk in case of stochastic definition of Net Present Value. The class (3) of processes is large: it includes Gaussian processes $B(t) = N(\mu(t), \delta(t))$ – used in practice (see Białek, 2005b). All the presented calculations confirm the proper construction of $R(t, x_0)$. However, the presented method should be only used as a supplement to well known statistic methods (Var[NPV(T)], etc. – see Domański, Pruska (2000).

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Metoda pomiaru ryzyka w przypadku stochastycznej definicji aktualnej wartości netto

Metoda Aktualnej Wartości Netto (NPV) jest filarem nowoczesnej teorii finansów. Klasyczna definicja NPV opiera się na nierealistycznych założeniach: zakłada chociażby stały w czasie i deterministyczny charakter stopy dyskontowej. Prowadzi to do sytuacji, w których jej stosowanie wyznacza błędne kierunki inwestowania. Ponadto jej deterministyczna formuła nie pozwala mierzyć ryzyka lub też inaczej – szansy uzyskania konkretnej wartości NPV.

W niniejszej pracy rozważać będziemy stochastyczną, ogólną definicję Aktualnej Wartości Netto. Zaproponowana będzie pewna metoda pomiaru ryzyka w przypadku stosowania tej definicji. Ryzyko utożsamione zostanie z prawdopodobieństwem, że stochastyczny proces NPV(t), startując z pewnego ujemnego, rzeczywistego punktu B(0) (nakład inicjujący projekt) do końca czasu trwania projektu (T), nie wydostanie się z przedziału ($-\infty$, 0). Rozważania nad tym prawdopodobieństwem doprowadzą do równania różniczkowego Pontriagina, którego rozwiązaniem będzie wspomniane prawdopodobieństwo.