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FUNDAMENTAL RELATION ON H_vBE-ALGEBRAS

Abstract

In this paper, we are going to introduce a fundamental relation on H_vBE-algebra and investigate some of properties, also construct new $(H_v)BE$-algebras via this relation. We show that quotient of any H_vBE-algebra via a regular regulation is an H_vBE-algebra and this quotient, via any strongly relation is a BE-algebra. Furthermore, we investigate that under what conditions some relations on H_vBE-algebra are transitive relations.

Keywords: $(H_v, Hyper)BE$-algebra, fundamental relation, quotient.

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1. Introduction and preliminaries

Hyperstructures represent a natural extension of classical structures and they were introduced by French Mathematician F. Marty in 1934 [10]. A hyperstructure is a nonempty set H, together with a function $\circ : H \times H \longrightarrow P^*(H)$ called hyper operation, where $P^*(H)$ denotes the set of all nonempty subsets of H. Marty introduced hypergroups as a generalization of groups [4, 3]. Hyperstructures have many applications to several sectors of both pure and applied sciences as geometry, graphs and hypergraphs, fuzzy sets and rough sets, automata, cryptography, codes, relation algebras, C-algebras, artificial intelligence, probabilities, chemistry, physics, especially in atomic physic and in harmonic analysis [2, 7].

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H. S. Kim and Y. H. Kim introduced the notation of BE-algebra as a generalization of dual BCK algebra [9]. R. A. Borzooei et al. defined the notation of a hyper K-algebra, bounded hyper K-algebra and considered the zero condition in hyper K-algebras. They showed that every hyper K-algebra with the zero condition can be extended to a bounded hyper K-algebra [1].

A. Radfar et al. combined BE-algebra with hyperstructures and defined the notation of hyper BE-algebra. Also, they focused on some types of hyper BE-algebras and show that every dual hyper K-algebra is a hyper BE-algebra [11].

We know that the class of the H_v-structures, introduced by Vougiouklis in 1990 [13, 14], is the largest class of hyperstructures. In the classical hyperstructures, in any axiom where the equality is used, if we replace the equality by the nonempty intersection, then we obtain a corresponding H_v structures.

F. Iranmanesh et al. present the notation of the H_vBE-algebra as generalization of hyper BE-algebra [8]. They defined new H_v-structures and considered some of their useful properties. Also discuss H_v-filters and homomorphism on this structure. Furthermore, they got more results in H_vBE-algebras [8]. Fundamental relations are one of the main tools in algebraic hyperstructures theory.

Algebraic hyperstructures are extension of algebraic structures and for better understanding their properties we want some connections between algebraic hyperstructures and algebraic structures, a fundamental relation is an interesting concept in algebraic hyperstructures that makes this connection. In this paper, for obtain a relationship between BE, hyper BE and $(H_v)BE$-algebra, we define a fundamental relation on $(H_v)BE$-algebra that is called "δ" also, we study "δ^*" as a transitive closure of "δ" in such away that is the smallest equivalence relation that contains "δ". Finally, a BE-algebras which is quotient of H_vBE-algebra via "δ^*" is obtained, therefore we find a connection between algebraic structures and (H_v)hyper algebraic structures.

Definition 1.1 ([9]). Let X be a nonempty set, "$*$" be a binary operation on X and a constant $0 \in X$. Then $(X, *, 0)$ is called a BCK-algebra if for all $x, y, z \in X$ it satisfies the following conditions:

$(BCI\text{-}1)$ $((x*y)*(x*z))*(z*y)=0$,

$(BCI\text{-}2)$ $(x*(x*y))*y=0$,

$(BCI\text{-}3)$ $x*x=0$,

$(BCI\text{-}4)$ $x*y=0$ and $y*x=0$, imply $x=y$,

$(BCI\text{-}5)$ $0*x=0$.

We define a binary relation "$\leq$" on X by $x \leq y$ if and only if $x*y=0$.

DEFINITION 1.2 ([9]). Let X be a nonempty set, "$*$" be a binary operation on X and $1 \in X$. An algebra $(X,*,1)$ of type $(2,0)$ is called a BE-algebra if the following axioms hold:

$(BE1)$ $x*x=1$,

$(BE2)$ $x*1=1$,

$(BE3)$ $1*x=x$,

$(BE4)$ $x*(y*z)=y*(x*z)$, for all $x,y,z \in X$.

We introduce the relation "$\leq$" on X by $x \leq y$ if and only if $x*y=1$.
The BE-algebra $(X,*,1)$ is said to be commutative, if for all $x,y \in X$, $(x*y)*y=(y*x)*x$.

PROPOSITION 1.3 ([9]). Let X be a BE-algebra. Then

(i) $x*(y*x)=1$.

(ii) $y*((y*x)*x)=1$, for all $x,y \in X$.

Example 1.4 ([12]). Let $X=\{1,2,...\}$. Define the operation "$*$" as follows:

$$x*y=\begin{cases}1 & \text{if } y \leq x\\ y & \text{otherwise.}\end{cases}$$

then $(X,*,1)$ is a BE-algebra.

DEFINITION 1.5 ([4]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ)$ is called an H_v-group if it satisfies the following axioms:

$(H1)$ $x \circ (y \circ z) \bigcap (x \circ y) \circ z \neq \phi$,

$(H2)$ $a \circ H = H \circ a = H$, for all $x, y, z, a \in H$,

where $a \circ H = \bigcup\limits_{h \in H} a \circ h$, $H \circ a = \bigcup\limits_{h \in H} h \circ a$.

DEFINITION 1.6 ([11]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 0)$ is called a hyper K-algebra if satisfies the following axioms:

$(HK1)$ $(x \circ z) \circ (y \circ z) < x \circ y$,

$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$,

$(HK3)$ $x < x$,

$(HK4)$ $x < y$ and $y < x$ implies $x = y$,

$(HK5)$ $0 < x$, for all $x, y, z \in H$,

where the relation " $<$ " is defined by $x < y \iff 0 \in x \circ y$. For every $A, B \subseteq H$, $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$. Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H.

DEFINITION 1.7 ([11]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called a hyper $BE-$algebra if satisfies the following axioms:

$(HBE1)$ $x < 1$ and $x < x$,

$(HBE2)$ $x \circ (y \circ z) = y \circ (x \circ z)$,

$(HBE3)$ $x \in 1 \circ x$,

$(HBE4)$ $1 < x$ implies $x = 1$, for all $x, y, z \in H$.

$(H, \circ, 1)$ is called a dual hyper K-algebra if it satisfies $(HBE1), (HBE2)$ and the following axioms:

$(DHK1)$ $x \circ y < (y \circ z) \circ (x \circ z)$,

$(DHK4)$ $x < y$ and $y < x$ imply that $x = y$,

where the relation " $<$ " is defined by $x < y \iff 1 \in x \circ y$.

Definition 1.8 ([8]). Let H be a nonempty set and $\circ : H \times H \longrightarrow P^*(H)$ be a hyperoperation. Then $(H, \circ, 1)$ is called an H_vBE-algebra if satisfies the following axioms:

(H_vBE1) $x < 1$ *and* $x < x$,

(H_vBE2) $x \circ (y \circ z) \bigcap y \circ (x \circ z) \neq \phi$,

(H_vBE3) (H_vBE3) $x \in 1 \circ x$,

(H_vBE4) $1 < x$ implies $x = 1$, for all $x, y, z \in H$,

where the relation "$<$" is defined by $x < y \iff 1 \in x \circ y$.

Also $A < B$ if and only if there exist $a \in A$ and $b \in B$ such that $a < b$.

Proposition 1.9 ([6]). Every dual hyper K-algebra is a hyper BE-algebra.

2. On H_vBE-algebras and some results

In this section, we consider H_vBE-structure with some results on its.

Example 2.1.

(i) Let $(H, *, 1)$ be a BE-algebra and we know that $\circ : H \times H \longrightarrow P^*(H)$ with $x \circ y = \{x * y\}$ is a hyperoperation. Then $(H, \circ, 1)$ is a trivial hyper BE-algebra and an H_vBE-algebra.

(ii) Let $H = \{1,a,b\}$. Define hyperoperation " $\circ$ " as follows:

$\circ$	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a}	{1,b}
b	{1}	{1,a,b}	{1}.

Then $(H, \circ, 1)$ is an H_vBE-algebra.

(iii) Define the hyperoperation " $\circ$ " on $\mathbb{R}$ as follows:

$$x \circ y = \begin{cases} \{y\} & \text{if } x = 1 \\ \mathbb{R} & \text{otherwise,} \end{cases}$$

then $(\mathbb{R}, \circ, 1)$ is an H_vBE-algebra.

PROPOSITION 2.2 ([8]). Any hyper BE-algebra is an H_vBE-algebra.

Example 2.3 shows that the converse of Proposition 2.2 does not hold in general.

Example 2.3. Define a hyperoperation " $\circ$ " on the set H = {1,a,b} as follows:

$\circ$	1	a	b
1	{1}	{a}	{b}
a	$\{1, b\}$	{1}	{1,a,b}
b	{1}	{1,b}	{1,b}.

Then $(H, \circ, 1)$ is an H_vBE-algebra. And we have that:

$$a \circ (b \circ b) = a \circ (\{1, b\}) = \{1, a, b\} \neq \{1, b\} = b \circ (\{1, a, b\}) = b \circ (a \circ b).$$

So $(H, \circ, 1)$ does not satisfy $(HBE2)$, therefore $(H, \circ, 1)$ is not a hyper BE-algebra.

THEOREM 2.4. *Let* $(H, \circ, 1)$ *be an* H_vBE*-algebra. Then*

(i) $A \circ (B \circ C) \bigcap B \circ (A \circ C) \neq \phi$ *for every* $A, B, C \in P^*(H)$,

(ii) $A < A$,

(iii) $1 < A$ *implies* $1 \in A$,

(iv) $1 \in x \circ (x \circ x)$,

(v) $x < x \circ x$.

PROOF: (i) Let $a \in A, b \in B$ and $c \in C$. We have $a \circ (b \circ c) \subseteq A \circ (B \circ C)$, $b \circ (a \circ c) \subseteq B \circ (A \circ C)$, Then by (H_vBE2), $a \circ (b \circ c) \bigcap b \circ (a \circ c) \neq \phi$, therefore $A \circ (B \circ C) \bigcap B \circ (A \circ C) \neq \phi$.

Other cases are similar. □

DEFINITION 2.5 ([6]). Let F be a nonempty subset of H_vBE-algebra H and $1 \in F$. Then F is called

(i) a weak H_v-filter of H if $x \circ y \subseteq F$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

(ii) an H_v-filter of H if $x \circ y \approx F(i.e\ \phi \neq (x \circ y) \bigcap F)$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

Example 2.6. Let $H = \{$1,a,b$\}$. Define the hyperoperation "$\circ_1$" and "$\circ_2$" as follows:

$\circ_1$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

$\circ_2$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b\}$.

We see that $(H, \circ_1, 1)$ is an H_vBE-algebra and $F_1 = \{1, a\}$ is a weak H_v-filter of H. Also $(H, \circ_2, 1)$ is an H_vBE-algebra and $F_2 = \{1, a\}$ is an H_v-filter of H.

In Example 2.6, F_1 is not an H_v-filter, because $a \circ_1 b \approx F_1$ and $a \in F_1$, but $b \notin F_1$.

THEOREM 2.7. *Every H_v-filter is a weak H_v-filter.*

Notation. By Example 2.6, we can see that the notion of weak H_v-filter and H_v-filter are different in H_vBE-algebra.

THEOREM 2.8 ([8]). *Let F be a subset of an H_vBE-algebra H and $1 \in F$. For all $x, y \in H$, if $x \circ y < F$ and $x \in F$ implies $y \in F$, then F=H.*

3. Relations on H_vBE-algebras

In this section, let $(H, \circ, 1)$ be a H_vBE-algebra and presents in summary with H. We show that there exists a connection between hyper algebraic structures and algebraic structures by strongly regular relations.

Definition 3.1. Let $(H, \circ, 1)$ be an H_vBE-algebra and R be an equivalence relation on H. If A, B are nonempty subsets of H, then

(i) $A\ \bar{R}\ B$ means that, for all $a \in A$, there exists $b \in B$ in such away that aRb and for all $b' \in B$, there exists $a' \in A$ in such away that $b'Ra'$,

(ii) $A\ \bar{\bar{R}}\ B$ means that, for all $a \in A$ and $b \in B$, we have aRb,

(iii) R is called right regular(left regular), if for all $x \in H$, from aRb, it follows that $(a \circ x)\bar{R}(b \circ x)((x \circ a)\bar{R}(x \circ b))$.

(iv) R is called strongly right regular(strongly left regular), if for all $x \in H$, from aRb, it follows that $(a \circ x)\bar{\bar{R}}(b \circ x)((x \circ a)\bar{\bar{R}}(x \circ b))$.

(v) R is called (strongly) regular, if it is (strongly) right regular and (strongly) left regular,

(vi) R is called good, if $(a \circ b)\ R\ 1$ and $(b \circ a)\ R\ 1$ imply aRb, for all $a, b \in H$.

It is clear that $(a \circ b)\ R\ 1$ means that there exists $x \in a \circ b$ in such a way that $xR1$.

Example 3.2. Let $H = \{1, a, b\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a, b\}$	$\{b\}$
b	$\{1, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that

$$R = \{(1,1), (a,a), (b,b), (a,b), (b,a), (1,b), (b,1), (a,1), (1,a)\}$$

is a good strongly regular relation on H and for any $A, B \in P^*(H)$, $A\ \bar{\bar{R}}\ B$.

Example 3.3. Let $H = \{1, d, b, c\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	b	c	d
1	$\{1\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
c	$\{1\}$	$\{b\}$	$\{1\}$	$\{d\}$
d	$\{1\}$	$\{b\}$	$\{1, c\}$	$\{1\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that

$$R = \{(1, 1), (d, d), (b, b), (c, c), (c, b), (b, c), (d, c), (c, d)\}$$

is not regular and strongly regular relation on H.

Notation. Let R be regular relation on H. We denote the set of all equivalence classes of R by H/R. Hence $H/R = \{\bar{x} : x \in H\}$. For any $\bar{x}, \bar{y} \in H/R$, define a hyperoperation "$*$" on H/R by

$$\bar{x} * \bar{y} = \{\bar{z} : z \in x \circ y\}$$

and a binary relation " $<$ " on H/R by

$$\text{"}\bar{x} < \bar{y}\text{"} \Longleftrightarrow \bar{1} \in \bar{x} * \bar{y}.$$

LEMMA 3.4. *Let R be a regular relation on H. Then $(H/R; *)$, is a hypergroupoid.*

PROOF: We must show that $*$ be well defined. Let $\bar{x_1}, \bar{x_2}, \bar{y_1}, \bar{y_2} \in H/R$ such that $\bar{x_1} = \bar{x_2}$, $\bar{y_1} = \bar{y_2}$. Then $x_1\ R\ x_2$ and $y_1\ R\ y_2$. Since R is a regular relation, we have $(x_1 \circ y_1)\bar{R}(x_2 \circ y_2)$ [5]. Let $\bar{r} \in \bar{x_1} * \bar{y_1}$. Then there exists $z \in x_1 \circ y_1$ in such a way that $\bar{r} = \bar{z}$. Now $z \in x_1 \circ y_1$ and $(x_1 \circ y_1)\bar{R}(x_2 \circ y_2)$, then there exists $u \in (x_2 \circ y_2)$ such that zRu then $\bar{z} = \bar{u}$ and $\bar{r} = \bar{u}$, thus $\bar{x_1} * \bar{y_1} \subseteq \bar{x_2} * \bar{y_2}$ and in a similar way we get $\bar{x_2} * \bar{y_2} \subseteq \bar{x_1} * \bar{y_1}$, i.e $\bar{x_1} * \bar{y_1} = \bar{x_2} * \bar{y_2}$ therefore $*$ is well defined and $(H/R; *)$ is a hypergroupoid. □

THEOREM 3.5. *If R is a regular relation on H then $(H/R; *; \bar{1})$ is a H_vBE-algebra.*

PROOF: Let R be a regular relation on H. If $x \in H$ then $\bar{x} \circ \bar{1} = \{\bar{t} : t \in x \circ 1\}$. Since H is an H_vBE- algebra by (H_vBE1) we conclude that $1 \in x \circ 1$ and so $\bar{1} \in \bar{x} * \bar{1}$. Therefore $\bar{x} < \bar{1}$. Also $1 \in x \circ x$ and $\bar{x} \circ \bar{x} = \{\bar{t} : t \in x \circ x\}$, then $\bar{1} \in \bar{x} * \bar{x}$ and $\bar{x} < \bar{x}$.

(H_vBE2) Let $x, y, z \in H$. Since $(H, \circ, 1)$ is an H_vBE- algebra, then $x \circ (y \circ z) \bigcap y \circ (x \circ z) \neq \phi$. If $t \in x \circ (y \circ z) \bigcap y \circ (x \circ z)$, then there exists $s_1 \in y \circ z$ in such away that $t \in x \circ s_1$ by a similar way there exists $s_2 \in x \circ z$ in such away that $t \in y \circ s_2$. We get the $\bar{t} \in \bar{x} * \bar{s}_1 \subseteq \bar{x} * (\bar{y} * \bar{z})$ and $\bar{t} \in \bar{y} * \bar{s}_2 \subseteq \bar{y} * (\bar{x} * \bar{z})$. Therefore $\bar{x} * (\bar{y} * \bar{z}) \bigcap \bar{y} * (\bar{x} * \bar{z}) \neq \phi$.

(H_vBE3) if $x \in H$ then $\bar{1} \circ \bar{x} = \{\bar{t} : t \in 1 \circ x\}$. Since H is a H_vBE-algebra, we have $x \in 1 \circ x$ and $\bar{x} \in \bar{1} * \bar{x}$.

(H_vBE4) $x \in H$ and $\bar{1} < \bar{x}$ then $\bar{1} \in \bar{1} * \bar{x}$. Hence $1 \in 1 \circ x$ and $1 < x$. Since H is a H_vBE- algebra, we have $x = 1$ and so $\bar{x} = \bar{1}$. □

COROLLARY 3.6. Let $(H, \circ, 1)$ be a dual hyper K-algebra and R be an equivalence relation on H. If R is a regular relation on H, then $(H/R; *; \bar{1})$ is an H_vBE-algebra.

THEOREM 3.7. *If R is strongly regular relation on H, then $(H/R; *; \bar{1})$ is a BE-algebra.*

PROOF: If $\bar{z}_1, \bar{z}_2 \in \bar{x} * \bar{y}$, for any $\bar{x}, \bar{y} \in H/R$, then $z_1, z_2 \in x \circ y$. Since R is strongly regular, for all $x, y \in H$, yRy then $(x \circ y)\bar{\bar{R}}(x \circ y)$ and $z_1, z_2 \in x \circ y$, we have $z_1 \ R \ z_2$, therefore $\bar{z}_1 = \bar{z}_2$ and $|\bar{x} * \bar{y}| = 1$ and so by Theorem 3.5, $(H/R; *; \bar{1})$ is a BE-algebra. □

Example 3.8. Let $H = \{1, a, b, c, d, e\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b	c	d	e
1	$\{1, c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
a	$\{1, c\}$	$\{1, c\}$	$\{a\}$	$\{1, c\}$	$\{c\}$	$\{d\}$
b	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{c\}$	$\{c\}$
c	$\{1, c\}$	$\{a\}$	$\{b\}$	$\{1, c\}$	$\{a\}$	$\{b\}$
d	$\{1, c\}$	$\{1, c\}$	$\{a\}$	$\{1, c\}$	$\{1, c\}$	$\{a\}$
e	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$	$\{1, c\}$

Then $(H, \circ, 1)$ is an H_vBE-algebra. It is easy to see that R=$\{(1, 1)$, (a, a), (b, b), (c, c), (d, d), (e, e), $(1, c)$, $(c, 1)$, (e, b), $(b.e)$, (a, d), $(d, a)\}$ is a good strongly regular relation on H and

$$H/R = \{\{1, c\}, \{a, d\}, \{e, b\}\} = \{R(1), R(a), R(b)\}.$$

Now we have:

$*$	$R(1)$	$R(a)$	$R(b)$
$R(1)$	$R(1)$	$R(a)$	$R(b)$
$R(a)$	$R(1)$	$R(1)$	$R(a)$
$R(b)$	$R(1)$	$R(1)$	$R(1)$

Clearly, $(H/R; *; R(1))$ is a BE-algebra.

In this place, we present some results and examples about dual hyper K-algebras and hyper BE-algebras that are useful.

LEMMA 3.9 ([6]). *Let $(X;\circ,1)$ be a dual hyper K-algebra and R be a regular relation on X. Then for any $\bar{x},\bar{y},\bar{z}\in X/R$, $(\bar{x}*\bar{y})<(\bar{y}*\bar{z})*(\bar{x}*\bar{z})$.*

THEOREM 3.10 ([6]). *Let $(X,\circ,1)$ be a dual hyper K-algebra and R be a regular relation on X. If R is a good relation, then $(X/R;*,\bar{1})$ is a dual hyper K-algebra.*

THEOREM 3.11 ([6]). *Let $(X,\circ,1)$ be a dual hyper K-algebra and R be a strongly regular relation on X. If R is a good relation, then $(X/R;*,\bar{1})$ is a dual BCK-algebra.*

Example 3.12. Let $X=\{1,a,b,c,d,e\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b	c	d	e
1	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
a	$\{1,e\}$	$\{1,e\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$
b	$\{1,e\}$	$\{a\}$	$\{1,e\}$	$\{c\}$	$\{d\}$	$\{e\}$
c	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{1,e\}$	$\{d\}$	$\{e\}$
d	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{1,e\}$	$\{e\}$
e	$\{1,e\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{1,e\}$

Then $(X,\circ,1)$ is a dual hyper K-algebra (H_vBE-algebra). It is easy to see that R={(1,1),(a,a),(b,b),(c,c),(d,d),(e,e),(1,c),(c,1),(e,c),(c,e)} is a good strongly regular relation on X and

$$X/R=\{\{1,e\},\{a\},\{b\},\{c\},\{d\}\}=\{R(1),R(a),R(b),R(c),R(d)\}.$$

Now we have:

$*$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(d)$
$R(1)$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(d)$
$R(a)$	$R(1)$	$R(1)$	$R(b)$	$R(c)$	$R(d)$
$R(b)$	$R(1)$	$R(a)$	$R(1)$	$R(c)$	$R(d)$
$R(c)$	$R(1)$	$R(a)$	$R(b)$	$R(1)$	$R(d)$
$R(d)$	$R(1)$	$R(a)$	$R(b)$	$R(c)$	$R(1)$

Clearly $(X/R;*,R(1))$ is a dual BCK-algebra.

4. δ- relation on H_vBE-algebra

Let $(H;\circ,1)$ be a H_vBE-algebra and A be a subset of H. The set of all finite combinations of A with hyperoperation $\circ$ and $\bigodot_{i=1}^{n} a_i = a_1 \circ a_2 \circ ... a_n$, is denoted by L(A) [5].

Definition 4.1. Let $(H;\circ,1)$ be a H_vBE-algebra. Consider:

$$\delta_1 = \{(x,x) : x \in H\}$$

and for every natural number $n \geq 1$, δ_n is defined as follows:

$x\delta_n y \Longleftrightarrow \exists(a_1, a_2, ..., a_n) \in H^n, \exists u \in L(a_1, a_2, ..., a_n)$ such that $\{x, y\} \subseteq u$.

Obviously for every $n \geq 1$ the relations $\delta\ _n$ are symmetric and no reflexive and transitive, but the relation $\delta = \bigcup_{n\geq 1} \delta_n$ is a reflexive and symmetric relation. Let δ^* be *transitive closure* of δ (the smallest transitive relation such that contains δ).

In the following theorem we show that δ^* is a strongly regular relation.S

Example 4.2. Let $H = \{1, a, b\}$. Define the hyperoperation "$\circ$" as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a,b\}$	$\{b\}$
a	$\{1\}$	$\{1,a,b\}$	$\{b\}$
b	$\{1,b\}$	$\{1,a,b\}$	$\{1,a,b\}$

Then $(H,\circ,1)$ is an H_vBE-algebra. $\delta\ _1 = \{(x,x) : x \in H\} = \{(1,1),(a,a),(b,b)\}$.

Since $\{1,a\},\{1,b\},\{a,b\} \subseteq b\circ a$ then $1\delta_2$a, $1\delta_2$b, $a\delta_2$b. Also, we know that $\{1,a\} \subseteq (1\circ a)\circ b = \bigcup_{x\in 1\circ a} (x\circ b)$ therefore $1\delta_3$a.

Similarly, $1\delta_3$b, $a\delta_3$b. Obviously, $1\delta_n$a, $1\delta_n$b and $a\delta_n$b, since $\delta = \bigcup_{n\geq 1} \delta_n$, then $1\delta a$, $1\delta b$ and $a\delta b$.

THEOREM 4.3. *Let $(H, \circ, 1)$ be a $H_v BE$-algebra. Then δ^* is a strongly regular relation on H.*

PROOF: Let $x, y \in H$ and $x\ \delta^*\ y$. Then we show that for any $s \in H$:

$$(x \circ s)\overline{\overline{\delta^*}}(y \circ s).$$

Since $\delta = \bigcup\limits_{n \geq 1} \delta\ _n$ and δ^* is the smallest transitive relation such that contains δ, then there exist $a_0, a_1, ..., a_n \in H$ such that $a_0 = x, a_n = y$ and there exist $q_1, q_2, ..., q_n \in \mathbb{N}$ such that

$$x = a_0 \delta_{q_1} a_1 \delta_{q_2} a_2 \ldots a_{n-1} \delta_{q_n} a_n = y,$$

where $n \in \mathbb{N}$. Since for any $1 \leq i \leq n$, $a_{i-1}\ \delta_{q_i}\ a_n$, then there exists $z_t^j \in H$ such that

$$\{a_i, a_{i+1}\} \subseteq z_1^{i+1} \circ z_2^{i+1} \circ \ldots \circ z_{q_{i+1}}^{i+1},$$

where for $1 \leq m \leq n-1$, we have $1 \leq t \leq q_m$, and $1 \leq j \leq n-1$. Now, since $s \in H$, then for all $0 \leq i \leq n-1$,

$$a_i \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s.$$

In a similar way, we get that

$$a_{i+1} \circ s \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s.$$

Then for all $1 \leq i \leq n$, and for all $u \in a_i \circ s, v \in a_{i+1} \circ s$, We have $\{u, v\} \subseteq z_1^{i+1} \circ z_2^{i+1} \circ ... \circ z_{q_{i+1}}^{i+1} \circ s$. Therefore $u\ \delta_{q_{i+1}}\ v$, and so for all $z \in a_0 \circ s = x \circ s, w \in a_n \circ s = y \circ s$, We have $z\ \delta^*\ w$. Then δ^* is a strongly right regular and similarly is a strongly left regular relation, therefore δ^* is a strongly regular relation on H. □

COROLLARY 4.4. Let $(H, \circ, 1)$ be a hyper BE-algebra. Then δ^* is a strongly regular relation on H.

THEOREM 4.5. *Let $(H, \circ, 1)$ be a $H_v BE$-algebra. $(H/\delta^*; *, \bar{1})$ is a BE algebra.*

PROOF: By Theorem 3.7 and 4.3, the proof is obvious. □

Example 4.6. Let $H = \{1, x, y, z, t\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z	t
1	$\{1,t\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{t\}$
x	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$
y	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$
z	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$
t	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$	$\{1,t\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. We have $(x \circ y) \circ x = \{1, x, t\}$, $(x \circ y) \circ y = \{1, y, t\}$, $(x \circ y) \circ t = \{1, t\}$, $(x \circ y) \circ z = \{1, z, t\}$. Then for any $u \in H$, $1\ \delta^* u$ and so $\delta^*\ (1) = \{u \in H : 1\ \delta^*\ \text{u}\} = H = \delta^*(u)$. Therefore H/δ^*=$\{\delta^*(1)\}$ and we see that $(H/\delta^*; *, \delta^*(1))$ is a trivial BE-algebra.

Example 4.7. Let $H = \{1, x, y, z\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z
1	$\{1\}$	$\{x\}$	$\{y\}$	$\{z\}$
x	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
y	$\{1\}$	$\{x\}$	$\{1\}$	$\{z\}$
z	$\{1\}$	$\{x\}$	$\{1, y\}$	$\{1\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. We conclude that $H/\delta^* = \{\{1, y\}, \{x\}, \{z\}\} = \{\delta^*(1), \delta^*(x), \delta^*(z)\}$ and then:

$*$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(z)$
$\delta^*(1)$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(z)$
$\delta^*(x)$	$\delta^*(1)$	$\delta^*(1)$	$\delta^*(1)$
$\delta^*(z)$	$\delta^*(1)$	$\delta^*(x)$	$\delta^*(x)$

Now, by Theorem 4.3, $(H/\delta^*; *, \delta(1))$ is a BE-algebra.

Notation. We know that δ is reflexive and symmetric but is not transitive on H. If R is an equivalence relation on H, then H/R is defined and we have the following theorem;

THEOREM 4.8 ([6]). *Let $(H, \circ, 1)$ be a hyper BE-algebra and R be an equivalence relation on H. Then, R is a regular relation on H if and only if $(H/R; *, \bar{1})$ is a hyper BE algebra.*

DEFINITION 4.9. Let M be a nonempty subset of H. M is called δ-part if for any $n \in \mathbb{N}$, $a_i \in H$, and $L(a_1, a_2, \ldots, a_n) \cap M \neq \emptyset$, then $L(a_1, a_2, \ldots, a_n) \subseteq M$.

Example 4.10. Let $H = \{1, x, y, z\}$. Define hyperoperation "$\circ$" as follows:

$\circ$	1	x	y	z
1	$\{1, x\}$	$\{1, x\}$	$\{y\}$	$\{z\}$
x	$\{1, x\}$	$\{1, x\}$	$\{y\}$	$\{z\}$
y	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$	$\{z\}$
z	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$	$\{1, x\}$

Then $(H, \circ, 1)$ is a H_vBE-algebra. It is easy to verify that for any $M \subseteq H$ that $M \neq \{1\}$ and $M \neq \{a\}$, M is a δ-part.

COROLLARY 4.11. Let $(H, \circ, 1)$ be a H_vBE-algebra and M, N are δ-part of H. Then $M \cap N$ is a δ-part of H.

PROOF: For any $n \in \mathbb{N}$, $a_i \in H$, if $L(a_1, a_2, ..., a_n) \cap (M \cap N) \neq \emptyset$, then $L(a_1, a_2, ..., a_n) \cap M \neq \emptyset, L(a_1, a_2, ..., a_n) \cap N \neq \emptyset$. Since M, N are δ-part, we have $L(a_1, a_2, ..., a_n) \subseteq M, L(a_1, a_2, ..., a_n) \subseteq N$ and then $L(a_1, a_2, ..., a_n) \subseteq M \cap N$. Therefore $M \cap N$ is a δ-part of H. $\square$

LEMMA 4.12 ([6]). *Let M be a non-empty subset of a dual hyper K-algebra H. Then the following conditions are equivalent:*

(i) *M is a δ-part of H,*

(ii) *$x \in M$, $x\ \delta\ y$ imply $y \in M$,*

(iii) *$x \in M$, $x\ \delta^*\ y$ imply $y \in M$.*

THEOREM 4.13. *Let $(H, \circ, 1)$ be a H_vBE-algebra. If H be a dual hyper K-algebra and for any $x \in H$, $\delta^*(x)$ is a δ-part, then δ is transitive relation.*

PROOF: Let $x\ \delta\ y$ and $y\ \delta\ z$. Then there exist $m, n \in \mathbb{N}$, $a_i, b_j \in H$ such that $\{x, y\} \subseteq (\bigodot_{i=1}^{n} a_i)$ and $\{y, z\} \subseteq (\bigodot_{j=1}^{m} b_j)$. Now, $\delta^*(x)$ is a δ-part, $x \in \delta^*(x) \cap (\bigodot_{i=1}^{n} a_i)$ and $y \in (\bigodot_{i=1}^{n} a_i) \cap (\bigodot_{j=1}^{m} b_j)$. Since $\delta^*(x)$ is a δ-part, then $(\bigodot_{i=1}^{n} a_i) \subseteq \delta^*(x)$ therefore $y \in \delta^*(x) \cap (\bigodot_{j=1}^{m} b_j)$. Since $\delta^*(x)$ is a δ-part, then $(\bigodot_{j=1}^{m} b_j) \subseteq \delta^*(x)$ therefore $z \in \delta^*(x)$. But $z \in \delta(z)$ by above $z\ \delta^*\ x$, set $M = \delta(z)$ and know that $\delta^*(x) = \delta(z)$ then by Lemma 4.12 , $x\ \delta\ z$, therefore δ is transitive relation. $\square$

Open problem: Under what conditions converse of above theorem is true?

5. Conclusion

In the present paper, we have introduced new H_vBE-algebras and BE-algebras based on equivalence relations.

This work focused on fundamental relations on H_vBE-algebras and we investigated some of their properties. The relations δ^* and δ are constructed and studied, they are one of the most main tools for better understanding the algebraic hyperstructures. In future, we try to find an answer to above open problem.

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