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Antonio Piccolomini d'Aragona

CLASSICAL LOGIC, UNIFORMITY, AND WEAK EXCLUDED MIDDLE IN NON-MONOTONIC PROOF-THEORETIC SEMANTICS

Abstract

Non-monotonic base-extension semantics (nB-eS), a kind of *non-monotonic proof-theoretic semantics* (nPTS), is known to validate classical logic when its meta-logic is classical. Schroeder-Heister has remarked that classical meta-logic is as problematic for the project of modelling intuitionistic logic, as an intuitionistic proof of incompleteness would be. It may be unclear, though, whether Schroeder-Heister's remark holds for *non-monotonic proof-theoretic validity* (nP-tV) as well, i.e., for Prawitz's original version of nPTS. We only know that, with classical meta-logic again, classical logic is sound over a variant of nP-tV, which I shall call *liberal non-monotonic proof-theoretic validity* (LnP-tV). The latter, in turn, differs from nP-tV in that reductions for the rewriting of proof-structures are not required to be *uniform*. After drawing attention to a number of divergences between nB-eS, nP-tV and LnP-tV, I show that Schroeder-Heister's remark might

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after all apply to nP-tV too. In particular, Weak Excluded Middle (WEM) is logically valid via uniform reductions (with a meta-logic which is non-intuitionistic, but non-classical either).

Keywords: classical logic, uniformity, weak excluded middle, non-monotonic proof-theoretic semantics.

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1. Introduction

By *non-monotonic proof-theoretic semantics* (nPTS) I shall understand in what follows the kind of constructive semantics introduced by Prawitz in [23].

Prawitz's original approach, called today (non-monotonic) *proof-theoretic validity* (nP-tV), is based on the notion of *valid argument structure*. An argument structure is a Natural Deduction derivation with arbitrary inferences, and it is said to be valid when it reduces, modulo a set of proof-rewriting functions called *reductions* (and possibly modulo substitution of unbound assumptions with closed valid arguments for these assumptions), to an argument structure which ends by a Natural Deduction introduction, and whose immediate sub-structures are also valid—the prior role of introductions stems from Gentzen's claim in [6], that introductions define the meaning of the logical constants, while eliminations (or better, inferences which are not in introduction-form, given that nP-tV appeals to arbitrary inferences) are consequences of this definition. Validity of arguments is first relativised to atomic proof-systems, called *atomic bases*. An argument is *logically* valid when it is valid over all atomic bases. Consequence over an atomic base means existence of an argument which is valid on the given base, whereas logical consequence means existence of a logically valid argument. The approach is non-monotonic in the sense that both validity and consequence may hold on a given atomic base, while failing on extensions of this base—a monotonic variant can be developed too, and is in fact more investigated nowadays, but I will leave it aside here.

In a more recent approach, called (non-monotonic) *base-extension semantics* (nB-eS), argument structures and reductions are left out. After introducing atomic bases, one defines a notion of consequence on an atomic base by direct induction on the complexity of the formulas of the underlying language. Logical consequence is just consequence over all atomic bases. Once again, the framework is non-monotonic since consequence might hold on a given atomic base, but fail on some of its extensions—and, also in this case, a more investigated monotonic picture is available, but I will not discuss it here.

Classical logic (`CL`) has been proved by Schroeder-Heister to be sound and complete on nB-eS, provided the meta-logic is classical [30]. Schroeder-Heister remarks that his result is as fatal as an intuitionistic proof of incompleteness would be for Prawitz's project of a semantics which intuitionistic logic (`IL`) be complete over—so-called *Prawitz's conjecture*, see also [17, 19]. Schroeder-Heister's remark appeals to the joint facts that an intuitionistic proof of completeness would imply a classical contradiction, that the proofs in question can be coded in first-order arithmetic, and finally that classical arithmetic and Heyting arithmetic are equi-consistent.

As said, however, nB-eS differs from Prawitz's original nP-tV, since it does without argument structures and reductions. This means, in particular, that the nP-tV and the nB-eS notions of consequence may not coincide, so the answer to the question whether Schroeder-Heister's remark applies to nP-tV too is not straightforward.

In this paper, I aim to shed a bit of light on this issue. I shall start by recalling the proof in [12] that `CL` is sound with classical meta-logic on a variant of nP-tV, called *liberal* (non-monotonic) *proof-theoretic validity* (LnP-tV). I note that from this one cannot infer that Schroeder-Heister's remark applies to nP-tV, since the proof at issue, besides classical meta-logic, forces reductions and the reduction sequences they induce to be *non-uniform*—namely, the rewriting of the argument structures depends on non-invariant features of the input-values, specifically, the atomic bases which these values are valid over. Prawitz's original nP-tV seems to take reductions to be uniform instead.

Next, I refine and improve some recent results established in [14]. One of these is that nB-eS and LnP-tV are in fact equivalent. This is not so relevant for the question whether `IL` is complete over LnP-tV which, as said, had been already settled negatively in [12] by proving `CL` to be sound over it—although it permits one to infer that `CL` is also complete over LnP-tV, via Schroeder-Heister's result for nB-eS mentioned above. Rather, the equivalence shows that nB-eS and LnP-tV share some structural principles, in particular, that the derivability of a rule on an atomic base is tantamount to the admissibility of that rule on that atomic base, and that logical consequence is tantamount to consequence on every atomic base. I highlight the connection between the potential failure of these principles in nP-tV, and the requirement that reductions and reduction sequences be uniform. On the other hand, I provide a result of completeness of the implication-free fragment of `IL` over nP-tV. The interest of this proof of completeness stems from the fact that it is obtained via the *iuxta propria principia* of nP-tV, namely, without using the disputed structural principles.

I also prove the equivalence between the nB-eS and the nP-tV notions of consequence on an atomic base when the meta-logic is classical. This, in turn, shows two things. First, that the non-uniform reading of reductions and reduction sequences is morally equivalent to a "local" usage of classical meta-logic—as I shall specify below, a similar phenomenon is encountered by Barroso Nascimento, Pereira and Pimentel in [2] in the somewhat different context of a proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25]. Second, that the ‘‘local" equivalence is not enough for incompleteness of `IL`, due to the fact that, when reductions and reduction sequences are uniform, logical consequence is not reducible to a collection of ‘‘local" consequence relations. One potential conclusion one may draw from this is that, unless some stronger result will be proved in the future, the requirement of uniformity on reductions and reduction sequences is in a sense stronger than classical meta-logic.

From what said, one can positively conclude that whether one can apply to nP-tV Schroeder-Heister's argument that a classical proof of incompleteness is enough for refuting Prawitz's conjecture, boils down to whether one can give a non-intuitionistic proof of incompleteness of `IL` over nP-tV, while insisting at the same time on the uniform character of reductions or reduction sequences. This is indeed the case, as shown by the final result of the paper, which establishes the logical validity of Weak

Excluded Middle (WEM) with a meta-logic where WEM itself holds, and in a framework which might be said to cope with most of the requirements of nP-tV. Note that this is not in contrast with the claim that the requirement of uniformity is stronger than classical meta-logic, since the proof of the logical validity of WEM employs as said a less-than-classical meta-logic—nor is it in contrast with [9] and [16], two sources I shall also touch upon below whose counter-examples to completeness of `IL` over nP-tV are not closed under replacements of atoms with arbitrary formulas.

The reason why I say that Schroeder-Heister's remark *might be*—rather than just *is*—applicable to nP-tV is that, while improving the soundness result for `CL` from [12], my proof of the logical validity of WEM over nP-tV highlights that there might be further features besides uniformity, especially concerning how *falsum* (noted $\bot$) is semantically dealt with, which are compatible with the nP-tV approach, and which the reductions that I shall put forward for the proof itself *do not* enjoy. Concerning the semantic treatment of $\bot$, I shall work in what follows under the assumption that atomic bases are always consistent, i.e., never prove $\bot$—a convention whose importance has been recently unfolded by Barroso Nascimento, Pereira and Pimentel in [2], as I remark below. These issues are dealt with in the concluding remarks.

Before starting, it may be useful to locate this work within the broader field of researches into completeness of `IL` over the kind of semantics which nPTS belongs to. Some crucial results proved by Sandqvist [26] and, in a more general framework, by de Campos Sanz, Piecha and Schroeder-Heister [5, 18, 19], showed that `IL` is incomplete over a monotonic variant of nB-eS. This was later on extended to a monotonic variant of LnP-tV by [13], via an equivalence theorem similar to the one proved in [14] for nB-eS and LnP-tV[1]. Incompleteness of `IL` over a monotonic variant of nP-tV was finally established in [15]. All these achievements, however, pertain as said to the monotonic picture which, again, I shall not be interested in what

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[1] The equivalence theorem between the monotonic variants of nB-eS and LnP-tV is expressly used in [13] to establish incompleteness of `IL` over LnP-tV, but it had been already achieved by Stafford in [31]. A more general proof of this result has been recently provided also by [16].

follows[2]. As for the non-monotonic picture, besides the results I have already mentioned [12, 14, 30], an early proof of incompleteness of `IL` over nB-eS was provided by Piecha and Schroeder-Heister in [19]. The latter, however, may not carry over to nP-tV, precisely in the same way as was observed above for the other related findings. This was remarked in [16]—and, as anticipated, it will be discussed also below—following similar analyses carried out in [10, 11, 12, 14]. Further insights into the non-monotonic behaviour induced by classical connectives in the framework of a proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25] have been recently provided by Barroso Nascimento, Pereira and Pimentel in [2]. As I shall hint at below, these insights are related to the topics at issue here mostly relative to the semantic treatment of $\bot$. Finally, [15, 16] provide proofs of incompleteness of `IL` over nP-tV which differ from the one presented in this paper in that, while sticking more strictly to Prawitz's "pure" version of nP-tV, they rely upon counterexamples to completeness which, as already pointed out, are not closed under replacements of atoms with formulas.

In what follows, I will limit myself to a propositional language $\mathscr{L}$ with connectives $\wedge, \vee$ and $\rightarrow$. The set of the formulas of $\mathscr{L}$ is written $\texttt{FORM}_{\mathscr{L}}$, while the set of the atoms of $\mathscr{L}$, written $\texttt{ATOM}_{\mathscr{L}}$, is $\{p_i \mid i \in \mathbb{N}\} \cup \{\bot\}$. Negation is not primitive, i.e., $\neg A$ is interpreted as $A \rightarrow \bot$.

Although I am concerned with the relation between nP-tV and nB-eS, I shall not work with the latter directly, but with its variant LnP-tV. This depends on what I said above, i.e., that a proof of soundness of `CL` with classical meta-logic is available for LnP-tV, that nP-tV obtains from LnP-tV by restricting the notion of reduction, and that the issue about reductions boils down to structural differences between nB-eS and nP-tV.

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[2] `IL` has been proved by Sandqvist to be complete over a monotonic variant of nB-eS with an elimination-like clause for disjunction and atomic bases of level ≥ 2 [27]. This was later on used by Gheorghiu and Pym to prove that `IL` is complete over a monotonic variant of nP-tV with elimination rules as primitive and atomic bases of level ≥ 2 [7], and further extended by Barroso Nascimento, Pereira and Pimentel in the broader context of Prawitz's ecumenical logic [2, 25]. By modifying the notion of atomic base, moreover, `IL` was proved to be complete on a monotonic variant of nB-eS by Schroeder-Heister [30] and, with a different strategy, by Stafford and Barroso Nascimento [1, 32].

2. Atomic bases, nPTS and classical logic

Let us first of all deal with LnP-tV and nB-eS, and their relation with `CL`.

2.1. Atomic bases

The starting notion is that of atomic base. The latter, as well as the notion of derivability in atomic bases, requires however the preliminary definitions of the concepts of atomic rule and atomic derivation.

Definition 2.1. *Atomic rules of level* n are defined by induction as follows:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$

  $$\frac{}{A}$$

  is an atomic rule of level 0

- $A_1, ..., A_n, A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$

  $$\frac{A_1 \qquad \dots \qquad A_n}{A}$$

  is an atomic rule of level 1

- $A_1, ..., A_n, A \in \mathtt{ATOM}_{\mathscr{L}}$ and $\mathfrak{C}_i$ are atomic rules of level at most k ($i \leq n$) $\Longrightarrow$

  $$\frac{\begin{matrix}[\mathfrak{C}_1] & & [\mathfrak{C}_n] \\ A_1 & \dots & A_n\end{matrix}}{A}$$

  is an atomic rule of level $k + 2$.

Brackets indicate dischargement of assumptions or of assumed rules—via Schroeder-Heister's *higher-level rules* [28].

Definition 2.2. *Atomic derivations* are defined by standard induction on the length of applications of atomic rules, starting from the basic case of a single-node derivation consisting of an application of a rule of level 0.

Definition 2.3. An *atomic base of level n* is a set of atomic rules whose maximal level is n.

Definition 2.4. A is *derivable* from $\mathfrak{C}$ in the atomic base $\mathfrak{B}$, written $\mathfrak{C} \vdash_{\mathfrak{B}} A$, iff there is an atomic derivation $\mathscr{D}$ such that, for every atomic rule $\mathfrak{c}$ applied and not discharged in $\mathscr{D}$, it holds that $\mathfrak{c} \notin \mathfrak{C} \Longrightarrow \mathfrak{c} \in \mathfrak{B}$.

I assume the following convention of consistency of atomic bases.

Convention 2.5. For every $\mathfrak{B}$, $\nvdash_{\mathfrak{B}} \bot$.

As we shall see in Section 5, Convention 2.5 will play a crucial role in the proof of the logical validity of WEM in nP-tV. The importance of this convention has been recently unfolded by Barroso Nascimento, Pereira and Pimentel in [2], although in a somewhat different framework from the one at stake in this paper. They provide some proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25], the latter being, roughly, a logic where classical and intuitionistic connectives coexist. Also, they work in a context which is more akin to—albeit eventually richer than—that of Sandqvist in [27], i.e., a monotonic variant of nB-eS with an elimination-like clause for disjunction. In spite of this, they provide a number of new results which seem to be of interest for the kind of non-monotonic approach that I am interested in here. In particular, they show that Convention 2.5, when combined with a "Hilbertian" understanding of the semantics of classical proofs as consistency of formulas over an atomic base, might have as an effect that formulas with classical connectives may behave non-monotonically—whereas purely intuitionistic formulas are always monotonic. They end up with a motto to the effect that, in the specific kind of semantics where such a phenomenon occurs, monotonic classicality equals intuitionistic double negation[3]. Now, as we shall prove below, nPTS entertains strict connections

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[3] This specific kind of semantics is the one that Barroso Nascimento, Pereira and Pimentel qualify as *weak*. Here, a distinction between *local* and *global validity* over an atomic base in the sense of Cobreros [4] is at play. Let me additionally remark that

with classical or quasi-classical logics, whence the interest for nPTS of Barroso Nascimento, Pereira and Pimentel's results is crystal-clear. In fact, it is an interesting open question what the precise interactions are between the "standard" picture provided by nB-eS, LnP-tV and nP-tV, on the one hand, and the one provided by Barroso Nascimento, Pereira and Pimentel—possibly enriched with argument structures and reductions—on the other. This question can be addressed in future works, but the general impression is that Barroso Nascimento, Pereira and Pimentel have singled out a sufficiently broad frame where "standard" variants of proof-theoretic semantics in Prawitz's style (both monotonic and non-monotonic) can be studied with the same peaceful coexistence as the one showed by the classical and the intuitionistic connectives in Prawitz's ecumenical logic.

2.2. LnP-tV

Let us now introduce LnP-tV. As said, the main notion here is that of valid argument structure.

Definition 2.6. An *argument structure* is a pair $\langle T, f\rangle$ where:

- T is a finite rooted tree with order relation ω, nodes labelled by formulas of $\mathscr{L}$, and top-nodes partitioned into two groups, i.e., axiomatic and non-axiomatic;
- f is a function mapping onto lower nodes elements of: (a) a sub-set of the non-axiomatic top-nodes of T; (b) a sub-set of the axiomatic top-nodes of T labelled by atoms; (c) a sub-set of $\wp(\omega)$, all the elements of which contain only the pairs linking a node labelled by an atom to all its children, labelled by atoms too, with no non-axiomatic top-node

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Barroso Nascimento, Pereira and Pimentel also show, very convincingly, that an approach with consistent atomic bases is able to work out a number of conceptual problems related to other existing semantic treatments of $\perp$ in the monotonic variant of nB-eS, see [5, 17, 18, 19, 27, 33], but see also the concluding remarks below. The idea of using consistent atomic bases is moreover historically very faithful to the "spirit" of Prawitz's original semantic project, since Prawitz himself has always understood in such terms a proof-theoretic treatment of the semantics of intuitionistic logic, e.g., [22, 24], but in fact already [21].

of T mapped by f onto its argument, or onto the image of the latter through f.

The non-axiomatic top-nodes and the root of T are called, respectively, the *assumptions* and the *conclusion* of $\langle T, f\rangle$.

f is to be understood as a discharge function which operates, not only on assumptions, but also on atomic rules of any level.

Definition 2.7. With $\mathscr{D} = \langle T, f\rangle$, $\mathscr{D}$ is *closed* iff the domain of f contains all the non-axiomatic top-nodes of T, otherwise $\mathscr{D}$ is *open*. When $\mathscr{D}$ has undischarged assumptions Γ and conclusion A, it is called an argument structure *from* Γ *to* or *for* A.

An argument structure $\mathscr{D}$ from Γ to or for A is indicated by the figure

$$\begin{array}{c}\Gamma\\ \mathscr{D}\\ A\end{array}$$

The notion of (*immediate*) *sub-structure* of an argument structure is defined in a standard way. The same happens for the notion of *replacement* of a sub-structure $\mathscr{D}^*$ of an argument structure $\mathscr{D}$ with an argument structure $\mathscr{D}^{**}$—written $\mathscr{D}[\mathscr{D}^{**}/\mathscr{D}^*]$—although in this case one must take care of re-indexings of the discharge functions of $\mathscr{D}$ and $\mathscr{D}^{**}$. I will abstract from these details here.

Definition 2.8. Given $\mathscr{D}$ from $\Gamma = \{A_1, ..., A_n\}$ $(n \geq 0)$ to A, and a function σ from $\mathtt{FORM}_{\mathscr{L}}$ to argument structures such that $\sigma(A_i)$ is an argument structure for A_i $(i \leq n)$, $\mathscr{D}^\sigma = \mathscr{D}[\sigma(A_1), ..., \sigma(A_n)/A_1, ..., A_n]$ is an *instance* of $\mathscr{D}$.

The notions of inference and inference rule can be thoroughly defined via the previous definitions—see [14, 23]. For what concerns us here, though, we can content ourselves with conceiving of an inference as an argument structure looked at, so to say, from below, i.e., as an argument structure $\mathscr{D}$ obtained by conjoining certain argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ $(n \geq 0)$ for some elements $A_1, ..., A_n \in \mathtt{FORM}_{\mathscr{L}}$ (the premises of the inference) through a new root node labelled by $A \in \mathtt{FORM}_{\mathscr{L}}$ (the conclusion of the inference), possibly plus dischargements δ that extend the discharge functions involved in $\mathscr{D}_1, ..., \mathscr{D}_n$. We also say that $\mathscr{D}$ ends by the inference identified by premises $A_1, ..., A_n$, conclusion A, and possibly dischargements δ. An inference rule is a set of inferences. It is understood that inference rules are recursive, in the sense that they can be identified by meta-linguistic descriptions in familiar Natural Deduction style. The elements of the rule are also called its instances, or applications. We shall also say that these instances or applications end by instantiating or applying the corresponding rule.

Definition 2.9. $\mathscr{D}$ is *canonical* iff it ends by applying a Natural Deduction introduction rule.

It should not be difficult to see that an argument structure can be morally understood as a derivation-tree in the Natural Deduction style, where arbitrary inferences occur, or where arbitrary inference rules are instantiated or applied—with potential dischargements. Observe also that atomic derivations are argument structures of a special kind—but I defined them separately because of the special role they play, also in the nB-eS approach, relative to the notions of (logical) validity and (logical) consequence.

We can now define the notion of reduction and the reducibility relation. For doing this, I shall follow Schroeder-Heister's approach in [29].

Definition 2.10. A *reduction* is a pair $\langle \mathscr{D}, \mathscr{D}^* \rangle$ with $\mathscr{D}$ from Γ to A and $\mathscr{D}^*$ from $\Delta \subseteq \Gamma$ to A. A *reduction system* is a set $\mathfrak{S}$ of reductions such that, for every $\langle \mathscr{D}, \mathscr{D}^* \rangle \in \mathfrak{S}$ and every σ, $\langle \mathscr{D}^\sigma, (\mathscr{D}^*)^\sigma \rangle \in \mathfrak{S}$. A *reduction sequence* is a sequence $\langle \mathscr{D}_1^1, \mathscr{D}_2^1 \rangle, ..., \langle \mathscr{D}_1^n, \mathscr{D}_2^n \rangle$ $(n \geq 0)$ of reductions such that $\mathscr{D}_1^{i+1} = \mathscr{D}_2^i$ $(i \leq n)$. The sequence is said to go *from* $\mathscr{D}_1^1$ *to* $\mathscr{D}_2^n$.

Observe in passing that I put no constraint on reduction systems. In particular, they may contain reduction sequences which send one and the same argument structure onto distinct values, e.g., $\langle \mathscr{D}, \mathscr{D}^* \rangle$ and $\langle \mathscr{D}, \mathscr{D}^{**} \rangle$, with $\mathscr{D}^* \neq \mathscr{D}^{**}$—what Schroeder-Heister calls *alternative justifications* [29]. Also, a reduction system has no complexity bound—e.g., the system might be very "big", say, the set $\mathbb{S}$ of *all* the reductions is a reduction system. I shall come back to this later on.

Definition 2.11. $\mathscr{D}$ reduces to $\mathscr{D}^*$ *modulo* a reduction system $\mathfrak{S}$, written $\mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$, iff $\mathfrak{S}$ contains a reduction sequence from $\mathscr{D}$ to $\mathscr{D}^*$.

Definition 2.12. $\langle \mathscr{D}, \mathfrak{S} \rangle$ is *valid* on $\mathfrak{B}$ iff

- $\mathscr{D}$ is closed for $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$ where $\mathscr{D}^*$ witnesses $\vdash_{\mathfrak{B}} A$
- $\mathscr{D}$ is closed for $A \notin \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathscr{D} \leq_{\mathfrak{S}} \mathscr{D}^*$ where $\mathscr{D}^*$ is canonical with immediate sub-structures $\mathscr{D}^{**}$ such that $\langle \mathscr{D}^{**}, \mathfrak{S} \rangle$ is valid on $\mathfrak{B}$
- $\mathscr{D}$ is open from $A_1, ..., A_n \Longrightarrow$ for every σ, every $\mathfrak{T} \supseteq \mathfrak{S}$, if $\langle \sigma(A_i), \mathfrak{T} \rangle$ is valid on $\mathfrak{B}$ with $\sigma(A_i)$ closed, then $\langle \mathscr{D}^\sigma, \mathfrak{T} \rangle$ is valid on $\mathfrak{B}$.

Definition 2.13. $\langle \mathscr{D}, \mathfrak{S} \rangle$ is *logically valid* iff it is valid on every $\mathfrak{B}$.

From the notion of argumental (logical) validity we can extract a notion of (logical) consequence.

Definition 2.14. $\Gamma \models_{\mathfrak{B}} A$ iff there is $\langle \mathscr{D}, \mathfrak{S} \rangle$ valid on $\mathfrak{B}$, with $\mathscr{D}$ from Γ to A.

Definition 2.15. $\Gamma \models A$ iff there is $\langle \mathscr{D}, \mathfrak{S} \rangle$ logically valid, with $\mathscr{D}$ from Γ to A.

2.3. nB-eS

As said, the nB-eS approach does without argument structures and reductions. The nB-eS notion of (logical) consequence is defined, not as existence of a (logically) valid argument, but directly by induction on the complexity of formulas.

Definition 2.16. $\Gamma \Vdash_{\mathfrak{B}} A$ iff

- $\Gamma = \emptyset \Longrightarrow$
  - $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \vdash_{\mathfrak{B}} A$
  - $A = B \wedge C \Longrightarrow \Vdash_{\mathfrak{B}} B$ and $\Vdash_{\mathfrak{B}} C$
  - $A = B \vee C \Longrightarrow \Vdash_{\mathfrak{B}} B$ or $\Vdash_{\mathfrak{B}} C$
  - $A = B \rightarrow C \Longrightarrow B \Vdash_{\mathfrak{B}} C$
- $\Gamma \neq \emptyset \Longrightarrow (\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A)$
  (where $\Vdash_{\mathfrak{B}} \Gamma$ means $\Vdash_{\mathfrak{B}} B$ for every $B \in \Gamma$).

Definition 2.17. $\Gamma \Vdash A$ iff, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$.

2.4. Relations with CL

Schroeder-Heister has proved in [30] that CL is sound and complete over $\Vdash$ with classical meta-logic.

Theorem 2.18. *$\Gamma \vdash_{\texttt{CL}} A \Longleftrightarrow \Gamma \Vdash A$ with classical meta-logic.*

Proof: ($\Longrightarrow$) Since we know IL is sound over $\Vdash$, see e.g. [18], it is enough if we prove $\Vdash A \lor \neg A$. For any $\mathfrak{B}$, by classical meta-logic, either $\Vdash_{\mathfrak{B}} A$ or $\nVdash_{\mathfrak{B}} A$. In both cases, $\Vdash_{\mathfrak{B}} A \lor \neg A$. ($\Longleftarrow$) By classical meta-logic, the right-hand side condition for $\Gamma \Vdash_{\mathfrak{B}} A$ with $\Gamma \neq \emptyset$, namely, $\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A$, can be understood as ($\nVdash_{\mathfrak{B}} \Gamma$ or $\Vdash_{\mathfrak{B}} A$), where $\nVdash_{\mathfrak{B}} \Gamma$ means that, for some $B \in \Gamma$, $\nVdash_{\mathfrak{B}} B$. So, on every $\mathfrak{B}$, $A \to B$ can be interpreted as $\neg A \lor B$. Put $\llbracket A \rrbracket = \{\mathfrak{B} \mid\ \Vdash_{\mathfrak{B}} A\}$ and, when $\mathbb{B}$ is the set of all the $\mathfrak{B}$-s, put $\overline{\llbracket A \rrbracket} = \mathbb{B} - \llbracket A \rrbracket$. By Convention 2.5, we have $\overline{\llbracket A \rrbracket} = \llbracket \neg A \rrbracket$. By the same convention, $\llbracket \bot \rrbracket = \emptyset$. So $\langle \wp(\mathbb{B}), \emptyset, \mathbb{B}, \cap, \cup, \overline{\llbracket\ \rrbracket} \rangle$, i.e., the algebra of the semantic values of formulas via $\Vdash_{\mathfrak{B}}$, is a Boolean algebra. □

The situation with LnP-tV seems to be similar since, as showed in [12], a proof of soundness of CL with classical meta-logic is available also for $\models$.

Proposition 2.19. For every $\mathfrak{B}$, $\models_{\mathfrak{B}} A \lor \neg A$ with classical meta-logic.

Proof: For any $\mathfrak{B}$, by classical meta-logic, $\models_{\mathfrak{B}} A$ or $\not\models_{\mathfrak{B}} A$. If $\models_{\mathfrak{B}} A$, then there is $\langle \mathscr{D}, \mathfrak{S}_{\mathfrak{B}} \rangle$ valid on $\mathfrak{B}$ with $\mathscr{D}$ for A closed. But then, if we put

$$\mathscr{D}^*_{\mathfrak{B}} = \cfrac{\begin{matrix}\mathscr{D}\\ A\end{matrix}}{A \lor \neg A}$$

we have that $\langle \mathscr{D}^*_{\mathfrak{B}}, \mathfrak{S}_{\mathfrak{B}} \rangle$ is valid on $\mathfrak{B}$. If $\not\models_{\mathfrak{B}} A$, then the one-step argument structure

$$\frac{A}{\bot}$$

is (vacuously) valid on $\mathfrak{B}$ when paired with $\emptyset$. Hence, when we put

$$\mathscr{D}^{**} = \cfrac{\cfrac{\cfrac{[A]_1}{\bot}}{\neg A}\,1}{A \vee \neg A}$$

we have that $\langle \mathscr{D}^{**}, \emptyset \rangle$ is valid on $\mathfrak{B}$. □

Theorem 2.20. $\Gamma \vdash_{\texttt{CL}} A \Longrightarrow \Gamma \models A$ *with classical meta-logic.*

Proof: Since `IL` is known to be sound over $\models$, see [23, 29], it is enough if we prove $\models A \vee \neg A$. Let

$$\texttt{EM} = \frac{}{A \vee \neg A}$$

and

$$\mathfrak{T} = \{\langle \texttt{EM}, \mathscr{D}^{*}_{\mathfrak{B}} \rangle \mid \mathfrak{B} \in \|A\|\} \cup \{\langle \texttt{EM}, \mathscr{D}^{**} \rangle\}$$

where $\|A\| = \{\mathfrak{B} \mid \ \models_{\mathfrak{B}} A\}$, and where the $\mathscr{D}^{*}_{\mathfrak{B}}$-s and $\mathscr{D}^{**}$ are as in the proof of Proposition 2.19, and

$$\mathfrak{U} = \mathfrak{T} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{S}_{\mathfrak{B}}$$

where the $\mathfrak{S}_{\mathfrak{B}}$-s are as in the proof of Proposition 2.19. It is then easy to see that $\langle \texttt{EM}, \mathfrak{U} \rangle$ is logically valid. □

Clearly, in view of Proposition 2.19, the proof of Theorem 2.20 would have gone through also by taking $\mathbb{S}$ as above, i.e., the set of all the reductions, in place of $\mathfrak{U}$. However, $\mathfrak{U}$ constitutes a more informative choice.

3. Schroeder-Heister's remark and uniformity

From the above we can obviously conclude that, if one takes the meta-logic to be classical, then completeness of intuitionistic logic just fails for nB-eS and LnP-tV. A natural question one may now ask is whether it makes sense to use classical meta-logic to prove or disprove completeness of `IL` in a proof-based framework. This is of course relevant for Prawitz's original nP-tV too, since one may want to extend to it the incompleteness theorems for nB-eS and LnP-tV proved above.

If we limit ourselves to nB-eS, one cannot expect that, in view of Theorem 2.18, an intuitionistic proof of completeness of `IL` over $\Vdash$ can ever be found. This is because of a remark put forward by Schroeder-Heister:

> claiming that completeness can nonetheless be proved by intuitionistic (and hence also by classical) means implies claiming a classical contradiction. Given that these proofs can be coded in first-order arithmetic and that classical arithmetic and Heyting arithmetic are equiconsistent, such a claim cannot be upheld. In simpler terms, inconsistency is a negative result, and on the negative side classical and intuitionistic logics coincide. [30, p. 501]

Can we use the remark to settle (in)completeness of `IL` also relative to nP-tV? Classical meta-logic implies soundness of `CL` (and hence incompleteness of `IL`) over $\models$, via Theorem 2.20. But this is not enough for applying Schroeder-Heister's remark directly to nP-tV. While certainly Prawitzian in spirit, due to the presence of argument structures and reductions, and to the idea that (logical) consequence means existence of a (logically) valid argument, LnP-tV is, as said, only a *variant* of nP-tV. However, it is now important to establish where LnP-tV and nP-tV precisely differ since, based on this, a better assessment of whether Schroeder-Heister's remark holds for nP-tV can be achieved.

Via Definition 2.10, reduction systems in LnP-tV can be seen as broad proof-rewriting systems where, as said above, one and the same argument structure can be transformed in different alternative ways, and where there is essentially no upper bound on the computational complexity of the proof-rewriting itself. In nP-tV, reduction sequences are instead given in a much more constrained way, namely, as induced by functions ϕ, also called reductions, which go from and to argument structures, and which are defined on a sub-set $\mathbb{D}$ of an inference rule (i.e., a set of argument structures) in such a way that, for every $\mathscr{D} \in \mathbb{D}$ and every σ as in Definition 2.8: $\mathscr{D}$ is from Γ to $A \Longrightarrow \phi(\mathscr{D})$ is from $\Delta \subseteq \Gamma$ to A; $\mathscr{D}^\sigma \in \mathbb{D}$; $\phi(\mathscr{D}^\sigma) = \phi(\mathscr{D})^\sigma$. Also, it seems to be part of Prawitz's understanding in [23] that ϕ, and the reduction sequences induced by a given set of reductions, are *uniform*, i.e., the outputs that ϕ produces for given inputs, as well as the reduction sequences made up of given inputs and outputs of reductions, can be specified independently of the potential validity of the inputs or outputs relative to specific sets of reductions or specific atomic bases. The prototype reduction of nP-tV is hence the kind of function used for removing detours in proofs of normalisation for Natural Deduction calculi, and the prototype reduction sequences of nP-tV are the sequences of Natural Deduction derivations induced by functions of that kind.

The validity of an argument structure in nP-tV is not relative to a reduction system as in Definition 2.10, but to a set of reductions $\mathfrak{J}$ as specified above. Apart from that, the definitions of validity of $\langle \mathscr{D}, \mathfrak{J} \rangle$ on $\mathfrak{B}$ and of logical validity of $\langle \mathscr{D}, \mathfrak{J} \rangle$ are in all ways similar, *mutatits mutandis*, to what happens in Definitions 2.12 and 2.13.

It is easy to see that, if $\langle \mathscr{D}, \mathfrak{J} \rangle$ is valid (logically or on some $\mathfrak{B}$) in nP-tV, then there is $\mathfrak{S}$ such that $\langle \mathscr{D}, \mathfrak{S} \rangle$ is valid (logically or on $\mathfrak{B}$) in LnP-tV: take $\mathfrak{S}$ to be the set of the reduction sequences induced by $\mathfrak{J}$ on $\mathscr{D}$. Generally, however, the inverse fails: $\mathfrak{S}$ gives rise to a $\mathfrak{J}$ only when special conditions obtain.

E.g., the presence in $\mathfrak{S}$ of what, following Schroeder-Heister [29], I have called above alternative justifications, may speak against $\mathfrak{S}$ being the graph of a function. Suppose for example that

$$\mathfrak{S} = \{\langle \mathscr{D}, \mathscr{D}^* \rangle, \langle \mathscr{D}, \mathscr{D}^{**} \rangle\}$$

where $\langle \mathscr{D}, \mathscr{D}^* \rangle$ and $\langle \mathscr{D}, \mathscr{D}^{**} \rangle$ are reductions, $\mathscr{D}$ is closed, and $\mathscr{D}^* \neq \mathscr{D}^{**}$. So, $\mathfrak{S}$ is a reduction system, but clearly $\mathfrak{S}$ cannot be the graph of a function defined on some sub-set of the rule which $\mathscr{D}$ belongs to. However, this may not be a big problem, thus I shall not discuss it further in what follows. One potential solution might be, roughly, that of considering the sub-sets of a given reduction system which coincide with reduction sequences induced by a set of reductions in Prawitz's sense, and then associating to each such sub-set as many functions from and to argument structures as are required for generating the reduction sequences themselves. In our example above, the sub-sets are of course $\{\langle \mathscr{D}, \mathscr{D}^* \rangle\}$ and $\{\langle \mathscr{D}, \mathscr{D}^{**} \rangle\}$, and to them we associate the functions $\phi_1(\mathscr{D}) = \mathscr{D}^*$ and $\phi_2(\mathscr{D}) = \mathscr{D}^{**}$, so to have $\mathfrak{J} = \{\phi_1, \phi_2\}$.

Much more seriously, reduction systems in the sense of Definition 2.10 may violate the uniformity constraint, i.e., from a reduction system we may not be able to extract a set of reductions, nor reduction sequences induced by these reductions, which are uniform in the sense hinted at above.

This is best seen with the reduction system $\mathfrak{U}$, especially its sub-set $\mathfrak{T}$, that I used for Theorem 2.20 for proving classically soundness of `CL` over $\models$. $\mathfrak{T}$ contains the (unproblematic) reduction $\langle \mathtt{EM}, \mathscr{D}^{**} \rangle$, where `EM` is as in the proof of Theorem 2.20, while $\mathscr{D}^{**}$ is as in the proof of Proposition 2.19, plus the pairs $\langle \mathtt{EM}, \mathscr{D}^{*}_{\mathfrak{B}} \rangle$ for every $\mathfrak{B} \in \|A\|$, where $\|A\|$ is as in the proof of Theorem 2.20, while $\mathscr{D}^{*}_{\mathfrak{B}}$ is as in the proof of Proposition 2.19. Had we to extract from $\mathfrak{U}$ a reduction for `EM` in Prawitz's sense, this would have to be a function ϕ onto argument structures from argument structures *and atomic bases*, such that

$$\phi(\mathfrak{B}, \mathtt{EM}) = \begin{cases} \mathscr{D}^{**} & \mathfrak{B} \notin \|A\| \\ \mathscr{D}^{*}_{\mathfrak{B}} & \mathfrak{B} \in \|A\| \end{cases}$$

Alternatively—with a strategy similar to the one hinted above for alternative justifications—we may consider the set of reductions for `EM` in Prawitz's sense

$$\mathfrak{F} = \phi_{\perp} \cup \textstyle\bigcup_{\mathfrak{B} \in \|A\|} \phi_{\mathfrak{B}}$$

where

$$\phi_{\perp}(\mathtt{EM}) = \mathscr{D}^{**}$$

and

$$\phi_{\mathfrak{B}}(\mathtt{EM}) = \mathscr{D}^{*}_{\mathfrak{B}}.$$

If we assume that each $\mathscr{D}^{*}_{\mathfrak{B}}$ is obtained by appending introduction of disjunction to an argument structure which is valid on $\mathfrak{B}$ relative to some set of reductions $\mathfrak{J}_{\mathfrak{B}}$ in Prawitz's sense, we would have then to associate to `EM` either the set of reductions

$$\{\phi\} \cup \textstyle\bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{J}_{\mathfrak{B}}$$

or the set of reductions

$$\mathfrak{F} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{I}_{\mathfrak{B}}.$$

But neither of these solutions works in nP-tV. As concerns the first, ϕ is non-uniform (and so too are the reduction sequences it generates), because its outputs depend on the atomic base relative to which the validity of `EM` is to be evaluated. As for the second solution, although each element of

$$\bigcup_{\mathfrak{B} \in \|A\|} \phi_{\mathfrak{B}}$$

might well be seen as uniform, the reduction sequences generated by

$$\mathfrak{F} \cup \bigcup_{\mathfrak{B} \in \|A\|} \mathfrak{I}_{\mathfrak{B}}$$

are also non-uniform, as we cannot describe the output values without referring to the actual $\mathscr{D}^*_{\mathfrak{B}}$-s which exist on each $\mathfrak{B} \in \|A\|$—where of course there might be a distinct $\mathscr{D}^*_{\mathfrak{B}}$ for each distinct $\mathfrak{B} \in \|A\|$.

4. nB-eS and nP-tV

Schroeder-Heister's remark points out that a classical proof of incompleteness of `IL` over $\Vdash$ is enough for ruling out the existence of an intuitionistic proof of completeness of `IL` over $\Vdash$. On the other hand, as I already said, LnP-tV is much in the spirit of nP-tV, as it uses argument structures and reductions, and it defines (logical) consequence as existence of a (logically) valid argument; also, we know that in the case of LnP-tV too, we can prove classically the incompleteness of `IL` over $\models$. From the latter two facts, one may be tempted to infer that Schroeder-Heister's remark might also apply to nP-tV.

We saw this is wrong, though, since classical meta-logic is not the only thing needed to prove the incompleteness of `IL` for a version of nPTS with argument structures, reductions, and (logical) consequence defined as existence of a (logically) valid argument. For, we must also use non-uniform reductions and reduction sequences, which are *not* allowed in nP-tV.

The issue of uniformity of reductions and reduction sequences is crucial for understanding the difference, not only between nP-tV and LnP-tV, but also and above all between nP-tV and nB-eS. Since we are seeking whether a remark formulated for the nB-eS picture is applicable to nP-tV, it is thus time we turn to this latter issue.

4.1. Admissibility and choice

In [14], nB-eS and LnP-tV are proved to be actually equivalent, both relative to consequence over an atomic base, and relative to logical consequence.[4]

THEOREM 4.1. $\Gamma \Vdash_{\mathfrak{B}} A \Longleftrightarrow \Gamma \models_{\mathfrak{B}} A$.

THEOREM 4.2. $\Gamma \Vdash A \Longleftrightarrow \Gamma \models A$.[5]

Although the proofs of these result do not require classical meta-logic, the results cannot be used as such to establish a direct connection between nB-eS and nP-tV since, again, they force a reading of reductions and reduction sequences which breaks the uniformity constraint. The results also suggest, though, that the difference between LnP-tV and nP-tV must somehow be at play when comparing nB-eS and nP-tV too.

This is actually the case, but is easily overlooked as nB-eS does not involve argument structures and reductions at all. In the case of consequence over an atomic base, however, the issue lurks out when trying to prove that, if $\Gamma \Vdash_{\mathfrak{B}} A$ with $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$, then there is an argument structure from Γ to A which is valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences.

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[4] Actually, for the equivalence to hold, sets of assumptions in nB-eS must always be finite—which is not surprising, nor harmful, since so are they in LnP-tV. Moreover, in [14] I did not use the kind of reductions that, following [29], I am employing here. Instead, I introduced a version of nP-tV where reductions and reduction sequences generated by them, understood in a strictly Prawitzian sense, are allowed to be non-constructive and non-uniform. In view of this, however, those results are easily adapted to the LnP-tV framework developed in this paper.

[5] Observe that, via Theorem 2.18, from this result it also follows that, besides soundness, we also have completeness of `CL` over $\models$. I could have proved soundness and completeness of `CL` over $\models$ as a corollary of Theorem 2.18 plus Theorem 4.2, but the proof of soundness I gave in Theorem 2.20 is to my mind more informative about the kind of non-uniform reductions and reduction sequences we need, and will be thus more relevant when I will prove the logical validity of WEM over LnP-tV (and nP-tV) in Section 5.

Let us see this more clearly. Assume that $\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A$, and concede that $\Vdash_{\mathfrak{B}} \Gamma$ and $\Vdash_{\mathfrak{B}} A$ imply the existence of closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n, \mathscr{D}$ for $A_1, ..., A_n, A$ valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences. Whenever we are given $\mathscr{D}_1, ..., \mathscr{D}_n$ for $A_1, ..., A_n$ of the kind described, we can find, depending on these, a closed argument structure for A of the kind described, call it $f(\mathscr{D}_1, ..., \mathscr{D}_n)$. To prove that from this we can infer the existence of an open argument structure from Γ to A as required, we seem to have no other means than associating the one-step argument structure

$$\mathscr{D} = \frac{A_1 \qquad \dots \qquad A_n}{A}$$

to a mapping ϕ which sends any instance $\mathscr{D}^\sigma$ of $\mathscr{D}$ obtained by replacing the unbound assumptions $A_1, ..., A_n$ by closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ for them of the kind described, onto $f(\mathscr{D}_1, ..., \mathscr{D}_n)$, i.e.,

$$\phi(\mathscr{D}^\sigma) = f(\mathscr{D}_1, ..., \mathscr{D}_n).$$

So, $\mathscr{D}$ is valid on $\mathfrak{B}$ modulo ϕ, plus the sets of reductions relative to which the images of f are valid on $\mathfrak{B}$—this is a simplified version of the full proof, which can be found in [14, Proposition 3].

The point is now that ϕ is non-uniform, as its outputs cannot be specified independently of the validity of the inputs on the given atomic base, that is, independently of the specific closed valid argument for A which exists by assumption on the atomic base, in connection with specific closed valid arguments for $A_1, ..., A_n$ whose validity on the atomic base is assumed. More generally, the problem concerns the nB-eS clause for consequence on an atomic base with non-empty set of assumptions, i.e.,

(A) $\Gamma \Vdash_{\mathfrak{B}} A \Longleftrightarrow (\Vdash_{\mathfrak{B}} \Gamma \Longrightarrow \Vdash_{\mathfrak{B}} A)$, with $\Gamma \neq \emptyset$.

Observe that, with argument structures and reductions, (A) says that every inference from Γ to A which is *admissible* on $\mathfrak{B}$, is also *derivable* on $\mathfrak{B}$, i.e., if the existence of a proof for every element of Γ on $\mathfrak{B}$ implies the existence of a proof for A on $\mathfrak{B}$, then A is provable from Γ on $\mathfrak{B}$.

In an approach *à la* nP-tV, i.e., with argument structures, and uniform reductions and reduction sequences, only the left-to-right direction of (A) can be taken for granted. In fact, one can show that the following conditions are all satisfied by a notion of consequence over an atomic base extracted from the nP-tV notion of argumental validity over an atomic base hinted at above—which I shall write $\models^u$ to highlight that it is in all ways similar to $\models$, except for the requirement of uniformity on reductions and reduction sequences.

PROPOSITION 4.3. The following facts hold for $\models^u_{\mathfrak{B}}$:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A)$
- $A = B \wedge C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \models^u_{\mathfrak{B}} B$ and $\models^u_{\mathfrak{B}} C)$
- $A = B \vee C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \models^u_{\mathfrak{B}} B$ or $\models^u_{\mathfrak{B}} C)$
- $A = B \rightarrow C \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow B \models^u_{\mathfrak{B}} C)$
- $\Gamma \models^u_{\mathfrak{B}} A$ with $\Gamma \neq \emptyset \Longrightarrow (\models^u_{\mathfrak{B}} \Gamma \Longrightarrow \models^u_{\mathfrak{B}} A)$

With Proposition 4.3, and classical meta-logic again, we can actually prove that the nB-eS and nP-tV notions of consequence on an atomic base coincide. This is interesting in itself, but also for introducing the next topic I want to deal with.

PROPOSITION 4.4. $\Gamma \models^u_{\mathfrak{B}} A \Longleftrightarrow \Gamma \Vdash_{\mathfrak{B}} A$ with classical meta-logic.

PROOF: Suppose $\Gamma = \emptyset$. We proceed by induction on A:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow (\models^u_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A \Longleftrightarrow \Vdash_{\mathfrak{B}} A)$
- $A = B \wedge C \Longrightarrow (\models^u_{\mathfrak{B}} B$ and $\models^u_{\mathfrak{B}} C \overset{\text{i.h.}}{\Longleftrightarrow} \Vdash_{\mathfrak{B}} B$ and $\Vdash_{\mathfrak{B}} C)$
- $A = B \vee C \Longrightarrow (\models^u_{\mathfrak{B}} B$ or $\models^u_{\mathfrak{B}} C \overset{\text{i.h.}}{\Longleftrightarrow} \Vdash_{\mathfrak{B}} B$ or $\Vdash_{\mathfrak{B}} C)$
- $A = B \rightarrow C \Longrightarrow$ we split the two cases. Suppose $B \models^u_{\mathfrak{B}} C$ and $B \nVdash_{\mathfrak{B}} C$. By classical meta-logic, $\Vdash_{\mathfrak{B}} B$ and $\nVdash_{\mathfrak{B}} C$. By i.h., $\models^u_{\mathfrak{B}} B$ hence, by Proposition 4.3, $\models^u_{\mathfrak{B}} C$. By i.h., we also have $\not\models^u_{\mathfrak{B}} C$. Contradiction. By classical meta-logic, $B \models^u_{\mathfrak{B}} C \Longrightarrow B \Vdash_{\mathfrak{B}} C$. Suppose $B \Vdash_{\mathfrak{B}} C$ and $B \not\models^u_{\mathfrak{B}} C$. By classical meta-logic, the following

obtains: for every $\mathscr{D}$ from B to C, for every $\mathfrak{J}$, there is σ as in Definition 2.8 such that, for some $\mathfrak{H} \supseteq \mathfrak{J}$, $\langle \sigma(B), \mathfrak{H} \rangle$ is valid on $\mathfrak{B}$ with $\sigma(B)$ closed, and $\langle \mathscr{D}^\sigma, \mathfrak{H} \rangle$ is not valid on $\mathfrak{B}$. Now, from the fact that $\langle \sigma(B), \mathfrak{H} \rangle$ is valid on $\mathfrak{B}$, i.e., $\models^u_{\mathfrak{B}} B$, by i.h., we have $\Vdash_{\mathfrak{B}} B$, hence $\Vdash_{\mathfrak{B}} C$, since we were assuming $B \Vdash_{\mathfrak{B}} C$. From $\Vdash_{\mathfrak{B}} C$ we infer by i.h. again that $\models^u_{\mathfrak{B}} C$, i.e., there is $\langle \mathscr{D}^*, \mathfrak{J}^* \rangle$ valid on $\mathfrak{B}$, with $\mathscr{D}^*$ closed for C. Consider now

$$\mathscr{D}^{**} = \frac{B}{C}$$

and consider

$$\mathfrak{J}^{**} = \{\phi\} \cup \mathfrak{J}^*$$

where ϕ is such that, for every σ,

$$\phi((\mathscr{D}^{**})^\sigma) = \mathscr{D}^*$$

Then, $\langle \mathscr{D}^{**}, \mathfrak{J}^{**} \rangle$ is valid on $\mathfrak{B}$, contradicting the assumption $B \not\models^u_{\mathfrak{B}} C$. Hence, $B \Vdash_{\mathfrak{B}} C \Longrightarrow B \models^u_{\mathfrak{B}} C$.

The case with $\Gamma \neq \emptyset$ is proved in a way similar to the implication case with $\Gamma = \emptyset$. $\square$

Note that ϕ in the proof of Proposition 4.4 is perfectly uniform—it is just a constant function. Now, since we have proved that $\Vdash$ and $\models^u$ coincide over atomic bases when the meta-logic is classical, we may be led to think that the same holds for the respective notions of logical consequence. But this is not so again, because of another difference between nP-tV and nB-eS.

Logical consequence is defined in nB-eS simply as consequence over all atomic bases, namely, as per Definition 2.16,

(C) $\Gamma \Vdash A$ iff, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$.

If argument structures and reductions are brought in, (C) becomes: A is a logical consequence of Γ iff, for every $\mathfrak{B}$, there is $\langle \mathscr{D}, \mathfrak{J} \rangle$ valid on $\mathfrak{B}$ with $\mathscr{D}$

from Γ to A. But this is not how things go in nP-tV. The quantifiers of the previous reading are here inverted, i.e., we do not require a valid argument for each atomic base, but a logically valid argument, namely, an argument which is valid over all atomic bases.

By Definition 2.15, this is how things go in LnP-tV too. As said, though, if the uniformity constraint on reductions and reduction sequences is given up, one can prove (constructively) Theorem 4.2, that is, the equivalence of $\Vdash$ and $\models$. As before, let us begin by assuming that, for every $\mathfrak{B}$, $\Gamma \Vdash_{\mathfrak{B}} A$, with $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$, and let us concede that this implies that, for every $\mathfrak{B}$, there is $\mathscr{D}_{\mathfrak{B}}$ valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences—with $\mathscr{D}_{\mathfrak{B}}$ from Γ to A, i.e.,

$$\begin{array}{ccc} A_1 & \dots & A_n \\ & \mathscr{D}_{\mathfrak{B}} & \\ & A & \end{array}$$

To infer from this that there is a logically valid argument from Γ to A, we seem to have again no other means than associating the one-step argument structure

$$\mathscr{D} = \frac{A_1 \qquad \dots \qquad A_n}{A}$$

to a mapping ϕ as follows. Let $\mathscr{D}^\sigma$ be any instance of $\mathscr{D}$ obtained by replacing the unbound assumptions $A_1, ..., A_n$ by closed argument structures $\mathscr{D}_1, ..., \mathscr{D}_n$ for them which are valid on $\mathfrak{B}$ modulo some set of reductions—under a suitable reading of reductions and reduction sequences. Then

$$\phi(\mathscr{D}^\sigma) = \begin{array}{ccc} \mathscr{D}_1 & & \mathscr{D}_n \\ A_1 & \dots & A_n \\ & \mathscr{D}_{\mathfrak{B}} & \\ & A & \end{array}$$

So, $\mathscr{D}$ is valid on every $\mathfrak{B}$ modulo ϕ, plus the sets of reductions relative to which the $\mathscr{D}_{\mathfrak{B}}$-s are valid—this is a simplified version of the full proof, which can be found in [14, Proposition 4].

Note that ϕ behaves like a sort of choice-function, which picks out the right argument structure relative to each $\mathfrak{B}$. So, it is similar to the reduction system $\mathfrak{U}$ as in the proof of Theorem 2.20, or to the function ϕ extracted from it in Section 3. As I observed in [12] already, the fact that we do not need classical meta-logic to prove the result above shows that the non-uniform character of reductions has, on the general constructivist spirit of the approach, essentially the same effects as classical meta-logic. Conversely, if we just require uniform reductions, *even without giving up* classical meta-logic, we can no longer prove Proposition 2.19, hence Theorem 2.20—the most we can obtain is the equivalence of nP-tV and nB-eS relative to consequence on an atomic base. There seems to be here another interesting, potential connection with the work of Barroso Nascimento, Pereira and Pimentel [2], as they show that, in one of the proof-theoretic semantics for (a variant of) Prawitz's ecumenical logic [25] they develop[6], metalinguistic excluded middle is valid in a "local monotonic" sense when classical proofs are defined in the "Hilbertian" sense specified in Section 2.1. Once again, this topic can be investigated in future works.

4.2. Completeness of the implication-free fragment on nP-tV

Due to (A) and (C) from the previous section, it is therefore difficult to see how to prove (in)completeness results for nP-tV via the results that we have for nB-eS. One could wonder how far one can go by allowing only for the left-to-right direction of (C), i.e.,

$$\Gamma \models^{u}_{\mathfrak{B}} A \Longrightarrow (\models^{u}_{\mathfrak{B}} \Gamma \Longrightarrow \models^{u}_{\mathfrak{B}} A)$$

and by reading (A) with quantifiers inverted, i.e., there is $\mathscr{D}$ from Γ to A logically valid modulo some set of uniform reductions.

By way of example, I show in this section that one can go as far as establishing completeness of the implication-free fragment of `IL`, written `IL`*—a similar result has been proved by Humberstone for a similar notion

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[6] Again, the weak kind with a distinction between local and global validity on an atomic base in the sense of Cobreros [4].

of validity [8, Theorem 4.13.3]. In the next section, however, I shall also show that, in certain relevant cases, one *cannot* go further than that.

Let us first of all introduce a preliminary definition and some preliminary, though simple results—as said, in what follows, up to the end of this section but not beyond, (sets of) formulas have to be taken as implication-free.

Definition 4.5. To any A we associate the inductively defined set $\mathtt{E}_A$:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow \mathtt{E}_A = \{A\}$
- $A = B \wedge C \Longrightarrow \mathtt{E}_A = \{x \wedge y \mid \langle x, y\rangle \in \mathtt{E}_B \times \mathtt{E}_C\}$
- $A = B \vee C \Longrightarrow \mathtt{E}_A = \{x \mid x \in \mathtt{E}_B \cup \mathtt{E}_C\}$.

Proposition 4.6. $A \vdash_{\mathtt{IL}^*} \bigvee_{x \in \mathtt{E}_A} x$ and $\bigvee_{x \in \mathtt{E}_A} x \vdash_{\mathtt{IL}^*} A$.

Proposition 4.7. $\vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A \Longleftrightarrow \vdash_{\mathfrak{B}} A$ with $A \in \mathtt{ATOM}_{\mathscr{L}}$.

Proof: Apply standard results from normalisation theory for Natural Deduction, see [3, 20, 22]. ☐

Proposition 4.8. $\models^{u}_{\mathfrak{B}} A \Longleftrightarrow \vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A$.

Proof: By induction on A—using the disjunction property. ☐

Now some additional definitions—inspired by the *import* and *export* functions that de Campos Sanz, Piecha and Schroeder-Heister defined for atomic rules of any level, see [18, 19].

Definition 4.9. For any atomic rule $\mathfrak{R}$ of the form

$$\frac{}{A}$$

we set $\mathfrak{R}^* = A$ and, for any set of atomic rules $\mathfrak{S} = \{\mathfrak{R}_1, ..., \mathfrak{R}_n, ...\}$ of level 0, we set $\mathfrak{S}^* = \{\mathfrak{R}^*_1, ..., \mathfrak{R}^*_n, ...\}$.

Definition 4.10. For any disjunction-free formula A we set:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow A^\circ = \{\mathfrak{R}\}$, where $\mathfrak{R}$ is the atomic rule

$$\frac{}{A}$$

- $A = \bigwedge_{i \leq n} B_i \Longrightarrow A^\circ = \bigcup\{B_i^\circ \mid i \leq n\}$.

For any disjunction-free set $S = \{A_1, ..., A_n, ...\}$, we set $S^\circ = \bigcup\{A_1^\circ, ..., A_n^\circ, ...\}$. Observe that every $x \in \mathtt{E}_A$ is disjunction-free, therefore x° is defined, and is a set of atomic rules of level 0—so $(x^\circ)^*$ is defined too.

PROPOSITION 4.11. With $x \in \mathtt{E}_A$, $(x^\circ)^* \vdash_{\mathtt{IL}^*} x$ and, for every $y \in (x^\circ)^*$, $x \vdash_{\mathtt{IL}^*} y$.

PROOF: By induction on A:

- $A \in \mathtt{ATOM}_{\mathscr{L}} \Longrightarrow$ trivial
- $A = B \wedge C \Longrightarrow x$ is of the form $z \wedge w$ with $z \in \mathtt{E}_B$ and $w \in \mathtt{E}_C$, while

  $$(x^\circ)^* = ((z \wedge w)^\circ)^* = \{\bigcup\{z^\circ, w^\circ\}\}^* = \bigcup\{(z^\circ)^*, (w^\circ)^*\}$$

  By i.h., $(z^\circ)^* \vdash_{\mathtt{IL}^*} z$ and $(w^\circ)^* \vdash_{\mathtt{IL}^*} w$, so $\bigcup\{(z^\circ)^*, (w^\circ)^*\} \vdash_{\mathtt{IL}^*} z \wedge w$. Vice versa, by i.h., for every $y \in (z^\circ)^*$, $z \vdash_{\mathtt{IL}^*} y$ so, because $z \wedge w \vdash_{\mathtt{IL}^*} z$, $z \wedge w \vdash_{\mathtt{IL}^*} y$. We prove similarly that, for every $y \in (w^\circ)^*$, $z \wedge w \vdash_{\mathtt{IL}^*} y$. Therefore, for every $y \in \bigcup\{(z^\circ)^*, (w^\circ)^*\}$, $z \wedge w \vdash_{\mathtt{IL}^*} y$.

The case with $A = B \vee C$ is in all ways similar to the conjunction case. ☐

We moreover have this result—proved in [13] for full $\mathtt{IL}$ and atomic bases of any level.

PROPOSITION 4.12. $\Gamma \vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A \Longleftrightarrow (\Gamma, \Delta \vdash_{\mathtt{IL}^*} A$, for some finite $\Delta \subseteq \mathfrak{B}^*)$.

Finally, the following holds trivially.

PROPOSITION 4.13. $\Gamma \models^u A \Longrightarrow$ for every $\mathfrak{B}, (\models^u_{\mathfrak{B}} \Gamma \Longrightarrow \models^u_{\mathfrak{B}} A)$.

So, we have the following.

THEOREM 4.14. $\Gamma \models^u A \Longrightarrow \Gamma \vdash_{\mathtt{IL}^*} A$.

PROOF: Suppose $\Gamma = \emptyset$. By Proposition 4.13, for every $\mathfrak{B}$, $\models^u_{\mathfrak{B}} A$. Hence, by Proposition 4.8, for every $\mathfrak{B}$, $\vdash_{\mathtt{IL}^* \cup \mathfrak{B}} A$. With $\mathfrak{B} = \emptyset$, $\vdash_{\mathtt{IL}^*} A$. Suppose then $\Gamma = \{A_1, ..., A_n\}$ $(n > 0)$. By Proposition 4.13, for every $\mathfrak{B}$, $\models^u_{\mathfrak{B}}$

$\Gamma \Longrightarrow \models^u_{\mathfrak{B}} A$. Consider any $S = \{x_1, ..., x_n\}$ with $x_i \in \mathtt{E}_{A_i}$ $(i \leq n)$—there might be some $x_j \in \mathtt{E}_{A_j}$ such that $x_i = x_j$. As S is disjunction-free, S° is defined, and is a set of atomic rules of level 0—so $(S^\circ)^*$ is defined too. Suppose now that S is also $\bot$-free. Then, S° is an atomic base, so we have $\models^u_{S^\circ} \Gamma \Longrightarrow \models^u_{S^\circ} A$. We apply Proposition 4.8 again, and we obtain $\vdash_{\mathtt{IL}^* \cup S^\circ} \Gamma \Longrightarrow \vdash_{\mathtt{IL}^* \cup S^\circ} A$, where $\vdash_{\mathtt{IL}^* \cup S^\circ} \Gamma$ means $\vdash_{\mathtt{IL}^* \cup S^\circ} B$ for every $B \in \Gamma$. By Proposition 4.12,

$$(S^\circ)^* \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow (S^\circ)^* \vdash_{\mathtt{IL}^*} A.$$

The same implication holds trivially when S is not $\bot$-free, since in this case $\bot \in (S^\circ)^*$. Now, spelling the implication out, we have

$$\bigcup\{(x_1^\circ)^*, ..., (x_n^\circ)^*\} \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \bigcup\{(x_1^\circ)^*, ..., (x_n^\circ)^*\} \vdash_{\mathtt{IL}^*} A$$

By Proposition 4.11, we have then

$$\{x_1, ..., x_n\} \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \{x_1, ..., x_n\} \vdash_{\mathtt{IL}^*} A.$$

By arbitrariness of the choice of S, we thus have

$$\bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} \Gamma \Longrightarrow \bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} A.$$

But the antecedent of this implication holds by Proposition 4.6, so we have

$$\bigvee_{x \in \mathtt{E}_{A_1}} x, ..., \bigvee_{x \in \mathtt{E}_{A_n}} x \vdash_{\mathtt{IL}^*} A$$

which, again by Proposition 4.6, just means $\Gamma \vdash_{\mathtt{IL}^*} A$. □

5. Validity of WEM and analysis of the result

The existence of a (constructive) proof of completeness of `IL`* over nP-tV might lead one to hope that the result extends (possibly, in a constructive way) to full `IL`. However, this cannot be the case, unless we require reductions and reduction sequences to undergo more constraints than the kind of uniformity I have dealt with above. This is because of Theorem 5.1 and Corollary 5.2 below.

The proofs of these result will make crucial use of two principles. The first is Convention 2.5 above, i.e., the requirement that atomic bases never prove $\bot$ categorically, that is, that atomic bases are always consistent. The second is that I shall allow reductions (both in the LnP-tV and in the nP-tV sense) to be defined on canonical argument structures. I shall comment upon these choices in the concluding remarks.

Theorem 5.1. *$\models \neg A \vee \neg\neg A$ when* WEM *holds in the meta-logic.*

Proof: Since WEM holds in the meta-logic, for every $\mathfrak{B}$, either $\not\models_{\mathfrak{B}} A$ or (not $\not\models_{\mathfrak{B}} A$). If the first, then the one-step argument structure

$$\frac{A}{\bot}$$

is vacuously valid on $\mathfrak{B}$ when paired with $\emptyset$, and so is

$$\mathscr{D}^* = \cfrac{\cfrac{\cfrac{[A]_1}{\bot}}{\neg A}\,1}{\neg A \vee \neg\neg A}$$

If (not $\not\models_{\mathfrak{B}} A$), then $\not\models_{\mathfrak{B}} \neg A$. For, suppose $\models_{\mathfrak{B}} \neg A$. Then, by Convention 2.5, $\not\models_{\mathfrak{B}} A$, which contradicts our assumption. So, the one-step argument structure

$$\frac{\neg A}{\bot}$$

is vacuously valid on $\mathfrak{B}$ when paired with $\emptyset$, and so is

$$\mathscr{D}^{**} = \cfrac{\cfrac{\cfrac{[\neg A]_1}{\bot}}{\neg\neg A}\,1}{\neg A \vee \neg\neg A}$$

Let now

$$\texttt{WEM} = \frac{}{\neg A \vee \neg\neg A}$$

and consider the reduction system

$$\mathfrak{S} = \{\langle \texttt{WEM}, \mathscr{D}^* \rangle, \langle \mathscr{D}^*, \mathscr{D}^{**} \rangle\}.$$

With WEM in the meta-logic, for every $\mathfrak{B}$, $\langle \texttt{WEM}, \mathfrak{S} \rangle$ is valid on $\mathfrak{B}$. □

Observe that the reduction system $\mathfrak{S}$ in the proof of Theorem 5.1 is specified in a completely base-independent way. We do not need to bring in specific argument structures whose shape may change depending on specific atomic bases which `WEM` is to be evaluated over. The argument structures $\mathscr{D}^*$ and $\mathscr{D}^{**}$ remain invariant throughout all atomic bases. In fact, $\mathfrak{S}$ is even finite (for every instance of `WEM`), *contra* the potentially infinite cardinality of $\mathfrak{U}$ in the proof of Theorem 2.20.

That $\mathfrak{S}$ is uniform in the sense specified above (and is even the graph of a composite function) can be seen by observing that the proof of Theorem 5.1 can be given also for nP-tV, by associating `WEM` to a finite set of reductions—where the latter are understood now in Prawitz's sense, i.e., as functions from and to argument structures which respect the conditions stated in Section 3.

Corollary 5.2. $\models^u \neg A \vee \neg\neg A$ when WEM holds in the meta-logic.

Proof: Set

$$\phi_1(\texttt{WEM}) = \mathscr{D}^* \text{ and } \phi_2(\mathscr{D}^*) = \mathscr{D}^{**}.$$

Then, $\langle \texttt{WEM}, \{\phi_1, \phi_2\} \rangle$ is logically valid in the sense of nP-tV. □

Of course, while the definitions of $\mathfrak{S}$ in the proof of Theorem 5.1 and of $\{\phi_1, \phi_2\}$ in the proof of Corollary 5.2 are completely uniform, the computation process they give rise to for validating `WEM` on each atomic base is non-constructive. The proofs work only because we are assuming WEM in the meta-logic. Hence, one might say, the question of the completeness of `IL` over nP-tV is not settled yet, since we may still hope for a constructive proof of completeness.

But we can now make appeal to Schroeder-Heister's remark that a classical proof of incompleteness of `IL` is enough for ruling out an intuitionistic proof of completeness of `IL`, insofar as both proofs can be coded in first-order arithmetic. The only reason we had for claiming that we could not exploit the classical proof of incompleteness of `IL` on LnP-tV for applying Schroeder-Heister's remark directly to nP-tV, was that for LnP-tV we did not need classical meta-logic only, but non-uniform reductions and reductions sequences too. Now we have a meta-logic which is non-intuitionistic (and weaker than classical meta-logic), and uniform reductions and reduction sequences. So, provided we accept that the proofs of Theorem 5.1 and of Corollary 5.2 can be coded in first-order arithmetic, Schroeder-Heister's remark does apply, and we can rule out the existence of an intuitionistic proof of completeness of `IL` over nP-tV.

6. Concluding remarks

To conclude, I would like to make some remarks concerning Theorem 5.1 and Corollary 5.2. There are certain changes one can make to the nP-tV and LnP-tV pictures as provided in this paper. Two of these do not affect the validity of the proofs of Theorem 5.1 and Corollary 5.2, whereas other two turn out to be crucial. Let me start from the former:

- Prawitz's original approach in [23] is limited to atomic rules of level ≤ 1. Theorem 5.1 and Corollary 5.2 also hold in this case;
- as I defined it, a single-node atomic derivation will always be a derivation of an atom depending on no assumption-formulas, but only on an assumed rule of level 0. Therefore, atomic derivations never contain assumption-formulas, and we cannot speak of derivability of an atom from assumptions-formulas in an atomic base. It is possible, of course, to define atomic derivations so that they contain both assumption-formulas and assumed rules of level 0, or take assumption-formulas to be identical with assumed rules of level 0. Theorem 5.1 and Corollary 5.2 also hold in these cases.

Let us now discuss the changes which may affect Theorem 5.1 and Corollary 5.2:

- rather than reading $\bot$ as an atomic formula, one could take it to be a nullary connective. This issue is connected with how $\bot$ is semantically dealt with. I have assumed atomic bases to be always consistent, and this fact is crucially used in the proofs of Theorem 5.1 and of Corollary 5.2. But there are other options. One can, e.g., allow atomic bases to be inconsistent, but require them to contain all the atomic instances of *ex falso*—see, e.g., [18, 19]—or, especially when $\bot$ is seen as a nullary connective, explain $\bot$ at the semantic level by equating its validity on an atomic base with the validity of all atoms on that base—see, e.g., [27]. A thorough discussion of this alternatives, and of their connection with the assumption of consistency of atomic bases, is as said to be found in [2]. Be that as it may, a proof of incompleteness of `IL` over nP-tV which does not rely at all on how $\bot$ is understood (and so applies to minimal logic too) has been recently provided in [16]. The proof uses an instance of excluded middle (thus, a stronger meta-logic than the one used in the proofs of Theorem 5.1 and Corollary 5.2), and shows $p \rightarrow (q \vee r) \models^u (p \rightarrow q) \vee (p \rightarrow r)$ with p, q and r atoms, without however generalising to formulas obtained by replacing p, q and r with any $A, B, C \in \mathtt{FORM}_{\mathscr{L}}$. Via Schroeder-Heister's remark, this result is however enough for refuting Prawitz's conjecture. Since nP-tV is a semantics of proofs (as opposed to a semantic of formulas), there might be, more in general, reasons to stay content with a notion of logical consequence which is *not* closed under replacements. These topics can be discussed in future works;

- the reduction system $\mathfrak{S}$ used in the proof of Theorem 5.1, and the set of reductions $\mathfrak{J}$ used in the proof of Corollary 5.2, use reductions defined on $\mathscr{D}^*$. Now, one may not like the idea of applying reductions to argument structures in canonical form. If so, then one can set in the proof of Theorem 5.1

$$\mathfrak{S} = \{\langle \mathtt{WEM}, \mathscr{D}^* \rangle, \langle \mathtt{WEM}, \mathscr{D}^{**} \rangle\}$$

and in the proof of Corollary 5.2

$$\phi^1(\mathtt{WEM}) = \mathscr{D}^*,\ \phi^2(\mathtt{WEM}) = \mathscr{D}^{**}.$$

This solution would work perfectly fine in the context of LnP-tV, where alternative justifications are allowed, but may be problematic in the case of nP-tV, as here reductions might be required to be deterministic. However, a set of both uniform and deterministic reductions in the sense of nP-tV can be given in such a way as to have $\neg\neg p \rightarrow q \vee r \models^u (\neg\neg p \rightarrow q) \vee (\neg\neg p \rightarrow r)$—with Convention 2.5 and WEM in the meta-logic [9]. Once again, via Schroeder-Heister's remark, this refutes completeness of `IL` over nP-tV but, similarly to the incompleteness proof of [16] (which incidentally also uses uniform and deterministic reductions and reduction sequences), it does not hold under replacement of p, q, r with any formulas. The above remarks thus also apply here and, as said, these topics can be discussed in future works.

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Antonio Piccolomini d'Aragona
Eberhard Karls Universität Tübingen
Carl Friedrich von Weizsäcker Center
Doblerstraße 33, 72074
Tübingen, Germany

e-mail: antonio.piccolomini-daragona@uni-tuebingen.de

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