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TESTING FOR TAIL INDEPENDENCE IN EXTREME VALUE MODELS – APPLICATION ON POLISH STOCK EXCHANGE

Abstract. An estimate of the degree of association between assets is required in many financial activities. Especially dependencies of extreme events are attracting an increasing attention in modern risk management. After estimation of the tail dependence coefficient we compare and investigate the Neyman–Pearsons' and Kolmogorov–Smirnovs' tests for independence. We provide a discussion on how the concept of extreme value dependence can be made into useful portfolio management tools.

Key words: extreme-value theory (EVT), copula, tests for independence.

INTRODUCTION

Estimating dependence between risky asset returns is the cornerstone of portfolio theory and many other finance applications. Common dependence measures such as Pearson's correlation coefficient are not always suited for a proper understanding of dependencies in financial markets, Embrechts *et al.* (2002). In particular, dependencies between extreme events such as extreme negative stock returns or large portfolio losses cause the need for alternative dependence measures to support asset-allocation strategies. Several empirical surveys such as An´e, Kharoubi (2003) and Malevergne, Sornette (2004) exhibited that the concept of *tail dependence* is a useful tool to describe the dependence between extremal data. Tail dependence is described via the *tail-dependence coefficient* introduced by Sibuya (1960). Extreme value theory is the natural choice for inferences on extreme values. In this paper, we are concerned with *testing* for pairwise independence of maxima from empirical data, which seem to be absolutely mandatory for tail dependence *estimation*. The aim of the paper is presentation of test for tail independence, which is indispensable when

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working with tail dependence, since all estimators of the tail dependence coefficient are strongly misleading when the data does not stem from a tail dependent setting.

I. TAIL DEPENDENCE CONCEPT

The *tail dependence coefficient* is roughly speaking the probability that a random variable exceeds a certain threshold given that another random variable has already exceeded that threshold. The following approach, Sibuya (1960) and Joe (1997) among others, represents the most common definition of tail dependence. Let (X,Y) be a random pair with joint cumulative distribution function *F* and marginals F_x and F_y. The quantity $\lambda_u = \lim_{v \to 1-} P(X > F_X^{-1}(v) \mid Y > F_Y^{-1}(v))$ is the *upper tail-dependence coefficient* (upper TDC), provided the limit exists. We say that (X,Y) is upper tail dependent if $\lambda_u > 0$ and upper tail independent if $\lambda_u < 0$. Similarly, we define the lower tail-dependence coefficient λ_L.

The TDC can also be defined via the notion of copula, introduced by Sklar (1959). A copula *C* is a cumulative distribution function whose margins are uniformly distributed on [0, 1]. The joint distribution function *F* of any random pair (X,Y) can be represented as $F(x) = C(F_X(x), F_Y(y))$ (refer to Joe (1997) for more information on copulas). The coefficient of upper tail dependence can be written in terms of copula: $\lambda_U = \lim_{v \to 1-} \frac{1 - 2v + C(v,v)}{1 - v}$. Analogously, we have $\lambda_L = \lim_{v \to o+} \frac{C(v,v)}{v}$.

II. BIVARIATE EXTREME DISTRIBUTIONS

The classical extreme bivariate theory is concerned with the limit behaviour of $(M_n(X), M_n(Y)) = (\max_{i=1,\dots n} X_i, \max_{i=1,\dots n} Y_i)$ as $n \to \infty$. Because of the definition, the marginals of $(M_n(X), M_n(Y))$ belong to the generalized extreme value (GEV) distribution family. The general form of a generalized extreme value GEV distribution is $GEV_{\mu,\sigma,\xi}(x) = \exp(-[1 + \xi \frac{x - \mu}{\sigma}]^{-1/\xi})$ with $\mu \in R$, $\sigma > 0$, $\xi \in R$ (Coles 2001). To simplify the presentation, Coles (2001) assumes without loss of generality that $F_X \equiv F_Y \equiv F$, where $F(\cdot)$ is the unit Frechet distribution. The following theorem (de Haan and Resnick, 1977) characterizes the limit joint distribution of $(M_n(X), M_n(Y))$:

If $P(M_n(X) \le nx, M_n(Y) \le ny) \xrightarrow[n\to\infty]{} G(x,y)$ where G is a non-degenerate distribution function, then $G(\cdot,\cdot)$ takes the form $G(x,y) = \exp(-V(x,y))$ with $V(x,y) = 2\int_0^1 \max(\omega/x,(1-\omega)/y)dH(\omega)$ and H is a distribution on $[0,1]$ with mean $1/2$.

III. ESTIMATION OF THE TDC

There are two possibilities to use Extreme Value Theory for the estimation of the TDC. The first one is to develop estimators based on the assumptions of the Generalized Pareto Distribution. Therefore, one assumes convergence (over some threshold) to a bivariate Generalized Pareto Distribution. This model is called Peaks over threshold. The other possibility is to assume that the assumptions of the GEV are fulfilled. This conception in a financial application rarely be the case. Both methods come to the same estimation problem: the dependence function is to be estimated. The difference is the treatment of the data: in the first case, we choose the realizations that lie above a threshold, in the second case - block-maxima. Frahm *et al*. (2005) give estimators for the TDC under different assumptions: using a specific distribution (e.g. t-distribution), within a class of distributions (e.g. elliptically contoured distributions), using a specific copula (e.g. Gumbel), within a class of copulae (e.g. Archimedean) or a nonparametric estimation (without any parametric assumption). The authors compare the performance of the different estimators for different cases: whether the assumption is true or wrong and whether there is tail dependence or not. It turns out that some of the estimators perform well if there is tail dependence but bad if there is not. In practical applications, one will never know which copula model is the correct one. The estimation can only be under misspecification. So difficulties in selecting a copula model, brings us to the important issue of testing for tail dependence.

IV. HOW TO TEST DEPENDENCIES ? A DIFFERENT APPROACH FOR TESTING FOR TAIL INDEPENDENCE

One of the most interesting approach for testing for tail independence is given in Falk and Michel (2006). They prove the following theorem:

With $c \to 0$*, we have uniformly for* $t \in [0,1]$*:*

$$P(X+Y>ct \mid X+Y>c)=\begin{cases} t^2; & \textit{there is no tail dependence} \\ t; & \textit{else} \end{cases}.$$

Using this theorem, Falk and Michel propose four different tests for tail independence, which can be grouped into 2 different classes: a Neymann-Pearson test (NP) and three goodness of fit tests: Fisher's κ, Kolmogorov-Smirnov and χ^2. In the latter class, the Komolgorov-Smirnov-test (KS) turns out to be the best in the simulation study by Falk and Michel (2006). Therefore, in the following, only NP and KS tests are described.

Neyman- Pearson test

Assume we have a random sample $(X_1,..,X_n)$ $(Y_1,..,Y_n)$ of independent copies of (X,Y). The marginal distribution is assumed to be reverse exponential (i.e. $F(x,0)=F(0,x)=\exp(x)$). Now, fix a threshold $c<0$ and consider $E=\{C_i=X_i+Y_i; C_i>c\}$. Let $K(n)=\#E$ and define $V_i=C_i/c$ $\forall i=1,...,k(n)$. The NP test considers the distribution function of V_i and tests whether it is more likely from $F_{(0)}(t)=t^2$ or $F_{(1)}(t)=t$. The test statistic for testing $F_{(0)}$ (tail independence) against $F_{(1)}$ is (for fixed n):

$$T_{NP}:=\log\Big(\prod_{i=1}^{k(n)}\frac{1}{2V_i}\Big)=-\sum_{i=1}^{k(n)}\log(V_i)-k(n)\log(2).$$

$F_{(0)}$ is rejected when T_{NP} gets large precisely, if the approximate p-value $p_{NP}:=\Phi(k(n)^{-1/2}\sum_{i=1}^{k(n)}(2\log(V_i)+1))$ is too close to 0, typically if $p_{NP}\le .05$; Φ - standard normal df.

Kolmogorov Smirnov test

A different possibility of using Falk and Michel (2006) theorem is to carry out a goodness-of- fit test, in this case using the Kolmogorov Smirnov test. Therefore, define, conditional on $K(n)=m$:

$$U_i=F_c(C_i/c)=(1-(1-C_i)\exp(C_i))/(1-(1-c)\exp c),\ \forall i\in\{1,...m\}.$$

Denote $\hat{F}_m(t) = \frac{1}{m}\sum I_{[0,t]}C_i$ the ecdf of U_i, $i = 1,..,m$. The Kolmogorov test statistic is then: $T_{KS} := \frac{1}{m}\sup_{t\in[0,1]}\left|\hat{F}_m(t) - t\right|$.

The approximate p-value is $p_{KS} = 1 - K(T_{KS})$, where K is the cdf of the Kolmogorov distribution. According to a rule of thumb given by the authors: for $m > 30$, tail independence is rejected if $T_{KS} > c_{0,05} = 1{,}36$.

V. ESTIMATION OF THE TDC AND TESTING FOR TAIL INDEPENDENCE - EMPIRICAL ANALYSIS

Eight different data sets are analyzed, namely: MILLENIUM, GETIN, PKNORLEN, PGNIG, ASECOPOL, COMARCH, NETIA, TPSA[1]. Our sample period covers a total 601 observations from 08/08/2006 to 01/01/2009. The series are non Gaussian (Jarque-Bera test with 95% level). We use data on stock return pairs to estimate the tail dependences coefficients and to test hypotesis on tails independence. We proceed in the following steps:

1. *Fitting marginal distributions*: using the Peak Over Threshold method, we estimate the tail distribution of each series. The marginal tail estimation formulas ($\hat{F}_i(x) = 1 - k/n \cdot (1 + \xi_i/\sigma_i \cdot (x - \mu_i))^{-1/\xi_i}$, $i = 1,...,n$, where k is the number of data exceeding the fixed threshold u, ξ, σ, μ – parameters of shape, scale and mean) are estimated by choosing the thresholds for each series.
2. *Fitting a copula to all pairs of joint standardized data using maximum likelihood:* we consider three widely used Archimedean copulas, namely the Gumbel, Frank, Clayton and t-copulas[2]. It is difficult to compare the fit of the two copulas directly because they are non-nested models. However, we did compute Akaike's Information Criteria for each model. For the four models

[1] Our objective was to create portfolio with selected stocks which accurately refelect the daily returns of the sub-indexes: WIG – Banks, WIG - IT , WIG - Telecom, WIG - Fuel . We had pre-selected 8 stocks.

[2] Gumbel copula (Gumbel, 1960): $C_G(v,v) = \exp(-((\log v)^{\theta} + (\log v)^{\theta})$

Clayton copula (Clayton, 1978): $C_{CL}(v,v) = (v^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$

Frank copula (Frank, 1979): $C_F(v,v) = 1/\theta \cdot \log[(1 + \exp^{-\theta v} - 1)(\exp^{-\theta v} - 1)/(e^{-\theta} - 1)]$

t-copula: $C_t(v,v,\alpha,\rho) = t_\alpha(t_\alpha^{-1}(v), t_\alpha^{-1}(v);\rho)$

considered for upper and lower tail thresholds, the Frank copula model has the best performance (68% pairs).

3. *Computing for all* (i, j) $(i, j = 1,...8)$ *pairs of returns the upper and lower tail dependence coefficients using the copulas parameters estimates (Table 1).*

Table 1: Left (below diagonal) and upper tail independence coefficients (above diagonal)

	TPSA	NETIA	PKN ORLEN	PGNIG	ASEC OPOL	COMARCH	MILLE-NIUM	GETIN
TPSA	1	0,1437	0,3354	0,3176	0,2067	0,235	0,2967	0,2145
NETIA	0,1624	1	0,1534	0,1134	0,0978	0,1864	0,4142	0,1045
PKNORLEN	0,3689	0,1769	1	0,3641	0,2964	0,2835	0,3528	0,2741
PGNIG	0,3567	0,106	0,405	1	0,2372	0,2375	0,3056	0,2701
ASECOPOL	0,246	0,078	0,3067	0,2369	1	0,2402	0,3186	0,2183
COMARCH	0,2496	0,163	0,296	0,2432	0,2453	1	0,3692	0,3285
MILLENIUM	0,328	0,1486	0,4061	0,3647	0,3375	0,4064	1	0,4682
GETIN	0,208	0,114	0,312	0,2969	0,2666	0,3462	0,4694	1

4. *Testing for tail independence:* Because the perception that left tails are heavier than the right tails we consider tests for lower tail. According to NP test one can see among the same sector only TPSA-NETIA and ASECOPOL-COMARCH are strongly linked in a way that tail dependence can be observed. Bad news for TPSA and Comarch should mean bad news for Netia and ASECOPOL, respectively. The null hypothesis of tail-independence can be rejected for 34% of considered pair of stocks (Table 2). Interestingly, in times, where tail independence can be rejected, the estimates for the TDC are not necessarily higher than in ones where this is not the case. This emphasizes the importance of the test for tail independence. Choosing the threshold u is difficult in practice and makes some technical problems. The higher the threshold, the lower the variance but the higher the bias. Therefore, the threshold has to be chosen approximately such that for the Gaussian distribution we can accept the null hypothesis of tail independence, whereas for all others, we are able to reject it. According to Falk and Michel (2006) simulations with u close enough to 0 all tests are equally good for independent data (control Type I error). But Neyman–Pearson gets problems with a smaller c (it does not control Type I error then). So if the threshold is not close to 0, one should take Kolmogorov-Smirnov for testing for tail independence.

Table 2.Goodness of fit

pair of stocks	p_{NP}	p_{KS}	pair of stocks	p_{NP}	p_{KS}
TPSA NETIA	**0,000**	**0.00329**	PKNORLEN ASECOPOL	0,204	0,2857
TPSA PKNORLEN	**0,001**	**0.00460**	PKNORLEN COMARCH	**0,040**	**0,0503**
TPSA PGNIG	**0,026**	**0.02816**	PKNORLEN MILLENIUM	0,589	0,6732
TPSA ASECOPOL	0,528	0,7376	PKNORLEN GETIN	0,520	0,6106
TPSA COMARCH	0,528	0,7058	PGNIG ASECOPOL	0,944	0,9978
TPSA MILLENIUM	**0,000**	**0.01939**	PGNIG COMARCH	0,155	0,1867
TPSA GETIN	**0,025**	**0.02179**	PGNIG MILLENIUM	0,100	0,1135
NETIA PKNORLEN	0,093	0,1834	PGNIG GETIN	0,356	0,4721
NETIA PGNIG	**0,011**	**0.00542**	ASECOPOL COMARCH	**0,000**	**0.00285**
NETIA ASECOPOL	**0,000**	**0.00465**	ASECOPOL MILLENIUM	0,462	0,4664
NETIA COMARCH	**0,000**	**0.00305**	ASECOPOL GETIN	0,509	0,6017
NETIA MILLENIUM	0,472	0,6118	COMARCH MILLENIUM	0,174	0,2018
NETIA GETIN	0,180	0,1853	COMARCH GETIN	**0.000**	**0,003**
PKNORLEN PGNIG	0,873	0,9477	MILLENIUM GETIN	0,382	0,4058

5. *Incorporating tail dependence into Markowitz Mean-Variance Model:* For this 8-dimensional data, the sample correlation coefficients are given in Table 3. We observe they are all positive. Comparing values form Table 3 to those in Table 1 we observe that the tail adjusted estimates are smaller than the Pearson correlation coefficient based on the entire data set. Most important, the strength of correlations estimated for joint negative values are different from those estimated for joint positive returns, an asymmetry not detected in Table 3. Testing tail dependence permits to construct less risky investment portfolio than classsical one (Table 4).

Table 3: Standard sample correlation coefficients

	TPSA	NETIA	PKN ORLEN	PGNIG	ASECO-POL	COMARCH	MILLE-NIUM	GETIN
TPSA	1,0000							
NETIA	0,1781	1,0000						
PKNORLEN	0,4058	0,1823	1,0000					
PGNIG	0,3501	0,1380	0,4255	1,0000				
ASECOPOL	0,2534	0,0882	0,3843	0,2470	1,0000			
COMARCH	0,2762	0,1850	0,3146	0,2838	0,2812	1,0000		
MILLENIUM	0,3287	0,1740	0,4427	0,4127	0,4398	0,4183	1,0000	
GETIN	0,2926	0,1665	0,3887	0,3106	0,2847	0,3605	0,4829	1,0000

Table 4: Portfolios compositions

	EXPECTED RETURN	RISK	TPSA	NETIA	PKN ORLEN	PGNIG	ASECO-POL	COMARCH	MILLE-NIUM	GETIN
Classical	0.0400	0.9682	0,25	0,28	0,06	0,14	0,16	0,10	0,00	0,02
Upper-Adjusted	0.0400	0.8338	0,10	0,15	0,05	0,11	0,19	0,22	0,11	0,08
Lower-Adjusted	0.0400	0.8740	0,09	0,12	0,03	0,19	0,14	0,42	0,02	0,00

CONCLUSION

Testing tail independence is simple and transparent enough to be implemented and easily monitored. Omitting the test for tail independence would introduce a large bias in the estimation and make it difficult to decide whether there is just correlation or in fact tail dependence. One important feature of this paper is the implementation of the tests for tail independence, which is recognized to be indispensable but rarely utilized in a financial context. On the basis of tail dependence test we are able to improve allocation results based on daily returns.

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TESTOWANIE NIEZALEŻNOŚCI W OGONACH ROZKŁADÓW WARTOŚCI EKSTREMALNYCH – ZASTOSOWANIE NA POLSKIEJ GIEŁDZIE PAPIERÓW WARTOŚCIOWYCH

Określanie stopnia zależności między aktywami jest niezbędne w wielu obszarach rynku finansowego. Koncepcja zależności w ogonie rozkładu stanowi obecny trend w ocenie siły ekstremalnych zależności. Przeprowadzona została analiza zależności w ogonie rozkładu na podstawie stóp zwrotu wybranych spółek polskiej giełdy oraz analiza porównawcza wybranych testów niezależności: Neyman-Pearson i Kolmogorov-Smirnov. Celem pracy jest uwypuklenie potrzeby uwzględniania zagadnienia określania zależności w ogonach rozkładu stop zwrotu składników portfela w zarządzaniu portfelem inwestycyjnym.