ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 141, 1997

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APPLICATION OF THE SEQUENTIAL PROBABILITY RATIO TEST TO VERIFICATION OF STATISTICAL HYPOTHESES

Abstract. The paper deals with some problems concerning the sequential probability ratio tests (SPRT) and their application to verifying simple and composite statistical hypotheses.

Besides properties and examples of SPRT, there are presented advantages of this group of tests and reasons why we cannot always apply them in practice.

Key words: sequential probability ratio test, operating characteristic function, average sample size.

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Sequential probability ratio test may be applied to verifying simple and composite parametric statistical hypotheses. In this test, as in all sequential methods the sample size isn't fixed in advance and every sampling unit can cause the hypothesis to be accepted or rejected, or the investigation may be continued by sampling a new unit.

Let $X_1, X_2, \dots$ be a sequence of the independent and identically distributed random variables. Let $f(x, \theta)$ be density function of X, if X is a continuous variable or probability function if X is discrete. We wish to verify the simple hypothesis about unknown value of parameter θ:

$$H_0\colon\ \theta = \theta_0 \qquad H_0\colon\ \theta = \theta_1,\ \theta_0 \neq \theta_1. \tag{1}$$

For given probability errors – α and β, we can approximate constants A and B using the following formulae (see Govindarajulu 1985, p. 135):

$$B \cong \frac{\beta}{1-\alpha}, \quad A \cong \frac{1-\beta}{\alpha} \tag{2}$$

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We note that $0 < B < 1 < A$.

For all $n \in N$ we calculate $\Delta_n = \prod_{i=1}^{n} \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}$ and compare this value with two constans A and B.

If $\Delta_n \leqslant B$ then we accept H_0, if $\Delta_n \geqslant A$ H_0 is rejected, if $B < \Delta_n < A$ we continue sampling, and draw $(n+1)$-*th* unit.

In practice, instead of random variable Δ_n we often use variable:

$$z_t = \ln \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}.$$

Then the rule is the following:

when $\sum_{i=1}^{n} z_i \leqslant \ln B$ we accept H_0, when $\sum_{i=1}^{n} z_i \leqslant \ln A$ we reject H_0, when $\ln B < \sum_{i=1}^{n} z_i < \ln A$ – we continue sampling.

The sequential tests, similarly to the classical tests, are characterized by the power function and consequently by the operating characteristic function (OC).

Consider an SPRT for $H_0: f = f(x, \theta_0)$ against $H_1: f = f(x, \theta_1)$, $\theta_0 \neq \theta_1$. The following theorem is true (see Fisz 1967, p. 611):

Theorem 1. Let $z = \ln \frac{f(x, \theta_1)}{f(x, \theta_0)}$. Assume that:

(a) there exists $D^2(z)$

(b) there exists $\delta > 0$ that $P(z < \ln(1-\delta)) > 0$ and $P(z > \ln(1+\delta)) > 0$

(c) for each $h \in R$ there exists the expectation $E_\theta(e^{hz}) = g(h)$

(d) $g(h)$ is twice differentiable.

If $E(z) \neq 0$ then there exists exactly one h_0 that $E_\theta(e^{h_0 z}) = 1$ and if $E(z) = 0$ then $h_0 = 0$.

Assuming that $E(z) \neq 0$, we can define $f^*(x, \theta) = \left(\frac{f(x, \theta_1)}{f(x, \theta_0)}\right)^h \cdot f(x, \theta)$ and formulate the auxiliary hypotheses:

H: $f = f(x, \theta)$ against H^*: $f = f^*(x, \theta)$.

Let $P_\theta(accept\ H_0) = P_\theta(accept\ H) = 1 - \alpha^*$, where α^* denotes the probability of error type I for the auxiliary test.

Evaluating

$$\Delta_n^* = \prod_{i-1}^{n} \frac{f^*(x_i, \theta)}{f(x_i, \theta)} = \prod_{i=1}^{n} \left(\frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}\right)^h = \left(\prod_{i=1}^{n} \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)}\right)^h$$

and applying constans A i B (from the estimation of Δ_n) we obtain the estimates Δ_n^* by means of the constans A^h and B^h. Since $A^h = \frac{1-\beta^*}{\alpha^*}$, $B^h = \frac{\beta^*}{1-\alpha^*}$, we can get α^* and β^* by solving the equations.

Hence

$$OC(\theta) = 1 - \alpha^* = \frac{A^h - 1}{A^h - B^h} \tag{3}$$

Note $h = 1$ if $\theta = \theta_0$ and $h = -1$ if $\theta = \theta_1$. So $OC(\theta_1) = 1 - \alpha$ and $OC(\theta_0) = \beta$.

If $E_\theta(z) = 0$ then the equation $g(h) = 0$ has the solution $h = 0$ and with the help of de l'Hospital we obtain:

$$\lim_{h \to 0} OC(\theta_1) = \frac{\ln A}{\ln A - \ln B}.$$

Having $OC(\theta)$ we derive the power function and the average sample size. In sequential tests the number of observations needed to arrive at a decision is a random variable (N). We can calculate the expectation of this variable from the formula:

$$E_\theta(N) = \begin{cases} \dfrac{OC(\theta) \cdot \ln B + (1 - OC(\theta) \cdot \ln A}{E_\theta(z)} & \text{when } E_\theta(z) \neq 0 \\ \dfrac{-\ln A \cdot \ln B}{E_\theta(z^2)} & \text{when } E_\theta(z) = 0 \end{cases} \tag{4}$$

The formula of $E_\theta(N)$ when $E_\theta(z) \neq 0$ is implied by the following equations:

(a) $E_\theta(\ln \Delta_n) = E_\theta(N) \cdot E_\theta(z)$,

(b) $E_\theta(\ln \Delta_n) = \ln B \cdot P_\theta(\text{accept } H_0) + \ln A \cdot P_\theta(\text{reject } H_\theta)$,

what is equivalent with the equation:

(b') $E_\theta(\ln \Delta_n) = \ln B \cdot OC(\theta) + \ln A \cdot (1 - OC(\theta))$.

(b) and (b') follow from the approximation of $\ln \Delta_n$ by the binomial distribution with values $\ln A$ and $\ln B$ at the n-th stage where a decision is reached.

The second part of formula (2) follows from the analogous formulae for $E_\theta(\ln \Delta_n)^2$.

Consider an example of verifying two simple hypotheses by means of SPRT's. Let X be normally distributed having mean θ and known variance σ^2.

Let H_0: $\theta = \theta_0$ against H_1: $\theta = \theta_1$, $\theta_1 > \theta_0$, We investigate the value:

$$\ln\Delta_n = \sum_{i=1}^{n} z_i,$$

where

$$z_i = \ln\frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} = \frac{1}{2\sigma^2}[(x_i, -\theta_0)^2 - (x_i - \theta_1)^2] = \frac{1}{\sigma^2}(\theta_1 - \theta_0)x_i + $$

$$+\frac{1}{2\sigma^2}(\theta_0^2 - \theta_1^2),$$

in dependence on constants $\ln A$ and $\ln B$.

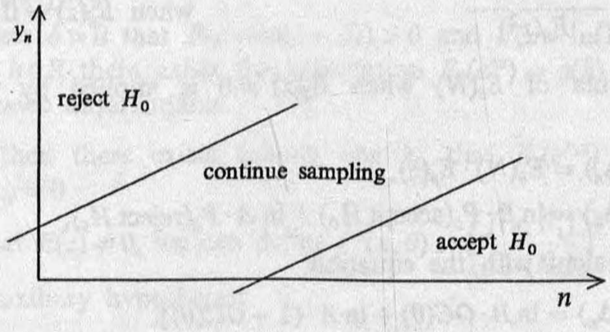
$$\ln B < \frac{1}{\sigma^2}(\theta_1 - \theta_0)\sum_{i=1}^{n} x_i + \frac{n}{2\sigma^2}(\theta_0^2 - \theta_1^2) < \ln A$$

$$\frac{\sigma^2 \ln B}{\theta_1 - \theta_0} + \frac{n}{2}(\theta_0 + \theta_1) < \sum_{i=1}^{n} x_i < \frac{\sigma^2 \ln A}{\theta_1 - \theta_0} + \frac{n}{2}(\theta_0 + \theta_1)$$

Denoting $b = \frac{\sigma^2 \ln B}{\theta_1 - \theta_0} < 0$, $a = \frac{\sigma^2 \ln A}{\theta_1 - \theta_0}$, $c = \frac{\theta_1 - \theta_0}{2} > 0$, $y_n = \sum_{i=1}^{n} x_i$

we obtain $b + nc < y_n < a + nc$.

Hence, we graphically determine acceptance and rejection regions, and the region of sampling continuation in dependence on n.



In this case, function OC has the form:

$$OC(\theta) = \frac{A^{ho} - 1}{A^{ho} - B^{ho}}, \quad \text{where } h(\theta) = \frac{\theta_1 + \theta_0 - 2\theta}{\theta_1 - \theta_0},$$

but the average sample size has the form:

$$E_\theta(N) = \sigma^2 \frac{OC(\theta)\cdot \ln B - (1 - OC(\theta)\cdot \ln A}{(\theta_1 - \theta_0)\theta + 0.5(\theta_0^2 - \theta_1^2)}$$

For $\theta = \theta_0$ and $\theta = \theta_1$ we obtain:

$$E_\theta(N) = \sigma^2 \frac{(1-\alpha)\cdot \ln B - \alpha \cdot \ln A}{-0.5(\theta_1 - \theta_0)^2}$$

$$E_\theta(N) = \sigma^2 \frac{\beta \cdot \ln B - (1-\beta)\cdot \ln A}{0.5(\theta_1 - \theta_0)^2}$$

Assuming, that $\alpha = \beta = 0.05$ and $\theta_0 = 0$, $\theta_1 = 1$ we have $E_{\theta_0}(N) = E_{\theta_1}(N) = 6$.

If we verify this hypothesis with a classical test with the fixed-sample size, the sample size ought to be 11 for the same error probabilities α and β.

It is not a coincidence. The following theorem is true (see Fisz 1967, p. 630–631):

Theorem 2. Among all tests (fixed-sample or sequential) for which P_{θ_0} (reject H_0) $\leqslant \alpha$, P_{θ_0} (accept H_0) $\leqslant \beta$, and for which $E_{\theta_i}(N) < \infty$ $(i = 0, 1)$, the SPRT with error probabilities α and β minimizes both $E_{\theta_0}(N)$ and $E_{\theta_1}(N)$.

In practice we hardly have to do with two simple hypotheses. There are often composite hypotheses which can also be verified by a sequential probability ratio test. But it is not always possible (see Silvey 1978, p. 168–176).

Consider the following hypotheses:

$$\begin{cases} H_0\colon \theta \leqslant \theta_1 \\ H_1\colon \theta \geqslant \theta_2, \ \theta_1 < \theta_2 \end{cases} \tag{5}$$

If for $\theta' > \theta$ the ratio $\dfrac{f(x_i, \theta')}{f(x_i, \theta)}$ is a nondecreasing function some statistic $t(x_i)$ then sequential probability ratio test for the hypothesis H'_0: $\theta = \theta_1$ against H'_1: $\theta = \theta_2$, $(\theta_1 < \theta_2)$ has the property:

$$P_\theta(\text{accept } H'_1) \leqslant \alpha \quad \text{for } \theta \leqslant \theta_1$$

$$P_\theta(\text{accept } H'_0) \leqslant \beta \quad \text{for } \theta \leqslant \theta_2$$

α and β are probabilities errors. Hence SPRT used for verification of H'_0 and H'_1 can be used to test the composite hypotheses H_0 and H_1. In this case to verify the composite hypotheses we can use the SPRT for the verification of simple hypotheses but it is possible only if these hypotheses refer to the individual parameter and if the likelihood ratio is monotone.

We can use SPRT to verify three or more hypotheses (see Govindarajulu 1985, p. 148–150. Let X be random variable normally distributed with unknown mean and θ known variance σ^2.

Consider the following hypotheses:

$$H_0:\ \theta=\theta_0$$
$$H_1:\ \theta=\theta_1, \qquad (6)$$
$$H_2:\ \theta=\theta_2,\ \theta_0<\theta_1<\theta_2$$

Assume, that P_{θ_0} (reject H_0) $\leqslant\gamma_0$, P_{θ_1} (reject H_1) $\leqslant\gamma_1$, and P_{θ_2} (reject H_2) $\leqslant\gamma_2$.

Having a sequence $\{X_n\}$ of independent random variables with the distribution $N(\theta,\sigma)$ we test the above hypotheses verifing each pair of neighbouring hypotheses.

Denote R_1 – the SPRT test to verify H_0 versus H_1, R_2 – the SPRT test to verify H_1 versus H_2. We can reach a decision at n-th stage if we reach a decision in both tests R_1 and R_2.

If R_1 – accept H_0, R_2 – accept H_1 then decision is to **accept** H_0.
If R_1 – accept H_1, R_2 – accept H_1 then decision is to **accept** H_1.
If R_1 – accept H_1, R_2 – accept H_2 then decision is to **accept** H_2.

The result R_1 – accept H_0, R_2 – accept H_2 is impossible. Let us assume that this is possible. Then the following inequalities are true:

$$\sum_{i=1}^{n}x_i\leqslant\frac{\ln B_1}{\theta_1-\theta_0}+\frac{n}{2}(\theta_0+\theta_1)$$

$$\sum_{i=1}^{n}x_i\geqslant\frac{\ln A_2}{\theta_2-\theta_1}+\frac{n}{2}(\theta_2+\theta_1),$$ where B_1 and A_2 are constans in tests R_1 and R_2 respectively.

Hence $\frac{\ln A_2}{\theta_2-\theta_1}+\frac{n}{2}(\theta_2+\theta_1)\leqslant\frac{\ln B_1}{\theta_1-\theta_0}+\frac{n}{2}(\theta_0+\theta_1)$, so $\frac{\ln A_2}{\theta_2-\theta_1}+\frac{n}{2}\theta_2-$ $-\frac{\ln B_1}{\theta_1-\theta_0}+\frac{n}{2}\theta_0\leqslant 0$, which contradicts our assumption about A_2 and B_1.

After n sampling, we always reach a decision of accepting one out of three hypotheses.

This test can be applied for testing a two-sided hypothesis for the mean of the normal distribution: H_0: $\theta=\theta_0$, H_1: $\theta\neq\theta_0$.

The probability ratio test (with the assumption $D(z)\neq 0$) has the following property: probability of making a decision of accepting or rejecting H_0 after a finite number of steps is equal 1. Average sample number of observations can be counted, and usually it is smaller than the sample size in classical tests with the same errors probabilities. Sometimes sample for sequential test exceeds considerably the sample size for classical tests, for example in the test for two variances (see Johnson 1954). The number of observations in this test depends on sampling procedure and sometimes is even twice larger.

The economics of tests, in spite of sampling procedure, is also affected by the constans A and B. In the above considerations A and B were determined approximately in dependence on the I and II kind error probabilities. In testing hypothesis about runs defection in quality control, constans A and B are computed in dependence on the costs of making a wrong decision. They are determined in such a way that the costs connected with making a decision of accepting or rejecting H_0 on n-stage are equal to the costs connected with further research and making a decision after $(n+k)$-steps (see Vagholkar, Wetherill 1960).

Besides SPRT, we can use truncated and generalized sequential probability ratio test. Truncated SPRT are applied if on N_0-stage, $N_0 < E_\theta(N)$, we must make a decision of accepting or rejecting H_0 although the value of statistic $\ln \Delta_n$ belongs to continuation region. We apply these tests in order to reduce costs or the number of observations on which we have to make a decision although it is smaller than the necessary but inaccessible number. We meet this kind of tests in medical researches most often. We divide the continuation area in this way: if $\ln B < \ln \Delta_n < 0$ we accept $0 < \ln \Delta_n < \ln A$ we accept H_1.

Generalized SPRT is characterized with the following property: constants A and B are not used on every stage of sampling. On n-th stage, value of statistic Δ_n is estimated in dependence on fixed values A_n and B_n.

Sequential tests are more economical than classical tests, but problems with determining function OC, average sample size or constants A and B cause them to be.

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ZASTOSOWANIE SEKWENCYJNYCH TESTÓW ILORAZOWYCH DO WERYFIKACJI HIPOTEZ STATYSTYCZNYCH

W pracy przedstawiono istotę sekwencyjnych testów ilorazu prawdopodobieństwa (SPRT) oraz ich zastosowania do weryfikacji prostych i złożonych hipotez statystycznych.

Oprócz własności i przykładów testów SPRT przedstawione są zalety tej grupy testów oraz powody, dla których nie zawsze łatwo można stosować je w praktyce.