ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 228, 2009

Bronisław Ceranka, Małgorzata Graczyk***

SOME REMARKS ABOUT VARIANCE BALANCED BLOCK DESIGNS

Abstract. Some construction methods of the variance balanced block designs for v and $v+1$ treatments are given. They are based on the incidence matrices of the balanced incomplete block designs.

Key words: balanced incomplete block design, variance balanced block design.

I. INTRODUCTION

In the paper we present some types of block designs, which are use full in practice as well as in the general theory of block design. The designs with repeated blocks with the equal replications and equal block sizes are widely used in several fields of research and they are available in the literature, see Foody and Hedayat (1977), Hedayat and Li (1979), Hedayat and Hwang (1984). However from the practical point of view, it may be not possible to construct the design with equal block sizes accommodating the equal replication of each treatment in all the blocks. Hence in the present paper we consider a class of block designs called variance balanced block designs which can be made available in unequal block sizes and for varying replications.

Let us consider v treatments arranged in b blocks in a block design with incidence matrix $\mathbf{N}=(n_{ij})$, $i=1,2,...,v, j=1,2,...,b$, where n_{ij} denotes the number of experimental units in the jth block getting the ith treatment, $n=\sum_{i=1}^{v}\sum_{j=1}^{b}n_{ij}$. When $n_{ij}=1$ or 0 for all i and j the design is said to be binary. Otherwise it is said to be nonbinary.

In this paper we consider binary block designs, only.

* Professor, Department of Mathematical and Statistical Methods, Agricultural University in Poznań.
** Ph.D., Department of Mathematical and Statistical Methods, Agricultural University in Poznań.

The following notation is used $\mathbf{r} = [r_1, r_2, ..., r_v]'$ is the vector of treatment replications, $\mathbf{k} = [k_1, k_2, ..., k_b]'$ is the vector of block sizes, $\mathbf{N1}_b = \mathbf{r}$, $\mathbf{N}'\mathbf{1}_v = \mathbf{k}$, $\mathbf{1}_a$ is the $a \times 1$ vector of ones. The information matrix $\mathbf{C}$ for treatment effects defined below as

$$\mathbf{C} = \mathbf{R} - \mathbf{NK}^{-1}\mathbf{N}', \tag{1}$$

where $\mathbf{R} = \mathrm{diag}(r_1, r_2, ..., r_v)$, $\mathbf{K} = \mathrm{diag}(k_1, k_2, ..., k_b)$ is very suitable in determining properties of block design.

For several reasons, in particular from the practical point of view, it is desirable to have repeated blocks in the design. For example, some treatment combinations may be preferable than the others and also the design implementation may cost differently according to the design structure admitting or not repeated blocks. The set of all distinct blocks in a block design is called the support of the design and the cardinality of the support is denoted by b^* and is referred to as the support size of the design.

Though there have been balanced designs in various sense (see Puri and Nigam (1977), Caliński (1977)), we will consider a balanced design of the following type. A block design is said to be balanced if every elementary contrast of treatment effects is estimated with the same variance (see Rao (1958)). In this sense the design is also called a variance balanced (VB) block design.

It is well known that block design is a VB if and only if it has

$$\mathbf{C} = \eta\left[\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right], \tag{2}$$

where η is the unique nonzero eigenvalue of the $\mathbf{C}$–matrix with the multiplicity $v-1$, $\mathbf{I}_v$ is the $v \times v$ identity matrix. For binary block design $\eta = \frac{\sum_{i=1}^{v} r_i - b}{v-1}$ (See Kageyama and Tsuji (1979)).

In particular case when block design is equireplicated then $\eta = \frac{vr-b}{v-1}$.

In the paper, we consider balanced incomplete block design (BIBD) with the parameters v, b, r, k, λ. It is binary block design for which $\mathbf{r} = r\mathbf{1}_v$, $\mathbf{k} = k\mathbf{1}_b$

and $\lambda = \dfrac{r(k-1)}{v-1}$, where λ is a scalar product of any two rows of incidence matrix $\mathbf{N}$. It is well known that

$$vr = bk, \quad \lambda(v-1) = r(k-1), \quad \mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda \mathbf{1}_v \mathbf{1}_v'.$$

Lemma 1. Any BIBD is VB.

Proof. For the information matrix for treatment effects given in (1) we have $\mathbf{C} = r\mathbf{I}_v - \dfrac{1}{k}\mathbf{NN}' = \dfrac{v\lambda}{k}\left(\mathbf{I}_v - \dfrac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right) = \dfrac{vr-b}{v-1}\left(\mathbf{I}_v - \dfrac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right)$. So, the Lemma is proven.

II. CONSTRUCTION FOR v TREATMENTS

Theorem 1. If $\mathbf{N}_h$ is BIBD with the parameters v, b_h, r_h, k_h, λ_h and $\mathbf{C}_h$ is respectively $\mathbf{C}$–matrix for $h = 1,2,\ldots,t$ then

$$\mathbf{N} = [\mathbf{N}_1 \ \mathbf{N}_2 \ \ldots. \ \mathbf{N}_t] \tag{3}$$

is incidence matrix of the VB block design.

Proof. For the design $\mathbf{N}$ in (3) we have $r = \sum_{h=1}^{t} r_h$, $\mathbf{k} = \left[k_1\mathbf{1}_{b_1}' \ k_2\mathbf{1}_{b_2}' \ldots k_t\mathbf{1}_{b_t}'\right]'$. Thus

$$\mathbf{C} = r\mathbf{I}_v - \mathbf{NK}^{-1}\mathbf{N}' = r\mathbf{I}_v - \sum_{h=1}^{t}\frac{1}{k_h}\mathbf{N}_h\mathbf{N}_h' = \sum_{h=1}^{t} r_h\mathbf{I}_v - \sum_{h=1}^{t}\frac{1}{k_h}\mathbf{N}_h\mathbf{N}_h' =$$

$$= \sum_{h=1}^{t}\left(r_h\mathbf{I}_v - \frac{1}{k_h}\mathbf{N}_h\mathbf{N}_h'\right) = \sum_{h=1}^{t}\mathbf{C}_h$$

The design $\mathbf{N}_h$ is VB as BIBD. Therefore from Lemma 1 we have

$$\mathbf{C} = \sum_{h=1}^{t}\eta_h\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right) = \eta\left(\mathbf{I}_v - \frac{1}{v}\mathbf{1}_v\mathbf{1}_v'\right),$$

where $\eta = \sum_{h=1}^{t}\eta_h$. Hence the claim of Theorem.

Example 1. For $t=2$ we consider BIBD with the parameters $v=5,\ b_1=10,\ r_1=4,\ k_1=2,\ \lambda_1=1,\ b_1^*=10$ and with the incidence matrix $\mathbf{N}_1$ and BIBD with the parameters $v=5,\ b_2=20,\quad r_2=12,\ k_2=3,\ \lambda_2=6,\ b_2^*=10$ and with the incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1=\begin{bmatrix}1&1&1&1&0&0&0&0&0&0\\1&0&0&0&1&1&1&0&0&0\\0&1&0&0&1&0&0&1&1&0\\0&0&1&0&0&1&0&1&0&1\\0&0&0&1&0&0&1&0&1&1\end{bmatrix},$$

$$\mathbf{N}_2=\begin{bmatrix}1&1&1&1&1&1&0&0&0&0&1&1&1&1&1&1&0&0&0&0\\1&1&1&0&0&0&1&1&1&0&1&1&1&0&0&0&1&1&1&0\\1&0&0&1&1&0&1&1&0&1&1&0&0&1&1&0&1&1&0&1\\0&1&0&1&0&1&1&0&1&1&0&1&0&1&0&1&1&0&1&1\\0&0&1&0&1&1&0&1&1&1&0&0&1&0&1&1&0&1&1&1\end{bmatrix}.$$

Based on the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ we form the design matrix $\mathbf{N}$ in the form (3) of the VB block design with repeated blocks and with the parameters $v=5,\ b=30,\ r=16,\ k=\begin{bmatrix}2\cdot\mathbf{1}_{10}\\3\cdot\mathbf{1}_{20}\end{bmatrix},\quad b^*=20$. The information matrix for treatment effects is equal $\mathbf{C}=\frac{25}{2}\left(\mathbf{I}_5-\frac{1}{5}\mathbf{1}_5\mathbf{1}_5'\right)$.

Now, we use the following specialized product of two matrices presented in Pal and Dutta (1979)

If $\mathbf{A}=(a_{st})_{m\times p}$ and $\mathbf{B}=(b_{zl})_{m\times q}$ then the specialized product of the matrices $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{D}=\mathbf{A}*\mathbf{B}=(d_{sl})_{m\times pq},$$

where $d_{sl}=a_{st}\times b_{iz}$, l being equal to $(t-1)q+z$ for $s=1,2,\ldots,m,\ t=1,2,\ldots,p,$ $z=1,2,\ldots,q$.

Let $\mathbf{N}_h$ be incidence matrix of the BIBD with the parameters $v,\ b_h,\ r_h,\ k_h,\ \lambda_h,\ h=1,2$. Let $\mathbf{C}_h$ be the $\mathbf{C}$–matrix of this design defined by $\mathbf{N}_h$. Now, we form the matrix $\mathbf{N}$ as

$$\mathbf{N}=\mathbf{N}_1 * \mathbf{N}_2 \tag{4}$$

Theorem 2. If $\mathbf{N}_1$ is incidence matrix of the BIBD with the parameters $v,\ b_1=\frac{v(v-1)}{2},\ r_1=v-1,\ k_1=2,\ \lambda_1=1$ and $\mathbf{N}_2$ is incidence matrix of the BIBD with the parameters $v=b_2,\ r_2=k_2=v-1,\ \lambda_2=v-2$ then $\mathbf{N}$ in the form (4) is incidence matrix of the VB block design with repeated blocks and with the parameters $v,\ b=\frac{v^2(v-1)}{2},\ r=(v-1)^2,\ k=\begin{bmatrix} 2\cdot\mathbf{1}_{\frac{v(v-1)(v-2)}{2}} \\ \mathbf{1}_{v(v-1)} \end{bmatrix},\ b^*=\frac{v(v+1)}{2}$.

Proof. For the product (4) we have $\mathbf{N}=\left[\mathbf{N}_1\otimes\mathbf{1}_{v-2}'\ \ \mathbf{I}_v\otimes\mathbf{1}_{v-1}'\right]$. Hence the information matrix $\mathbf{C}=(v-2)\mathbf{C}_1$. Therefore, taking into consideration Theorem 1, $\mathbf{N}$ is incidence matrix of the VB block design with repeated blocks. So, the Theorem is proven.

Example 2. Let us consider BIBD with the parameters $v=5,\ b_1=10,\ r_1=4,\ k_1=2,\ \lambda_1=1,\ b_1^*=10$ and with the incidence matrix $\mathbf{N}_1$ (Example 1) and BIBD with the parameters $v=5,\ b_2=5,\ r_2=4,\ \ k_2=4,\ \lambda_2=3,\ b_2^*=5$ and with the incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_2=\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Based on the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ we form the design matrix $\mathbf{N}$ in the form (4) of the VB block design with repeated blocks and with the parameters

$v=5,\ b=50,\ r=16,\ k=\begin{bmatrix}2\cdot\mathbf{1}_{30}\\ \mathbf{1}_{20}\end{bmatrix},\ b^{*}=15$ and $\mathbf{N}=\left[\mathbf{N}_1\otimes\mathbf{1}_3^{'}\ \ \mathbf{I}_5\otimes\mathbf{1}_4^{'}\right]$. The information matrix for treatment effects is given as $\mathbf{C}=\frac{15}{2}\left(\mathbf{I}_5-\frac{1}{5}\mathbf{1}_5\mathbf{1}_5^{'}\right)$.

III. CONSTRUCTION FOR $v+1$ TREATMENTS

Let $\mathbf{N}_h$ be incidence matrix of the BIBD with repeated blocks and with the parameters $v,\ b_h,\ r_h,\ k_h,\ \lambda_h,\ b_h^{*},\ h=1,2$. Let $\mathbf{C}_h$ be the $\mathbf{C}$-matrix of this design defined by $\mathbf{N}_h$. Now, we form the matrix $\mathbf{N}$ as

$$\mathbf{N}=\begin{bmatrix}\mathbf{N}_1\otimes\mathbf{1}_t^{'} & \mathbf{N}_2\otimes\mathbf{1}_u^{'}\\ \mathbf{1}_{b_1}^{'}\otimes\mathbf{1}_t^{'} & \mathbf{0}_{b_2}^{'}\otimes\mathbf{1}_u^{'}\end{bmatrix} \tag{5}$$

Theorem 3. Block design with the incidence matrix $\mathbf{N}$ in the form (5) is the VB block design and with the parameters $v+1,\ b=tb_1+ub_2$, $\mathbf{r}=\left[(tr_1+ur_2)\mathbf{1}_v^{'}\ \ tb_1\right]^{'}$, $\mathbf{k}=\left[(k_1+1)\mathbf{1}_{tb_1}^{'}\ \ k_2\mathbf{1}_{ub_2}^{'}\right]^{'}$, $b^{*}=b_1^{*}+b_2^{*}$ if and only if

$$tk_2(r_1-\lambda_1)=u\lambda_2(k_1+1). \tag{6}$$

Proof. For the block design with the matrix $\mathbf{N}$ given in (5) we have

$$\mathbf{C}=\begin{bmatrix}\left(t\left(r_1-\frac{r_1-\lambda_1}{k_1+1}\right)+u\left(r_2-\frac{r_2-\lambda_2}{k_2}\right)\right)\mathbf{I}_v-\left(\frac{t\lambda_1}{k_1+1}+\frac{u\lambda_2}{k_2}\right)\mathbf{1}_v\mathbf{1}_v^{'} & -\frac{tr_1}{k_1+1}\mathbf{1}_v\\ -\frac{tr_1}{k_1+1}\mathbf{1}_v^{'} & \frac{tb_1k_1}{k_1+1}\end{bmatrix}. \tag{7}$$

Comparing the diagonal elements of the matrix $\mathbf{C}$ of the form (7) we have $\frac{t(r_1(k_1-1)+r_1)}{k_1+1}+\frac{ur_2(k_2-1)}{k_2}=\frac{tb_1k_1}{k_1+1}$. Thus $\frac{t(\lambda_1(v-1)+r_1-vr_1)}{k_1+1}=-\frac{u\lambda_2(v-1)}{k_2}$ and we get (6).

Comparing the offdiagonal elements of the matrix $\mathbf{C}$ of the form (7) we have $\frac{t\lambda_1}{k_1+1}+\frac{u\lambda_2}{k_2}=\frac{tr_1}{k_1+1}$. If the equation (6) is fulfilled then the matrix $\mathbf{C}=\eta\left(\mathbf{I}_{v+1}-\frac{1}{v+1}\mathbf{1}_{v+1}\mathbf{1}_{v+1}^{'}\right)$, where $\eta=\frac{tr_1(v+1)}{k_1+1}$. So, the Theorem is proven.

Example 3. For the special case $t=u=1$ we consider BIBD with the parameters $v=5,\ b_1=10,\ r_1=4,\ k_1=2,\ \lambda_1=1,\ b_1^{*}=10$ and with the incidence matrix $\mathbf{N}_1$ (Example 1) and BIBD with the parameters $v=5,\ b_2=10,\ r_2=6,$ $k_2=3,\ \lambda_2=3,\ b_2^{*}=10$ and with the incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_2=\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Based on the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ we form the design matrix $\mathbf{N}$ in the form (5) of the VB block design with repeated blocks and with the parameters $v=6,\ b=20,\ r=10,\ k=3,\ b^{*}=20$. The information matrix for treatment effects equals $\mathbf{C}=8\left(\mathbf{I}_6-\frac{1}{6}\mathbf{1}_6\mathbf{1}_6^{'}\right)$.

REFERENCES

Caliński T. (1977), *On the notion of balance block designs*, [in:] G. Barra et al. eds. Recent Developments in Statistics, North-Holland Publishing Company. Amsterdam, 365–374.

Foody W., Hedayat, A. (1977), *On theory and applications of BIB designs with repeated blocks*, Ann. Statist. 5, 932–945.

Hedayat A., Li Shuo-Yen R. (1979), *The trade off method in the construction of BIB designs with repeated blocks*, Ann. Statist. 7, 1277–1287.

Hedayat A., Hwang H.L. (1984), *BIB(8,56,21,3,6) and BIB(10,30,9,3,2) designs with repeated blocks*, J. Comb. Th. (A) 36, 73–91.

Kageyama S., Tsuji T., (1979), *Inequality for equireplicated n-array block designs with unequal block sizes*, Journal of Statistical Planning and Inferences 3, 101–107.

Puri P.D., Nigam A.K., (1977), *Balanced block designs*, Common. Statist.-Theor. Meth. (A) 6, 1171–1179.

Rao V.R., (1958), *A note on balanced designs*, Ann. Math. Statist. 29, 290–294.

Bronisław Ceranka, Małgorzata Graczyk

UWAGI O ZRÓWNOWAŻONYCH W SENSIE WARIANCJI UKŁADACH BLOKÓW

W pracy zostały przedstawione metody konstrukcji zrównoważonych w sensie wariancji układów bloków dla v oraz $v + 1$ obiektów. Metody te są oparte na macierzach incydencji układów zrównoważonych o blokach niekompletnych.