A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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ESTIMATION OF THE STIELTJES TRANSFORM OF THE NORMALIZED SPECTRAL FUNCTION OF COVARIANCE MATRICES

Abstract. In the classical approach of the large sample theory estimator properties are examined basing on the assumption that *n* sample size tends towards infinity. From a mathematical point of view the results resemble limit theorems. In practice asymptotic results are applied as approximate in a case when n is finite.

In his works V. Girko describes a more general case when together with a growing number of observations the number of random vector coordinates also tends towards infinity.

The suggested approach i.e. optimization of asymptotic behaviour under the assumption that the ratio of parameter numbers to the number of observations is constant, is often applied in practice. Girko deals with this problem using the theory of random matrices, namely the General Statistical Analysis (GSA).

This article presents the G-estimator of the Stieltjes transform of the normalized spectral function of covariance matrices based on the GSA assumptions as described by Girko, and its properties on the simulation examples for multivariate normal distribution.

Key words: random matrices, covariance matrix, Stieltjes transform, spectral function.

I. SPECTRAL FUNCTIONS

Let Ξ_n be a complex random matrix of n order, and λ_i, $I = 1,\ldots,n$ are its eigenvalues. Spectral function of Ξ_n matrix is a function that can be characterized in the following way:

$$\mu_n(x,y) = \frac{1}{c_n}\sum_{i=1}^{n}\chi\left(x - re\left(\frac{\lambda_i}{b_n}\right)\right)\chi\left(y - im\left(\frac{\lambda_i}{b_n}\right)\right),$$

where

$$\chi(x) = \begin{cases} 0 & if\ x < 0 \\ 1 & if\ x \geq 0 \end{cases},$$

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c_n, b_n are constant. If $c_n = n$, then $\mu_n(x, y)$ is called a normalized spectral function of covariance matrices $\Xi_n b_n^{-1}$. If constant values of Ξ_n matrix are real, then a normalized spectral function of this matrix looks like this:

$$\mu_n(x, y) = \frac{1}{n} \sum_{i=1}^{n} \chi(x - \lambda_i).$$

A spectral function is a random function whose realization are distribution functions and it is also called a random spectral function or random distribution function.

The most general task concerning limit spectral functions is as follows: at the appropriate choice of normalizing constants a_n, b_n you have to find convergence conditions and a general form of asymptotic distributions for a sequence of random variables

$$\frac{\mu_n(x) - a_n}{b_n}, \; n = 1,2,\ldots.$$

The person who first examined the convergence of spectral functions of random matrices whose elements located on the main diagonal and above are independent, was E. Wigner. He proved that under certain assumptions the density of limit spectral function has a shape of a semicircle, precisely that $\lim_{n\to\infty} \mu_n(x) = \mu(x)$, where

$$\mu'(x) = \begin{cases} \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2} & \text{for} \quad -2\sigma < x < 2\sigma \\ 0 & \text{for} \quad x < -2\sigma \; or \; x > 2\sigma \end{cases}, \sigma > 0$$

under the following assumptions on the elements of symmetric matrix $\Xi_n = \left[\xi_{ij}^{(n)}\right]$: $\xi_{ij}^{(n)} = \frac{\eta_{ij}}{\sqrt{n}}$, where random variables η_{ij} for all $i \geq j$, $i, j = 1,\ldots, n$ are independent, $E\eta_{ij}^2 = \sigma^2$ and have bound moments of all rows.

Limit theorems for spectral functions can be proved using the Stieltjes transform:

$m(z) = \int \frac{1}{x - z} d\mu_n(x) = \frac{1}{n} tr\left(\Xi_n - z\mathbf{I}\right)^{-1} = -\frac{1}{n}\frac{\partial}{\partial z} \ln\det\left(\Xi_n - z\mathbf{I}\right)$, where z is a complex number, $imz \neq 0$.

II. G – ESTIMATOR OF THE STIELTJES TRANSFORM OF THE SPECTRAL FUNCTION OF COVARIANCE MATRICES

Let $\vec{\xi}$ be m-dimensional random vector with a vector of expectations $\mathbf{a} = E\vec{\xi}$ and covariance matrix $\mathbf{R}_m = E\left(\vec{\xi} - \mathbf{a}\right)\left(\vec{\xi} - \mathbf{a}\right)^T$.

Expression

$$\mu_m\left(x, \mathbf{R}_m\right) = \frac{1}{m}\sum_{p=1}^{m} \chi\left(x - \lambda_p\left(\mathbf{R}_m\right)\right), \tag{1}$$

where $\lambda_m\left(\mathbf{R}_m\right) \le ... \le \lambda_1\left(\mathbf{R}_m\right)$ are eigenvalues of $\mathbf{R}_{m_n}$ matrix is called a normalized spectral function of covariance matrices. As it has been mentioned before, the notion of a description of normalized spectral function limit can be examined using limit theorems for the Stieltjes transform. Below one can find the estimator of the Stieltjes transform of the normalized spectral function of covariance matrices under the assumption that m_n and n numbers satisfy the following so-called G-condition:

$$\lim_{n\to\infty} \frac{m_n}{n} = const < \infty \ ,$$

which means that together with a growing number of observations the number of random vector coordinates also tends towards infinity and in fact m is dependent on *n*.

Let $\mathbf{x}_1,\ldots,\ \mathbf{x}_n$ be independent observations of $\vec{\xi}$ random vector and let $\hat{\mathbf{R}}_{m_n}$ be empirical covariance matrix, i.e. $\hat{\mathbf{R}}_{m_n} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i - \overline{\mathbf{x}}\right)\left(\mathbf{x}_i - \overline{\mathbf{x}}\right)^T$, where $\overline{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i$.

Let's consider the following Stieltjes transform of a function (1):

$$\varphi\left(t, \mathbf{R}_{m_n}\right) = \frac{1}{m_n} tr\left(\mathbf{I} + t\mathbf{R}_{m_n}\right)^{-1} \tag{2}$$

and, analogically, the transform of empirical covariance matrix:

$$\hat{\varphi}(t)=\frac{1}{m_n}tr\left(\mathbf{I}+t\hat{\mathbf{R}}_{m_n}\right)^{-1}. \tag{3}$$

If we want to estimate a certain function of covariance matrices $f(R)$, then usually $f(R)$ is estimated by $f(\hat{R})$. Under certain regularity conditions when $\hat{R}\xrightarrow{N\to\infty}R$, then $f(\hat{R})\xrightarrow{N\to\infty}f(R)$. However, when both the number of observations and their dimension tend towards infinity, replacing the argument of empirical covariance matrix does not quarantine consistency.

$G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ estimator of the Stieltjes transform of covariance matrices is presented in the following way:

$$G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)=\hat{\varphi}\left(\hat{\theta}\right) \tag{4}$$

where $\hat{\theta}$ is nonnegative solution of the equation

$$\theta\left(1-\frac{m_n}{n}+\frac{m_n}{n}\hat{\varphi}(\theta)\right)=t\,,t>0. \tag{5}$$

The solution of equation (5) exists and is unique.

After meeting certain assumptions for $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ a central limit theorem occurs. Let's introduce the following symbols:

$$a_n(t)=\left[q_1(t)q_2(t)\right]^{-1}$$

$$q_1(t)=1-E\hat{\varphi}'(\theta_n)k_n\theta_n\left[1-k_n+k_n\theta_nE\hat{\varphi}'(\theta_n)+k_nE\hat{\varphi}'(\theta_n)\right]^{-1}$$

$$q_2(t)=\left[\frac{2}{m_n}tr\mathbf{A}^2+\frac{1}{m_n}\sum_{i=1}^{m_n}d_ia_{ii}^2\right]^{-\frac{1}{2}}$$

$$d_i=E\left[\left(\tilde{x}_{ij}^{(n)}\right)^2-1\right]-2\,,\ \tilde{x}_{ij}^{(n)}$$

are vector dimensions

$$\tilde{\mathbf{x}}_i=\mathbf{R}_{m_n}^{-\frac{1}{2}}\left(\mathbf{x}_i-\mathbf{a}\right),\ i=1,...,n\,,\ j=1,...,m_n\,,$$

$$\mathbf{A}=\left[a_{ij}\right]=\left[1+\frac{\theta_n}{n}Etr\mathbf{S}\mathbf{R}_{m_n}\right]^{-1}\theta_n\mathbf{S}^2\mathbf{R}_{m_n}+\theta_n^2\mathbf{S}\mathbf{R}_{m_n}E\frac{1}{n}tr\mathbf{S}^2\mathbf{R}_{m_n}\left[1+\frac{\theta}{n}Etr\mathbf{S}\mathbf{R}_{m_n}\right]^{-2}$$

$$\mathbf{S}=\left(\mathbf{I}+\theta_n\mathbf{Q}\right)^{-1},\ \mathbf{Q}=\frac{1}{n}\sum_{k=1}^{n}\left(\mathbf{x}_k-E\mathbf{x}_k\right)\left(\mathbf{x}_k-E\mathbf{x}_k\right)^T,\ k_n=\frac{m_n}{n},$$

θ_n is nonnegative solution of the equation $\theta_n\left(1-k_n+k_nE\hat{\varphi}\left(\theta_n\right)\right)=t,\ t>0$,

$$\hat{\varphi}\left(\theta_n\right)=\frac{1}{m_n}tr\mathbf{S},$$

$$c_n\left(t\right)=-a_n\left(t\right)\left[\frac{\theta_nE\left(\mathbf{S}^2\boldsymbol{\eta},\boldsymbol{\eta}\right)}{\sqrt{m_nn}\left[1-\frac{\theta_n}{n}E\left(\mathbf{S}\boldsymbol{\eta},\boldsymbol{\eta}\right)\right]}q_1\left(t\right)+\sqrt{\frac{m_n}{n}}\sum_{p=1}^{m_n}\left[-\frac{E\varepsilon_{2p}^2}{\left(1+\lambda_pt\right)^3}+\frac{E\varepsilon_{2p}}{\left(1+\lambda_pt\right)^2}\right]\right]$$

$$\boldsymbol{\eta}=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\mathbf{x}_i-E\mathbf{x}_i\right),$$

$$\varepsilon_{2p}=\theta_n\left[\sum_{l=1}^{n}b_{pl}^2-\lambda_p\right]-\theta_n^2\left[\left(\mathbf{B}_p^T\Gamma_p\mathbf{B}_p\mathbf{b}_p,\mathbf{b}_p\right)-\frac{\lambda_p}{n}tr\mathbf{R}_p\mathbf{B}_p\mathbf{B}_p^T\right]+$$

$$+\theta_n\lambda_p\left[\frac{1}{m_n}tr\Gamma_p-E\hat{\varphi}\left(\theta_n\right)\right]$$

,

$\mathbf{B}_p$ is a matrix obtained by crossing out a p-row of matrix $\mathbf{B}=\frac{1}{\sqrt{n}}\left(\vec{\xi}_1,...,\vec{\xi}_n\right)$,

$\vec{\xi}_k=\Lambda_{m_n}^{\frac{1}{2}}\vec{\zeta}_k$, $\vec{\zeta}_k=\mathbf{H}\mathbf{R}_{m_n}^{-\frac{1}{2}}\left(\mathbf{x}_k-E\mathbf{x}_k\right)$, $\Lambda_{m_n}=diag\left(\lambda_i,i=1,..m_n\right)$, $\mathbf{H}$ is a matrix of eigenvectors of matrix $\mathbf{R}_{m_n}$, $\Gamma_p=\left(\mathbf{I}+\theta_{\mathbf{n}}\mathbf{B}_p\mathbf{B}_p^T\right)^{-1}$, $\mathbf{b}_p$ is a row of $\mathbf{B}$ matrix.

If $0<t<c<\infty$, $0<\lambda_i\le c<\infty$, $Eb_{pl}^4<c<\infty$, then functions $a_n\left(t\right)$ and $c_n\left(t\right)$ satisfy inequality: $\sup_n\left[\left|a_n\left(t\right)\right|+\left|c_n\left(t\right)\right|\right]<\infty$.

Using the above symbols the following central limit theorem is true, Girko (1995):

Theorem. Let's assume that there are independent observations $\mathbf{x}_1,\ldots,\ \mathbf{x}_n$ of a random vector $\vec{\xi}$ and the following conditions are satisfied:

$$\overline{\lim_{n\to\infty}}\frac{m_n}{n}<1,\ 0<c_1\le\lambda_i\le c_2<\infty,i=1,...,m_n,$$

coordinates of a vector $\vec{\eta}_k = \left(\eta_{1k},...,\eta_{m_n k}\right)^T = \mathbf{H}\mathbf{R}_{m_n}^{-\frac{1}{2}}\left(\mathbf{x}_k - E\mathbf{x}_k\right)$ are independent and

$$\sup_n \sup_{k=1,...,n} \sup_{i=1,...,m_n} E\left|\eta_{ik}\right|^{4+\delta} < \infty,\ \delta > 0 .$$

Then, for every $t > 0$:

$$\lim_{n\to\infty} P\left\{\left[G\left(t,\hat{\mathbf{R}}_{m_n}\right) - \varphi\left(t,\mathbf{R}_{m_n}\right)\right]\sqrt{m_n n}a_n(t) - c_n(t) < x\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy .$$

III. SIMULATION EXAMPLES

In order to examine the G-estimator properties of the Stieltjes transform of a normalized spectral function of covariance matrices a few simulation experiments have been conducted. The transform estimator described by the formula (2) - $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right) = \hat{\varphi}\left(\hat{\theta}\right)$ has been compared to the classic one described by the formula (3).

Let a random vector $\left(X_1,...,X_p\right)$ be a p-dimensional random variable with a normal distribution $N_p\left(0,\mathbf{R}\right)$. According to this scheme n vectors have been generated and on their basis $\hat{\theta}$ has been determined as well as the value of estimator $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ and the classic maximum likelihood estimator $\hat{\varphi}(t) = \frac{1}{m_n} tr\left(\mathbf{I} + t\hat{\mathbf{R}}_{m_n}\right)^{-1}$. The experiments have been conducted for k=10 000 repetitions. Mean values of estimators and mean square errors have been calculated. The calculations have been conducted for different values of *t*, *p*, *n*, $\varphi\left(t,\mathbf{R}\right)$.

The results are presented in tables 1–6.

Mean values of $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ and $\varphi\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ estimators and their mean square errors for a multivariate normal distribution depending on random vector dimension and sample size

Table 1. Normal distribution, p=2, t=1, $\varphi(1,\mathbf{R})=0.5$

Sample size	$\hat{\theta}$	Mean estimator value		Mean square terror	
		$G(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	$\varphi(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	BSK(G)	BSK(φ)
5	1.235	0.530	0.566	0.013	0.014
10	1.108	0.514	0.533	0.006	0.006
15	1.070	0.511	0.524	0.003	0.003
20	1.052	0.508	0.518	0.002	0.002
25	1.041	0.505	0.513	0.002	0.002
30	1.034	0.504	0.510	0.001	0.001
35	1.029	0.503	0.509	0.001	0.001
40	1.026	0.502	0.507	0.001	0.001
45	1.023	0.503	0.509	0.001	0.001
50	1.020	0.503	0.507	0.001	0.001
55	1.018	0.503	0.507	0.001	0.001
60	1.017	0.501	0.504	0.001	0.001
65	1.016	0.502	0.505	0.001	0.001
70	1.015	0.501	0.504	0.001	0.001
75	1.014	0.501	0.504	0.001	0.001
80	1.013	0.501	0.504	0.001	0.001
85	1.012	0.500	0.503	0.001	0.001
90	1.011	0.500	0.502	0.000	0.000
95	1.011	0.501	0.503	0.000	0.000
100	1.010	0.501	0.503	0.000	0.000

Source: Own calculations.

Table 2. Normal distribution, p=5, t=1, $\varphi(1,\mathbf{R})=0.5\ 28$

Sample size	$\hat{\theta}$	Mean estimator value		Mean square terror	
		$G(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	$\varphi(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	BSK(G)	BSK(φ)
10	1.297	0.543	0.583	0.002	0.005
15	1.183	0.536	0.563	0.001	0.002
20	1.132	0.533	0.554	0.001	0.001
25	1.103	0.532	0.549	0.001	0.001
30	1.085	0.531	0.546	0.001	0.001
35	1.072	0.531	0.543	0.001	0.001
40	1.062	0.530	0.541	0.000	0.001
45	1.055	0.529	0.539	0.000	0.001
50	1.049	0.530	0.539	0.000	0.000
55	1.045	0.530	0.538	0.000	0.000
60	1.041	0.529	0.536	0.000	0.000
65	1.038	0.529	0.536	0.000	0.000
70	1.035	0.530	0.536	0.000	0.000
75	1.032	0.529	0.535	0.000	0.000
80	1.030	0.529	0.534	0.000	0.000
85	1.029	0.530	0.535	0.000	0.000
90	1.027	0.529	0.534	0.000	0.000
95	1.025	0.529	0.533	0.000	0.000
100	1.024	0.529	0.533	0.000	0.000

Source: Own calculations.

Table 3. Normal distribution, p=10, t=1, $\varphi(1,\mathbf{R})=0.5$

Sample size	$\hat{\theta}$	Mean estimator value		Mean square error	
		$G(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	$\varphi(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	BSK(G)	BSK(φ)
15	1.475	0.518	0.593	0.002	0.009
20	1.324	0.512	0.570	0.001	0.006
25	1.244	0.512	0.558	0.001	0.004
30	1.197	0.506	0.547	0.001	0.003
35	1.165	0.505	0.539	0.001	0.002
40	1.141	0.505	0.535	0.000	0.002
45	1.124	0.503	0.530	0.000	0.001
50	1.110	0.503	0.528	0.000	0.001
55	1.099	0.503	0.526	0.000	0.001
60	1.090	0.503	0.523	0.000	0.001
65	1.083	0.502	0.522	0.000	0.001
70	1.077	0.503	0.520	0.000	0.001
75	1.071	0.502	0.518	0.000	0.001
80	1.066	0.502	0.517	0.000	0.000
85	1.062	0.501	0.516	0.000	0.000
90	1.059	0.502	0.516	0.000	0.000
95	1.055	0.501	0.514	0.000	0.000
100	1.052	0.502	0.514	0.000	0.000

Source: Own calculations.

Mean values of $G\left(t,\hat{\mathbf{R}}_{\mathbf{m}_n}\right)$ and $\varphi\left(t,\hat{\mathbf{R}}_{\mathbf{m}_n}\right)$ estimators when there is interdependence n=2p between dimensions of observation matrix

Table 4. Normal distribution, t=5, $\varphi(5,\mathbf{R})=0.167$

Random p vector dimension	Sample size	$\hat{\theta}$	Mean estimator value	
			$G(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$	$\varphi(t,\hat{\mathbf{R}}_{\mathbf{m}_n})$
5	10	8.352	0.199	0.274
10	20	8.469	0.181	0.254
15	30	8.504	0.176	0.252
20	40	8.524	0.173	0.250
25	50	8.533	0.172	0.249
30	60	8.539	0.171	0.248
35	70	8.547	0.170	0.247
40	80	8.547	0.170	0.247
45	90	8.547	0.170	0.247
50	100	8.552	0.169	0.246

Source: Own calculations.

Mean values of $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ and $\varphi\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ estimators for different t values at fixed dimensions of observation matrix.

Table 5. Normal distribution, $p = 2$, $n = 10$

t	$\hat{\theta}$	$\varphi(t,\mathbf{R})$	Mean estimator value	
			$G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$	$\varphi\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$
0.5	0.538	0.636	0.647	0.659
1.0	1.107	0.500	0.517	0.536
1.5	1.691	0.419	0.437	0.460
2.0	2.284	0.364	0.380	0.404
2.5	2.884	0.322	0.336	0.361
3.0	3.484	0.290	0.306	0.332
3.5	4.093	0.263	0.277	0.303
4.0	4.695	0.241	0.261	0.286
4.5	5.313	0.223	0.236	0.261
5.0	5.920	0.207	0.224	0.249
10	12.085	0.122	0.138	0.158

Source: Own calculations.

Table 6. Normal distribution, p=2, n=20

t	$\hat{\theta}$	$\varphi(t,\mathbf{R})$	Mean estimator value	
			$G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$	$\varphi\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$
0.5	0.519	0.636	0.643	0.650
1.0	1.052	0.500	0.505	0.514
1.5	1.592	0.419	0.424	0.435
2.0	2.135	0.364	0.370	0.382
2.5	2.680	0.322	0.330	0.343
3.0	3.227	0.290	0.296	0.309
3.5	3.774	0.263	0.274	0.287
4.0	4.325	0.241	0.250	0.262
4.5	4.875	0.223	0.232	0.244
5.0	5.426	0.207	0.214	0.226
10	10.955	0.122	0.129	0.138

Source: Own calculations.

Estimator $G\left(t,\hat{\mathbf{R}}_{\mathbf{m_n}}\right)$ enables a better estimation of $\varphi(t,\mathbf{R})$ size than a classic estimator. Mean square errors are very small but for G-estimator the error is even smaller. For a determined dimension of a random vector together with the growing sample size $\hat{\theta}$ values tend towards t. The situation is different when

p changes together with n. In the presented case when $p = \frac{1}{2n}$, and when n is growing, $\hat{\theta}$ assumes the values located further and further from t, but other relations between estimators remain unaltered.

In all of the presented examples G-estimator enables a very good parameter estimation. It is extremely important when the number of observations is small in comparison to a random vector dimension, i.e. together with the growing number of observations, the random vector dimension is also growing. The experiment proved that contrary to the classic estimator, G-estimator maintained its basic property i.e. consistency.

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ESTYMACJA PRZEKSZTAŁCENIA STIELTJESA UNORMOWANEJ FUNKCJI SPEKTRALNEJ MACIERZY KOWARIANCJI

W pracy przedstawiono G-estymator przekształcenia Stieltjesa unormowanej funkcji spektralnej macierzy kowariancji otrzymany przez V. Girko i jego własności na przykładach symulacyjnych dla wielowymiarowego rozkładu normalnego.

W teorii dużych prób, w klasycznym podejściu, bada się własności estymatorów, przyjmując założenie, że liczebność próby n dąży do nieskończoności. Z matematycznego punktu widzenia wyniki mają charakter twierdzeń granicznych. W praktyce wyniki asymptotyczne stosowane są jako przybliżone, w sytuacji, gdy n jest skończone. W wielu zagadnieniach praktycznych problemem staje się ograniczenie liczebności próby. Wtedy estymatory największej wiarogodności tracą swoje optymalne własności (efektywność, zgodność). V. Girko w swoich pracach zajmuje się ogólniejszym przypadkiem, gdy wraz ze wzrostem liczby obserwacji również liczba współrzędnych wektora losowego dąży do nieskończoności.

Proponowane podejście: optymalizować zachowanie asymptotyczne, przy założeniu, że stosunek liczby parametrów do liczby obserwacji jest stały, często występuje w praktyce.