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MULTICRITERIA ANALYSIS BASED ON STOCHASTIC DOMINANCE AND ALMOST STOCHASTIC DOMINANCE RULES

Abstract. In the paper a new technique for discrete multiple criteria decision making problems under risk is presented. The procedure uses Stochastic Dominance and Almost Stochastic Dominance rules for comparing distributional evaluations of alternatives with respect to criteria. ELECTRE III technique is used for generating the final ranking of alternatives. An numerical example is presented to show applicability of the technique.

Key words: Stochastic Dominance, Almost Stochastic Dominance, Multicriteria Analysis, ELECTRE III.

I. INTRODUCTION

Most of real-world decision problems involve uncertainty. Two methods are frequently used for ranking uncertain outcomes: stochastic dominance (SD) and mean-risk models. In SD approach random variables are compared by point-wise comparison of some performance functions constructed from their distribution functions. Mean-risk analysis is based on two criteria: one measuring expected outcome and the second representing variability of outcomes. Markowitz (1952) uses mean and variance in his portfolio optimization model. According to this approach prospect X dominates prospect Y, if mean for X is not less then mean for Y, variance for X is not greater than variance for Y, and at least one condition is strict inequality.

Both SD and MV rules may fail to show dominance in cases where almost everyone would prefer one uncertain project to another. Leshno and Levy (2002) propose Almost Stochastic Dominance (ASD) rules which reveal preference for most decision makers (DM), but not all of them. The motivation for implementing such rule is the aspiration for ranking otherwise unrankable alternatives.

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Initially SD rules were used for solving single-criterion problems. In fact most decision problems involve multiple objectives. Huang et al. (1978) proposed the multiattribute stochastic dominance (MSD) rule. They showed that in the case of the probability independence and additive multiattribute utility function, the necessary condition for MSD is to verify stochastic dominance tests with respect to each attribute. Unfortunately, in typical multiattribute problem criteria are conflicting, and the MSD relation can be verified only in exceptional cases. Zaras and Martel (1994) suggest weakening unanimity condition and accepting a majority attribute condition. They solve multiattribute problem by verifying stochastic dominance tests for each pair of alternatives with respect to each attribute and multiattribute aggregation procedure based on the outranking synthesis. Nowak (2004) employs thresholds' concept in multicriteria decision problems under risk.

In this paper SD and ASD rules are used in multicriteria decision problem under risk. The problem is solved in two steps. First, relations between alternatives with respect to criteria are identified. Next, ELECTRE III technique is used for building the global outranking relations.

II. FORMULATION OF THE PROBLEM

The decision situation can be conceived as a (**A**, **X**, **E**) problem, in which we have

1. A finite set of alternatives: $\mathbf{A} = \{ a_1, a_2, ..., a_m \}$.
2. A finite set of attributes: $\mathbf{X} = \{ X_1, X_2, ..., X_n \}$; each attribute is defined in such a way that larger values are preferred to smaller ones.
3. A set of evaluations of alternatives with respect to attributes:

$$\mathbf{E} = \begin{bmatrix} X_1^1 & \cdots & X_j^1 & \cdots & X_n^1 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ X_1^i & \cdots & X_j^i & \cdots & X_n^i \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ X_1^m & \cdots & X_j^m & \cdots & X_n^m \end{bmatrix}$$

where X_k^i is a random variable with a cumulative distribution function $F_k^i(x) = \Pr(X_k^i \le x)$.

In order to build up an overall preference relation between two alternatives a_i and a_j one must compare two vectors of probability distributions. The construction of a local preference relation requires the comparison of two probability distributions. In our approach SD and ASD rules are employed for this comparison.

III. STOCHASTIC DOMINANCE AND ALMOST STOCHASTIC DOMINANCE

Let X and Y be two random variables, and F and G denote the cumulative distribution functions of X and Y, respectively. We assume that distributions have a finite support, say $[a,b]\,(-\infty < a < b < +\infty)$. First-Degree Stochastic Dominance (FSD) and Second-Degree Stochastic Dominance (SSD) can be defined as follows:

1. F dominates G by FSD ($F \succeq_{\text{FSD}} G$) if $F(t) \le G(t)$ for all $t \in [a,b]$ and a strict inequality holds for at least some t.

2. F dominates G by SSD ($F \succeq_{\text{SSD}} G$) if $\int_a^x [F(t)-G(t)]dt \le 0$ for all $x \in [a,b]$ and a strict inequality holds for at least some x.

Hadar and Russel (1969) show that the FSD rule is equivalent to the expected utility maximization rule for all decision makers preferring larger outcomes, while the SSD rule is equivalent to the expected utility maximization rule for risk-averse decision makers preferring larger outcomes. Thus, if U_1 is the set of all nondecreasing differentiable real-valued functions, then $F \succeq_{\text{FSD}} G$ iff $E_F u(X) \ge E_G u(Y)$ for all $u \in U_1$. If U_2 is the set of all nondecreasing twice differentiable real-valued functions such that $u'' \le 0$, then $F \succeq_{\text{SSD}} G$ iff $E_F u(X) \ge E_G u(Y)$ for all $u \in U_2$.

Unfortunately FSD and SSD rules may fail to show dominance in cases where almost everyone would prefer one gamble to another. These rules relate to all utility functions in a given class, even the ones that probably do not characterize the preference of any investor. Leshno and Levy (2002) propose modified stochastic dominance rules to show how to obtain decisions that reveal a preference for one prospect to another when SD rules fail.

Let us define the notation.

$$S_1(F,G) = \{t \in [a,b] : G(t) < F(t)\} \tag{1}$$

$$S_2(F,G) = \left\{ t \in S_1(F,G) : \int_a^t G(x)dx < \int_a^t F(x)dx \right\} \tag{2}$$

$$\|F - G\| = \int_a^b |F(t) - G(t)|dt \tag{3}$$

The definitions of Almost First-Degree Stochastic Dominance (AFSD) and Almost Second-Degree Stochastic Dominance (ASSD) are as follows:

Almost First-Degree Stochastic Dominance (AFSD) and Almost Second-Degree Stochastic Dominance (ASSD) are defined as follows:

1. AFSD. F dominates G by ε-FSD ($F \succeq_{\text{AFSD}(\varepsilon)} G$) if and only if,

$$\int_{S_1} [F(t) - G(t)]dt \le \varepsilon \|F - G\| \quad \text{where} \quad 0 < \varepsilon < 0.5 \tag{4}$$

2. ASSD. F dominates G by ε-SSD ($F \succeq_{\text{ASSD}(\varepsilon)} G$) if and only if,

$$\int_{S_2} [F(t) - G(t)]dt \le \varepsilon \|F - G\| \quad \text{where} \quad 0 < \varepsilon < 0.5 \quad \text{and}$$

$$E_F(X) \ge E_G(Y) \tag{5}$$

Leshno and Levy (2002) show that F dominates G by ε-AFSD if and only if for all u in $U_1^*(\varepsilon)$, $E_F u(X) \ge E_G u(Y)$, where $U_1^*(\varepsilon)$ is defined as follows:

$$U_1^*(\varepsilon) = \left\{ u \in U_1 : u'(x) \le \inf\{u'(x)\}\left[\frac{1}{\varepsilon} - 1\right], \forall x \in [a,b] \right\} \tag{6}$$

Analogously, F dominates G by ε-ASSD if and only if for all u in $U_2^*(\varepsilon)$, $E_F u(X) \ge E_G u(Y)$, where $U_2^*(\varepsilon)$ is defined as follows:

$$U_2^*(\varepsilon) = \left\{ u \in U_2 : -u''(x) \le \inf\{-u''(x)\}\left[\frac{1}{\varepsilon} - 1\right], \forall x \in [a,b] \right\} \tag{7}$$

These types of utility functions do not assign a relatively high marginal utility to very low values or a relatively low marginal utility to large values of x. The value of ε determines the set of utility functions which are permissible. As ε gets smaller the set of permissible utility functions gets larger.

IV. MULTICRITERIA TECHNIQUE BASED ON SD AND ASD RULES

The procedure is based on expected utility with respect to each attribute separately and on the outranking relation. The procedure includes following steps:

1. Verifying stochastic dominance tests for all pairs of alternatives with respect to attributes.

2. Computation of credibility indexes for each pair of alternatives (a_i, a_j):

$$\sigma(a_i, a_j) = \sum_{k=1}^{n} w_k \varphi_k (a_i, a_j) \tag{8}$$

where weighting coefficients w_k sum up to one, and

$$\varphi_k(a_i, a_j) = \begin{cases} 1 & \text{if } F_k^i \succeq_{\text{FSD}} F_k^j \text{ or } F_k^i \succeq_{\text{SSD}} F_k^j \\ (0.5 - \varepsilon^{\min})/0.5 & \text{if } F_k^i \succeq_{\text{AFSD}(\varepsilon^{\min})} F_k^j \text{ or } F_k^i \succeq_{\text{ASSD}(\varepsilon^{\min})} F_k^j \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$\varepsilon^{\min}$ – the minimal value of ε for which AFSD(ε) or ASSD(ε) is verified, $\varepsilon^{\min} < 0.5$.

3. The exploitation of the outranking relation by distillations.

The ranking procedure used in ELECTRE III is described, for example, in Roy, Bouyssou (1993). The basic principles are:

– construction of a complete preorder Z_1 – ranking alternatives from the best to the worst,

– construction of a complete preorder Z_2 – ranking alternatives from the worst to the best,

– construction of a partial preorder $Z = Z_1 \cap Z_2$.

V. NUMERICAL EXAMPLE

To illustrate the procedure let us consider the following example. Decision-maker has to set the order of nine alternatives taking into account four attributes. The evaluations of alternatives with respect to attributes are expressed in the form of probability distributions (table 1). Weighting coefficients are as follows: $w_1 = 0.09$, $w_2 = 0.55$, $w_3 = 0.27$, and $w_4 = 0.09$.

Table 1. Alternatives' evaluations

Value	Alternative								
X_1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1							1/7		
2	3/7	1/7							1/7
3	1/7				1/7				2/7
4		2/7							2/7
5	2/7	1/7	3/7	1/7			3/7	2/7	1/7
6		2/7	1/7		2/7	1/7	1/7	1/7	
7	1/7		1/7	1/7	2/7	2/7		3/7	1/7
8		1/7	2/7	1/7		2/7	1/7	1/7	
9				3/7	2/7				
10				1/7		2/7	1/7		
X_2	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1							1/7		
2	2/7						3/7		1/7
3	1/7			1/7		4/7	1/7	1/7	
4				1/7				1/7	
5	2/7				1/7		1/7		
6		1/7	1/7	1/7	2/7		1/7	1/7	
7		1/7			1/7	1/7		4/7	2/7
8	1/7	3/7	2/7	3/7	2/7	2/7			3/7
9	1/7	2/7	3/7	1/7	1/7				
10			1/7						1/7
X_3	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1									1/7
2							3/7		2/7
3	1/7			1/7			1/7	1/7	
4	3/7					1/7	1/7	2/7	
5		1/7				1/7	2/7	2/7	
6	1/7								2/7
7		1/7		1/7	2/7			2/7	2/7
8	1/7	2/7	4/7	2/7	3/7	2/7			
9	1/7	3/7	3/7	1/7	1/7	1/7			
10				2/7	1/7	2/7			
X_4	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
1									
2								1/7	
3	3/7						1/7		
4								1/7	
5	2/7					1/7	1/7		
6					1/7	1/7		3/7	3/7
7			1/7		1/7	1/7			1/7
8	1/7	4/7	3/7	3/7	3/7	2/7	3/7	1/7	1/7
9		2/7		1/7	1/7	1/7	1/7		1/7
10	1/7	1/7	2/7	3/7	1/7	1/7	1/7	1/7	1/7

First, SD and ASD relations are verified for each pair of alternatives with respect to each attribute. Then, credibility indexes are calculated (table 2). Finally, final ranking is generated (table 3). The best is alternative a_3.

Table 2. Credibility indexes

	a1	a2	a3	a4	a5	a6	a7	a8	a9
a1	0,000	0,000	0,000	0,000	0,000	0,000	0,820	0,162	0,270
a2	1,000	0,000	0,090	0,820	0,640	0,910	0,910	0,910	1,000
a3	1,000	0,910	0,000	0,820	0,910	0,910	1,000	0,910	1,000
a4	1,000	0,180	0,180	0,000	0,180	0,748	1,000	1,000	0,817
a5	1,000	0,360	0,039	0,820	0,000	0,910	1,000	0,910	0,529
a6	1,000	0,090	0,090	0,000	0,090	0,000	1,000	0,450	0,875
a7	0,169	0,064	0,000	0,000	0,000	0,000	0,000	0,090	0,077
a8	0,704	0,090	0,090	0,000	0,000	0,550	0,910	0,000	0,810
a9	0,730	0,000	0,000	0,000	0,000	0,000	0,856	0,090	0,000

Table 3. Distillations' results

Descending distillation		Ascending distillation		Final ranking	
Rank	Alternatives	Rank	Alternatives	Rank	Alternatives
1	a3	1	a3	1	a3
2	a2	2	a2	2	a2
3	a5	3	a5	3	a5
4	a4	4	a4	4	a4
5	a6	5	a6	5	a6
6	a8	6	a9	6	a8, a9
7	a9	7	a8	7	a1
8	a1	8	a1	8	a7
9	a7	9	a7		

VI. CONCLUSIONS

In the paper a new technique for discrete multiple criteria decision problems under risk was presented. The procedure uses Stochastic Dominance and Almost Stochastic Dominance rules for comparing distributional evaluations of alternatives with respect to attributes. SD is based on an axiomatic model of risk averse preferences. However, these rules may fail to show dominance in cases where almost everyone would prefer one uncertain project to another. In such cases ASD rules can be usefull.

Multiple criteria analysis based on stochastic dominance has been successfully applied in decision analysis during last thirty years. Initially, investments and savings, portfolio diversification, option evaluation and portfolio insurance were the main areas of application. Since 1990 various new areas of employment of SD concept has been proposed: production process control, investment projects' evaluation, measuring the quality of life. The methodology proposed in this paper can be employed in all these fields.

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ANALIZA WIELOKRYTERIALNA OPARTA NA DOMINACJACH STOCHASTYCZNYCH I REGUŁACH DOMINACJI STOCHASTYCZNYCH TYPU ALMOST

W pracy przedstawiono technikę wspomagania decyzji, która może być wykorzystywana do rozwiązywania dyskretnych wielokryterialnych problemów podejmowania decyzji w warunkach ryzyka. Do porównania rozkładów ocen wariantów decyzyjnych wykorzystywane są reguły dominacji stochastycznej oraz prawie-dominacji stochastycznej. Ranking końcowy uzyskiwany jest za pomocą procedur destylacji znanych z metody ELECTRE III. Zamieszczony w pracy przykład numeryczny opisuje sposób wykorzystania procedury do rozwiązywania problemu wielokryterialnego.