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Linear combination of projections in von Neumann factors

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ABSTRACT

It is shown that any self-adjoint operator in a finite discrete or infinite von Neumann factor can be written as a real linear combination of 4 projections. On the other hand, in any type II_1 algebra and in any type II_∞ factor there exists a self-adjoint operator that is not a linear combination of 3 projections.

1. Introduction

We prove here four results, two positive and two negative:

(1) Any self-adjoint operator A acting on a finite-dimensional (complex) Hilbert space can be written as a linear combination of 4 projections, with two of the coefficients chosen arbitrarily from the interval $[2\|A\|, \infty[$.
(2) Any self-adjoint operator A in an infinite von Neumann factor can be written as a linear combination of 4 projections, with two of the coefficients chosen arbitrarily from the interval $]2\|A\|, \infty[$.
(3) There is a self-adjoint operator in any type II_1 algebra that cannot be written as a linear combination of 3 projections.
(4) There is a self-adjoint operator in any type II_∞ factor that cannot be written as a linear combination of 3 projections.

Any self-adjoint operator acting in a (complex) finite-dimensional Hilbert space can be written as a linear combination of a finite number of projections — this follows easily from the spectral theorem. What is not immediately obvious is whether we can make the number of projections independent of the dimension of the Hilbert space. Is this also possible if the Hilbert space is infinite dimensional? The first positive results

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in this direction were obtained by Fillmore 1967 (see [3]), who was able to get down to 9 projections [4]. Pearcy and Topping [16] reduced it to 8 already in 1967, then the second-named author to 6 in 1980 (see [15]). Using the ideas from [15], Matsumoto showed in 1984 that 5 is enough. Also in 1984 Nakamura [12] proved that any Hermitian matrix is a linear combination of 4 projections.

In 1985 the second-named author proved that any self-adjoint operator in $\mathbb{B}(\mathbb{H})$ can be written as a linear combination of 4 projections, both for finite and infinite-dimensional Hilbert space $\mathbb{H}$, with a proof for Hermitian matrices differing from that of Nakamura [12]. The paper [14] with the results was sent to a journal and got a positive review, requiring only minor improvements, but for some inexplicable reason it has not been resent to the journal. This fact went unnoticed by the first-named author who was compiling the joint publication on linear combinations of projections in von Neumann algebras [5] while on leave from his *alma mater*, and the paper was advertised as accepted for publication. Even worse, some results from the joint publication depend on the unpublished paper, including the criterion for a von Neumann algebra to be a complex linear span of its projections.

In 2016, the first-named author met Viacheslav Rabanovich at the conference "Groups and Operators" in Gothenburg, Sweden. There he learned that [14] has never been published, and that Rabanovich [17] himself proved that 4 projections are sufficient in the case of $\mathbb{B}(\mathbb{H})$ with a separable infinite-dimensional Hilbert space $\mathbb{H}$.

We decided that there are good reasons for complementing our earlier paper [5] with this one, dealing with linear combinations of projections in von Neumann factors. Both Nakamura's [12] and Rabanovich's [17] proofs are elegant, and they show clearly that 4 projections are enough for self-adjoint operators in $\mathbb{H}$ both in finite- and infinite-dimensional cases. However, there are several reasons for our presentation:

(1) Theorem 1.1 (a) of [5] attributed to [14], and hence did not prove, a decomposition of self-adjoint operators in a type I_n factor as linear combinations of 4 projections, with a specific form of the coefficients. That form is not available from Nakamura's [12] but is required to obtain in Theorem 2.1 (a) of [5] the decomposition of self-adjoint operators in a finite discrete von Neumann algebra as a linear combination of 4 projections with coefficients in the center of the algebra. A proof based on the unpublished paper [14] is presented in Theorem 2.1 below.
(2) Theorem 1.1 (a) of [5] attributed to [14] a decomposition of self-adjoint operators in an infinite von Neumann factor as a linear combination of 4 projections. The proof presented by Rabanovich [17] holds for a separable Hilbert space and cannot be easily adapted to the case of an arbitrary infinite factor. An independent proof based on [14] is presented in Theorem 3.5.
(3) Given Theorem 2.1 and Theorem 3.5, all the results from [5] are now proved. The results have already been used by several authors throughout the years.

The interest in this field of research continues (see, for example, [7]), with more attention directed recently to a situation in C^*-algebras (see [10]). Nevertheless, as seen below, there are some open questions even in the von Neumann algebra case (see 'Open problems' at the end of the paper).

For a von Neumann algebra $\mathcal{A}$ we denote by $\mathcal{A}_h$ its self-adjoint part and by $\operatorname{Proj}\mathcal{A}$ the lattice of projections in $\mathcal{A}$. For a projection E, we denote by $\mathcal{A}_E$ the reduced von Neumann algebra $E\mathcal{A}E$ acting in $E\mathbb{H}$. For $A \in \mathcal{A}_h, e_A(\cdot)$ denotes the spectral measure of A. The unit of $\mathcal{A}$ is denoted by $1_\mathcal{A}$ or simply 1, if there is no danger of confusion. For projections $P, Q \in \mathcal{A}$, we write $P \sim Q$ if there is a partial isometry $U \in \mathcal{A}$ such that $P = U^*U$ and $Q = UU^*$, and then $P \precsim Q$ if $P \sim R \leq Q$ for a projection $R \in \mathcal{A}$, and $P \prec Q$ if $P \precsim Q$ and $P \nsim Q$.

2. Finite discrete factors

Let $\mathcal{A}$ be factor of type $I_n, n < \infty$, and τ the tracial state on $\mathcal{A}$. We assume (for convenience) that $\mathcal{A}$ is represented on a Hilbert space $\mathbb{H}$ in such a way that $\mathcal{A} = \mathbb{B}(\mathbb{H})$.

Theorem 2.1. *[1.1(a) in [5]] Let $A \in \mathcal{A}_h$ and $\alpha = \tau(A)$. For any $\beta, \gamma \geq 2\|A\|$, there are projections P, Q, R, S in $\mathcal{A}$ with $P \sim Q$ and $R \sim S$ such that*

$$A = (\beta + \alpha)P - \beta Q + (\gamma + \alpha)R - \gamma S \tag{1}$$

when $\alpha \geq 0$, and

$$A = \beta P - (\beta - \alpha)Q + \gamma R - (\gamma - \alpha)S \tag{2}$$

when $\alpha \leq 0$.

Proof. (2) follows from (1) applied to $-A$. We show (1). If $A = \alpha 1$, take $P = Q = 1, R = S = 0$. Consider $n \geq 2$ and $A \neq \alpha 1$. Assume that $\|A\| = 1$, otherwise use $(1/\|A\|)A$. Put $\tilde{A} = A - \alpha 1$. Then $\tilde{A} \neq 0$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis in H consisting of eigenvectors of A (and $\tilde{A}$), and let $\alpha_1, \ldots, \alpha_n$ be the corresponding eigenvalues of $\tilde{A}$. We order the eigenvalues (and the corresponding eigenvectors) in such a way that for each $j, j = 1, \ldots, n, \quad \beta_j := \alpha_1 + \cdots + \alpha_j \in [0, 2]$. By Lemma 1 in [12] (or Lemma 3.4 below), for any $\beta, \gamma \geq 2$, there exist rank one projections P_j, Q_j in $\mathcal{A}$ with $P_j, Q_j \leq E_j + E_{j+1}$, when E_j is a rank 1 projection onto the subspace generated by e_j, such that

$$\beta_j(E_j - E_{j+1}) + \alpha E_j = \begin{cases} (\beta + \alpha)P_j - \beta Q_j & \text{for } j \text{ odd,} \\ (\gamma + \alpha)P_j - \gamma Q_j & \text{for } j \text{ even.} \end{cases}$$

We have (noting that $\beta_n = 0$ and $\alpha_n = -\beta_{n-1}$) that for $1 \leq j \leq n-1$

$$\begin{aligned} A &= [\beta_1(E_1 - E_2) + \alpha E_1] + \cdots + [\beta_{n-1}(E_{n-1} - E_n) + \alpha E_{n-1}] + \alpha E_n \\ &= (\beta + \alpha)(\sum_{j \text{ odd}} P_j) - \beta(\sum_{j \text{ odd}} Q_j) + (\gamma + \alpha)(\sum_{j \text{ even}} P_j) - \gamma(\sum_{j \text{ even}} Q_j) + \alpha E_n. \end{aligned}$$

If n is odd, put $P = \sum_{j \text{ odd}} P_j + E_n, Q = \sum_{j \text{ odd}} Q_j + E_n$, $R = \sum_{j \text{ even}} P_j$, $S = \sum_{j \text{ even}} Q_j$; if n is even, add E_n to the sums over even indices. (For $n = 2$, put $R = S = E_2$.) $\square$

3. Infinite factors

The following lemma belongs to mathematical folklore. We give here its proof for completeness.

Lemma 3.1. *Let $\mathcal{A}$ be a von Neumann factor, and let $P, Q, R \in \operatorname{Proj} \mathcal{A}$ with $P + Q = R$, and R infinite. Then $R \lesssim P$ or $R \lesssim Q$.*

Proof. Let $E, F \in \operatorname{Proj} \mathcal{A}$ be such that $E + F = R$, $E \sim F \sim R$ (see Lemma 6.3.3 in [8]). Then $P \wedge E \lesssim Q \wedge F$ or $P \wedge E \gtrsim Q \wedge F$. Assume $P \wedge E \lesssim Q \wedge F$; then, using Kaplansky's Parallelogram Law, we get $R \sim E = P \wedge E + (E - P \wedge E) \lesssim Q \wedge F + (E \vee P - P) \leq Q$, so $R \lesssim Q$.

The proof in the case when $P \wedge E \gtrsim Q \wedge F$ is obtained by exchanging the roles of P and Q in the proof above. $\square$

The next four lemmas are essential.

Lemma 3.2. *Let $\mathcal{A}$ be an infinite factor, and let $A \in \mathcal{A}_h$. There exists $a \in [-\|A\|, \|A\|]$ such that $e_A\Big(]a - \varepsilon, a + \varepsilon[\Big) \sim 1$ for each $\varepsilon > 0$.*

Proof. Assume there is no such a. Then, for each $x \in [-\|A\|, \|A\|]$, there is $\varepsilon_x > 0$ such that $e_A\Big(]x - \varepsilon_x, x + \varepsilon_x[\Big) \prec 1$. From compactness of $[-\|A\|, \|A\|]$, there is a finite number of intervals $]x_n - \varepsilon_{x_n}, x_n + \varepsilon_{x_n}[$ covering the interval $[-\|A\|, \|A\|]$. Hence, there is a finite number of disjoint intervals I_n, each contained in $]x_n - \varepsilon_{x_n}, x_n + \varepsilon_{x_n}[$, with union $[-\|A\|, \|A\|]$, so that $1 = \sum p_n$ for $p_n = e_A(I_n) \leq e_A\Big(]x_n - \varepsilon_{x_n}, x_n + \varepsilon_{x_n}[\Big) \prec 1$, which contradicts Lemma 3.1. □

Note that a satisfies the conditions of the Lemma if and only if it belongs to the essential spectrum of A with respect to the largest closed ideal of $\mathcal{A}$ (the one generated by all the projections not equivalent to 1).

Lemma 3.3. *Let $\mathcal{A}$ be an infinite factor and let $A \in \mathcal{A}_h$. Assume that for each $\delta > 0, e_A(] - \delta, \delta[) \sim 1$. For any $\varepsilon > 0$, there are: a sequence $(\varepsilon_i), i \in \mathbb{Z}$ of positive numbers satisfying $\varepsilon_0 = \|A\|$ and $\sum_{i \neq 0} \varepsilon_i \leq \varepsilon$, and a sequence $(E_i), i \in \mathbb{Z}$ of projections from $\mathcal{A}$ such that $E_i \sim 1$ and $E_i A = A E_i$ for all $i, \sum_{i \in \mathbb{Z}} E_i = 1$ and $-\varepsilon_i E_i \leq A E_i \leq \varepsilon_i E_i$ for $i \in \mathbb{Z}$.*

Proof. Consider the following cases:

I $e_A(\{0\}) \sim 1$. Choose $E_i, i \in \mathbb{Z} \backslash \{0\}, E_0'$ so that $E_0' \sim E_i \sim 1$ and $\sum_{i \neq 0} E_i + E_0' = e_A(\{0\})$. Put $E_0 = E_0' + e_A(\mathbb{R} \backslash \{0\}), \varepsilon_0 = \|A\|$ and $\varepsilon_i = 0$ for $i \in \mathbb{Z} \backslash \{0\}$.

II $e_A\Big(\{0\}\Big) \nsim 1$. First observe that we can replace the set of indices $\mathbb{Z}$ by any other countable set. Define, for a sequence $(\varepsilon_n), n = 0, 1, \ldots$ with $\varepsilon_0 = \|A\|, \varepsilon_1 = \varepsilon/2$ and $\varepsilon_{n+1} \leq \varepsilon_n/2$ for $n \geq 2$, sets $I_n = [-\varepsilon_n, -\varepsilon_{n+1}[\cup]\varepsilon_{n+1}, \varepsilon_n]$ and $D_n = e_A(I_n)$. Since $\sum_{n=0}^{\infty} D_n \sim 1$ (by Lemma 3.1), we can choose ε_n so that $D_n \neq 0$ for each n, by assumption and again Lemma 3.1. There are again two cases:

(a) For infinitely many n, say $n \in M \subseteq \mathbb{N}, D_n \sim 1$. Then we can use them for forming E_i, with E_0 obtained as $1 - \sum_{i \neq 0} E_i$.

(b) If $D_n \sim 1$ for finitely many (or none) of indices n, we can assume that $D_n \nsim 1$ for each $n \neq 0$. Now, if 1 is σ-finite, we are necessarily in a semifinite factor (all non-zero projections in a σ-finite type III factor are equivalent to 1), and we can use a faithful normal semifinite trace τ on $\mathcal{A}$. We have $\sum_{n=1}^{\infty} \tau(D_n) = \tau\Big(e_A\Big(] - \varepsilon_1, \varepsilon_1[\backslash\{0\}\Big)\Big) = \tau(1) = +\infty$, by assumption and Lemma 3.1. Now we can find infinitely many disjoint infinite sets $M_i \subseteq \mathbb{N}$ such that $\sum_{n \in M_i} \tau(D_n) = +\infty$. Thus we can take $E_i = \sum_{n \in M_i} D_n, i \neq 0$, with (as before) $E_0 = 1 - \sum_{i \neq 0} E_i$.
If 1 is not σ-finite, we may assume that all projections $D_n (n \neq 0)$ are infinite (and $\prec 1$). For a projection $F \in \mathcal{A}$, denote by $\#F$ the largest cardinal number of a maximal orthogonal family of non-zero σ-finite projections majorized by F (this corresponds directly to the generalized dimension function of Tomiyama in the factorial case (see [19]). By Lemma 3.1, $e_A\Big(] - \varepsilon_1, \varepsilon_1[\backslash\{0\}\Big) \sim 1$. Since $\mathcal{A}$ is a factor, we have, by Tomiyama [19, Theorem 4 and its corollary] (see also Blackadar [1, III.1.7.1])

$$\sum_{n=1}^{\infty} \#D_n = \# \sum_{n=1}^{\infty} D_n = \#e_A\Big(] - \varepsilon_1, \varepsilon_1[\backslash\{0\}\Big) = \#1.$$

This means that there is no cardinal $\kappa < \lambda = \#1$ such that $\#D_n \leq \kappa$ for $n = 1, 2, \ldots$. Hence, λ is a limit cardinal of countable cofinality (see, for example, [11]), and we can form a subsequence (D_{m_n}) of

(D_n) such that $\#D_{m_n} \nearrow \lambda$. We use the subsequence to build countably many sets $M_i \subseteq \mathbb{N}$ such that $\sum_{n\in M_i} \#D_{m_n} = \#1$ for each κ. We finish the proof as in the σ-finite case. $\square$

Lemma 3.4. *Let $\mathcal{A}$ be a von Neumann algebra, $E, F \in \operatorname{Proj}\mathcal{A}$, $E \perp F$, $E = U^*U$, $F = UU^*$ for some $U \in \mathcal{A}$ (so that $E \sim F$). For any $\alpha, \beta \geq 0$ and $D \in \mathcal{A}_+$ satisfying $D \leq \beta E$, there exist $P, Q \in \operatorname{Proj}\mathcal{A}$ with $P \sim Q$ such that*

$$(\alpha+\beta)P - \beta Q = \alpha E + D - UDU^*$$

and that $P, Q \leq E + F$.

Proof. If $\beta = 0$, then $D = 0$ and one can take $P = Q = E$. Hence, we can assume that $\beta > 0$. We use the standard material on the relative position of two projections in a von Neumann algebra (see Halmos [6] or Takesaki [18, pp. 306–308]). We construct $P, Q \in \operatorname{Proj}\mathcal{A}$ in such a way that $P, Q \leq E + F$ and $P \sim E \sim Q$ (in $\mathcal{A}_{E+F}$). More specifically, we represent the operators as 2×2 matrices over $\mathcal{A}_E$, so that

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = A_{11} + UA_{22}U^* + A_{12}U^* + UA_{21}$$

for $A_{ij} \in \mathcal{A}_E$ (note that $1_{\mathcal{A}_E} = E$). Then

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} C'^2 & C'S' \\ C'S' & S'^2 \end{bmatrix}$$

for suitably chosen C, S, C', S' with $0 \leq C, S \leq 1, 0 \leq C', S' \leq 1$ and $C^2 + S^2 = 1, C'^2 + S'^2 = 1$.

Define first $H \in \mathcal{A}_E$ by $H := (2D + \alpha 1)^{-1}(D + \alpha 1)$ for $\alpha > 0$ and $H := (1/2)1_E$ for $\alpha = 0$. Check that $0 \leq (\alpha+\beta)^{-1}[D + (\alpha+\beta)1]H \leq 1$ and $0 \leq \beta^{-1}(\beta 1 - D)H \leq 1$. Let C and C' be the square roots of the two operators, respectively, and let $S = (1 - C^2)^{\frac{1}{2}}, S' = (1 - C'^2)^{\frac{1}{2}}$. Note that C, S, C' and S' belong to the abelian von Neumann algebra generated by D and E. We check that

$$(\alpha+\beta)P - \beta Q = \begin{bmatrix} \alpha 1 + D & 0 \\ 0 & -D \end{bmatrix},$$

which ends the proof. $\square$

Theorem 3.5. *[1.3(a) in [5]] Let $\mathcal{A}$ be an infinite factor, and let $A \in \mathcal{A}_h$. For any $\beta, \gamma > 2\|A\|$, there exist $\alpha \in \mathbb{R}$ with $|\alpha| \leq \|A\|$ and projections $P, Q, R, S \in \mathcal{A}$ such that, depending on α, (1) or (2) of Theorem 2.1 holds.*

Proof. The case $A = 0$ is trivial. We may assume that $\|A\| = 1$, otherwise use $(1/\|A\|)A$ instead of A. Let α be such that $e_A\Big(]\alpha - \varepsilon, \alpha + \varepsilon[\Big) \sim 1$ for any $\varepsilon > 0$. It is enough to consider the case $\alpha \geq 0$: if $\alpha < 0$, use $-A$ instead of A. Put $\tilde{A} = A - \alpha 1$. Take $\varepsilon > 0$ satisfying $2(1 + \varepsilon) < \beta, \gamma$. Let $(E_i)_{i\in\mathbb{Z}}$ and $(\varepsilon_i)_{i\in\mathbb{Z}}$ be as in Lemma 3.3, for $\tilde{A}$ and ε.

Let $U_j \in \mathcal{A}$ be any partial isometry satisfying $U_j^*U_j = E_j, U_jU_j^* = E_{j+1}$. Put $U = \sum_{j\in\mathbb{Z}} U_j$ and let h denote the mapping $X \mapsto UXU^*$ on $\mathcal{A}$. Note that U is unitary and h^{-1} exists and maps X to U^*XU. Let $A_j = E_j\tilde{A}E_j$ and $B_0 = (1 - \alpha + \varepsilon)E_0$. Use the recurrence relation $A_j = B_j - hB_{j-1}$ to define B_j for all $j \neq 0$. Then

$$(-1 - \alpha)E_0 \leq A_0 \leq (1 - \alpha)E_0;$$

for $j \in \mathbb{Z}, j \neq 0$,

$$-\varepsilon_j E_j \leq A_j \leq \varepsilon_j E_j;$$

finally,

$$0 \leq B_j \leq 2(1+\varepsilon)E_j.$$

Apply Lemma 3.4 to obtain

$$\alpha E_j + B_j - hB_j = \begin{cases} (\beta+\alpha)P_j - \beta Q_j & \text{for} \quad j \quad \text{odd,} \\ (\gamma+\alpha)R_j - \gamma S_j & \text{for} \quad j \quad \text{even.} \end{cases}$$

Then put

$$P = \sum_{j\ odd} P_j, \quad Q = \sum_{j\ odd} Q_j, \quad R = \sum_{j\ even} R_j, \quad S = \sum_{j\ even} S_j.$$

Note that

$$A = \tilde{A} + \alpha 1 = \sum_{j \in \mathbb{Z}} (A_j + \alpha E_j) = \sum_{j \in \mathbb{Z}} (B_j - hB_j + \alpha E_j)$$

to end the proof. □

4. Type II_1 algebras

Let $\mathcal{A}$ be a σ-finite type II_1 algebra and let τ_0 be a fixed faithful normal tracial state on $\mathcal{A}$. In this section τ denotes a fixed trace that is a scalar multiple of τ_0.

We will be constructing an operator $A \in \mathcal{A}_+$ with the property that whatever $\alpha, \beta \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}$, and whatever projections $P, Q, R \in \mathcal{A}$,

$$A + \gamma R \neq \alpha P + \beta Q. \tag{3}$$

We shall first describe the properties of the right-hand side of (3), introducing convenient terminology.

Definition 4.1. A *distribution function* is a non-decreasing, left-continuous function $F\colon \mathbb{R} \to [0, \infty[$ such that $\lim_{t\to-\infty} F(t) = 0$. For a distribution function F we denote by F^+ its right-continuous variant: $F^+(t) := \lim_{s\searrow t} F(s)$, and by $F^{(\delta)}$ and $F^{(\delta)+}$, for any $\delta \in \mathbb{R}$, the functions $F^{(\delta)}(t) := F(t) + F^+(\delta - t)$ and $F^{(\delta)+}(t) := F^+(t) + F(\delta - t)$, respectively.

For any $A \in \mathcal{A}_h$, the function F_A given by $F_A(t) := \tau(e_A(]-\infty, t[)$ is a distribution function, called *the distribution function of* A *w.r.t. the trace* τ. Note that $\lim_{t\to+\infty} F_A(t) = \tau(1)$ and $F_A^+(t) = \tau(e_A(]-\infty, t]))$.

Definition 4.2. A distribution function F is called (α, β)-*symmetric* (with $0 \leq \alpha \leq \beta$) if it is constant on each of the intervals $]-\infty, 0]$, $]\alpha, \beta]$, $]\alpha+\beta, \infty[$ and $F^{(\alpha+\beta)}$ is constant on $]0, \alpha] \cup]\beta, \alpha+\beta]$.

F is *symmetric* if it is (α, β)-symmetric for some $0 \leq \alpha \leq \beta$.

It is clear that the notions of symmetry just defined do not depend on the choice of the trace τ (being always a scalar multiple of τ_0). Similarly, if the distribution function of an operator is symmetric, then the distribution function of a positive multiple of this operator is symmetric as well.

Remark 4.3. If the distribution function F_A (w.r.t. τ) of an operator $A \in \mathcal{A}_h$ is (α, β)-*symmetric*, then $A \in \mathcal{A}_+$ and

$$F_A(0) = 0 \tag{4}$$

$$F_A^+(\alpha + \beta) = \tau(1) \tag{5}$$

$$F_A^+(\alpha) = F_A(\beta), \tag{6}$$

and

$$F_A^{(\alpha+\beta)} \text{ takes on at most three different values.} \tag{7}$$

In fact, positivity of A and (4) follow from the distribution F_A being constant on $]-\infty, 0[$. Then (5) and (6) follow from F_A being constant on $]\alpha + \beta, \infty[$ and $]\alpha, \beta]$, respectively. The definition of (α, β)-symmetry also implies that $F^{(\alpha+\beta)}$ is constant and equal to $\tau(1)$ on $]-\infty, 0] \cup]\alpha + \beta, \infty[$, and that $F^{(\alpha+\beta)}$ is constant (and equal to $2F_A(\beta)$) on $]\alpha, \beta]$, which yields (7).

We will now investigate properties of the distribution function of a linear combination $A = \alpha P + \beta Q$ of two projections $P, Q \in \mathcal{A}$ with $\alpha, \beta \geq 0$. The structure of such an operator A was described (for real coefficients) by Nishio [13]. The symmetry of the spectrum of A follows immediately from Corollary 3 of [13], as observed by Rabanovich, who adds a few more facts on the symmetry properties of A in section 2 of [17]. Nevertheless, we have not spotted any proof (or even statement) of the more general fact given below and concerning symmetry of the spectral measure of A, a result crucial for the sequel.

Lemma 4.4. *Let $\mathcal{A}$ be an arbitrary von Neumann algebra and let $P, Q \in \mathcal{A}$ be two projections in a generic position (i.e. $P \wedge Q = P \wedge Q^\perp = P^\perp \wedge Q = P^\perp \wedge Q^\perp = 0$). Then, for any $\alpha, \beta \geq 0$ and any Borel $Z \subseteq [0, \alpha + \beta]$,*

$$e_{\alpha P + \beta Q}(Z) \sim e_{\alpha P + \beta Q}(\alpha + \beta - Z).$$

Proof. We use 2×2 matrix representation of $\mathcal{A}$ over $\mathcal{A}_P$ (cf. references in the proof of Lemma 3.4). Then

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} C^2 & CS \\ CS & S^2 \end{bmatrix}$$

for some $0 \leq C, S \leq 1, C^2 + S^2 = 1$. Let $V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then

$$V^*(\alpha P + \beta Q)V = \alpha P^\perp + \beta Q^\perp = (\alpha + \beta)1 - (\alpha P + \beta Q),$$

hence the spectral projections

$$e_{\alpha P + \beta Q}(Z) \text{ and } e_{\alpha P + \beta Q}(\alpha + \beta - Z) = e_{(\alpha+\beta)1 - (\alpha P + \beta Q)}(Z)$$

are unitarily equivalent. □

Corollary 4.5. *Let $\mathcal{A}$ be an arbitrary von Neumann algebra, and let $P, Q \in \operatorname{Proj} \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$, $0 \leq \alpha \leq \beta$. Then*

$$e_{\alpha P + \beta Q}(]0, \alpha[) \sim e_{\alpha P + \beta Q}(]\beta, \alpha + \beta[) \text{ and } e_{\alpha P + \beta Q}(]\alpha, \beta[) = 0. \tag{8}$$

Moreover, if $0 < \alpha \le \beta$*, then*

$$e_{\alpha P+\beta Q}(\{0\}) = P^{\perp} \wedge Q^{\perp} \text{ and } e_{\alpha P+\beta Q}(\{\alpha+\beta\}) = P \wedge Q.$$

If, additionally, $\alpha < \beta$*, then*

$$e_{\alpha P+\beta Q}(\{\alpha\}) = P \wedge Q^{\perp} \text{ and } e_{\alpha P+\beta Q}(\{\beta\}) = P^{\perp} \wedge Q,$$

while if $\alpha = \beta$*,*

$$e_{\alpha P+\beta Q}(\{\alpha\}) = P \wedge Q^{\perp} + P^{\perp} \wedge Q.$$

(Trivially, for $0 = \alpha < \beta$ *we have* $e_{\alpha P+\beta Q}(\{0\}) = Q^{\perp}$ *and* $e_{\alpha P+\beta Q}(\{\beta\}) = Q$*, and for* $\alpha = \beta = 0$*,* $e_{\alpha P+\beta Q}(\{0\}) = 1$*.)*

Proof. Let $A = \alpha P + \beta Q$. Put (cf. Takesaki [18, pp. 306–308)])

$$P_0 = P - P \wedge Q - P \wedge Q^{\perp},\ Q_0 = Q - P \wedge Q - P^{\perp} \wedge Q, \tag{9}$$

and

$$A_0 = \alpha P_0 + \beta Q_0. \tag{10}$$

If $PQ = QP$, then $P_0 = Q_0 = 0$. If P and Q do not commute, then P_0 and Q_0 are both non-zero and in a generic position in $\mathcal{A}_{P_0 \vee Q_0}$. Note that

$$1 = P_0 \vee Q_0 + P \wedge Q + P \wedge Q^{\perp} + P^{\perp} \wedge Q + P^{\perp} \wedge Q^{\perp}.$$

As observed by Rabanovich in the first paragraph of section 2 of [17], it follows from Halmos [6] that the only parts of P and Q that contribute to the possibly non-zero values of the spectral measure $e_A(\cdot)$ at the one-point sets $\{0\}, \{\alpha\}, \{\beta\}$ and $\{\alpha+\beta\}$ are the 'commuting' ones, that is $P \wedge Q, P \wedge Q^{\perp}, P^{\perp} \wedge Q$ and $P^{\perp} \wedge Q^{\perp}$. By the above, $e_A(]0,\alpha[) = e_{A_0}(]0,\alpha[)$, $e_A(]\alpha,\beta[) = e_{A_0}(]\alpha,\beta[)$ and $e_A(]\beta,\alpha+\beta[) = e_{A_0}(]\beta,\alpha+\beta[)$. Note that $e_{A_0}(]\alpha,\beta[) = 0$. This follows, for example, from the formula in Corollary 3 in Nishio [13]. Hence also $e_A(]\alpha,\beta[) = 0$. The rest follows from Lemma 4.4. □

We return to our assumption that $\mathcal{A}$ is a σ-finite algebra of type II_1, and τ is a scalar multiple of a finite tracial state on the algebra. The following result is well known (see, for example, the proof or Lemma 2.5 (iii) in [2]), but we give here a simple proof for completeness.

Lemma 4.6. *For* $0 \le B \le A$ *and* $\lambda \in \mathbb{R}$ *we have* $F_A(\lambda) \le F_B(\lambda)$*.*

Proof. Assume that $F_A(\lambda) > F_B(\lambda)$ for some λ. This means that $\tau\big(e_B([\lambda,\infty[)\big) > \tau\big(e_A([\lambda,\infty[\big)$, so that there is a non-zero $\xi \in \big(e_B([\lambda,\infty[) \wedge e_A(]-\infty,\lambda[)\big)(\mathbb{H})$, for which $\langle A\xi,\xi\rangle < \lambda\|\xi\|^2$ and $\langle A\xi,\xi\rangle \ge \langle B\xi,\xi\rangle \ge \lambda\|\xi\|^2$, a contradiction. □

Lemma 4.7. *For any* $P, Q \in \operatorname{Proj}\mathcal{A}$ *in generic position and any* $0 \le \alpha \le \beta$*, the distribution function* $F := F_{\alpha P+\beta Q}$ *(w.r.t.* τ*) satisfies*

$$F(0) = 0, \tag{11}$$

$$F(t) = \frac{1}{2}\tau(1) \quad \textit{for } t \in]\alpha, \beta], \tag{12}$$

$$F^{+}(\alpha+\beta) = \tau(1), \tag{13}$$

$$F^{(\alpha+\beta)}(t) = \tau(1) \quad \textit{for } t \in \mathbb{R}. \tag{14}$$

In particular, F is (α, β)-symmetric.

Proof. Since P and Q are in a generic position, also P and $Q^{\perp}$ are in a generic position, which implies $Q \sim P \sim Q^{\perp}$, so that $\tau(Q) = (1/2)\tau(1)$. By Lemma 4.6, we have $F_{\alpha P+\beta Q}(0) \leq F_0(0) = 0$, which yields (11), and $F_{\alpha P+\beta Q}(\beta) \leq F_{\beta Q}(\beta) = \tau(1) - \tau(Q) = (1/2)\tau(1)$. Applying Lemma 4.4 with $Z =]-\infty, t[$ gives

$$\begin{aligned} F(t) &= \tau(e_{\alpha P+\beta Q}(]-\infty, t[)) \\ &= \tau(e_{\alpha P+\beta Q}(]\alpha+\beta-t, \infty[)) = \tau(1) - F^{+}(\alpha+\beta-t) \end{aligned} \tag{15}$$

for all $t \in \mathbb{R}$. Hence (14) and, by (11), $F^{+}(\alpha+\beta) = \tau(1) - F(0) = \tau(1)$, which shows (13). For (12) note that by (15) we have $F^{+}(\alpha) = \tau(1) - F(\beta) \geq \tau(1) - (1/2)\tau(1) = (1/2)\tau(1)$. □

Lemma 4.8. *For any commuting $P, Q \in \operatorname{Proj} \mathcal{A}$ and any $0 \leq \alpha \leq \beta$, the distribution function $F := F_{\alpha P+\beta Q}$ is constant on each of the intervals $]-\infty, 0]$, $]0, \alpha]$, $]\alpha, \beta]$, $]\beta, \alpha+\beta]$ and $]\alpha+\beta, \infty[$. Moreover,*

$$F^{(\alpha+\beta)}(t) = 2\tau(1) - \tau(P) - \tau(Q) \quad \textit{for } t \in]0, \alpha] \cup]\beta, \alpha+\beta]. \tag{16}$$

In particular, F is (α, β)-symmetric.

Proof. Note that for commuting P and Q, $\alpha P + \beta Q = \alpha P \wedge Q^{\perp} + \beta P^{\perp} \wedge Q + (\alpha+\beta) P \wedge Q$ is a linear combination of three mutually orthogonal projections, which shows that the distribution function $F_{\alpha P+\beta Q}$ has to be constant on each of the intervals $]-\infty, 0]$, $]0, \alpha]$, $]\alpha, \beta]$, $]\beta, \alpha+\beta]$ and $]\alpha+\beta, \infty[$. In particular,

$$\tau(P^{\perp} \wedge Q^{\perp}) = \begin{cases} F(t) & \text{for } t \in]0, \alpha], \\ F^{+}(\alpha+\beta-t) & \text{for } t \in]\beta, \alpha+\beta] \end{cases}$$

and

$$\tau((P \wedge Q)^{\perp}) = \begin{cases} F(t) & \text{for } t \in]\beta, \alpha+\beta], \\ F^{+}(\alpha+\beta-t) & \text{for } t \in]0, \alpha]. \end{cases}$$

A simple calculation gives now (16). □

The work done so far can be summed up in the following:

Theorem 4.9. *Let P, Q be projections from $\mathcal{A}$, and let $0 \leq \alpha \leq \beta$. Then the distribution function F of $A = \alpha P + \beta Q$ (w.r.t. τ) is (α, β)-symmetric, and*

$$F^{(\alpha+\beta)}(t) = 2\tau(1) - \tau(P) - \tau(Q) \quad \textit{for } t \in]0, \alpha] \cup]\beta, \alpha+\beta]. \tag{17}$$

Proof. Let us first extract commuting and generic parts of P and Q. We put

$$P_c := P \wedge Q + P \wedge Q^{\perp}, \qquad Q_c := P \wedge Q + P^{\perp} \wedge Q;$$
$$P_0 := P - P_c, \qquad Q_0 := Q - Q_c.$$

We consider P_0 and Q_0 in the reduced von Neumann algebra $\mathcal{A}_{P_0 \vee Q_0}$. Similarly, we treat P_c and Q_c as elements of $\mathcal{A}_{(P_0 \vee Q_0)^{\perp}}$. Put $F_0 := F_{\alpha P_0 + \beta Q_0}$ and $F_c := F_{\alpha P_c + \beta Q_c}$. By Lemmas 4.7 and 4.8, both F_0 and F_c are (α, β)-symmetric, and they satisfy, for $t \in]0, \alpha] \cup]\beta, \alpha + \beta]$,

$$F_0^{(\alpha+\beta)}(t) = \tau(P_0 \vee Q_0) = 2\tau(P_0 \vee Q_0) - \tau(P_0) - \tau(Q_0),$$
$$F_c^{(\alpha+\beta)}(t) = 2\tau((P_0 \vee Q_0)^{\perp}) - \tau(P_c) - \tau(Q_c).$$

Hence $F = F_0 + F_c$ is also (α, β)-symmetric and F satisfies (17). □

The next few lemmas show the existence of a positive operator A such that the distribution function of $A + \gamma R$ is not symmetric, whatever the choice of $\gamma \in \mathbb{R}$ and projection $R \in \mathcal{A}$.

Lemma 4.10. *Choose arbitrary $\lambda > 0$ and $0 \leq \Lambda < \tau(1)$. If $A = B + C$, $B, C \in \mathcal{A}_+$, $\|B\| < \lambda$ and $\tau(\operatorname{supp} C) \leq \Lambda$, then $F_A(\lambda) \geq \tau(1) - \Lambda$. Consequently, if $A = B_1 + \cdots + B_m + C_1 + \cdots + C_n$ with $B_i, C_j \in \mathcal{A}_+$, $\|B_1\| + \cdots + \|B_m\| < \lambda$ and $\tau(\operatorname{supp} C_1) + \cdots + \tau(\operatorname{supp} C_n) \leq \Lambda$, then $F_A(\lambda) \geq \tau(1) - \Lambda$.*

Proof. Suppose that $F_A(\lambda) < \tau(1) - \Lambda$. Then $\tau(e_A(]-\infty, \lambda[) \vee \operatorname{supp} C) < (\tau(1) - \Lambda) + \Lambda = \tau(1)$, so that $e_A([\lambda, \infty[) \wedge (\operatorname{supp} C)^{\perp} = (e_A(]-\infty, \lambda[) \vee \operatorname{supp} C)^{\perp} \neq 0$, and there is a non-zero vector $\xi \in \big(e_A([\lambda, \infty[) \wedge (\operatorname{supp} C)^{\perp}\big)(\mathbb{H})$. Hence $\langle A\xi, \xi \rangle = \langle B\xi, \xi \rangle \leq \|B\|\|\xi\|^2 < \lambda\|\xi\|^2$, while at the same time $\langle A\xi, \xi \rangle \geq \langle \lambda e_A([\lambda, \infty[)\xi, \xi \rangle = \lambda\|\xi\|^2$, a contradiction. □

In the rest of this section we assume that the trace τ satisfies $\tau(1) \geq 1071$.

Remark 4.11. Below you will find a few technical lemmas that lead to the construction of an operator that cannot be written as a linear combination of three projections. To this aim, we build in 4.19 an operator A such that the distribution function F of $A + \gamma R$ is not symmetric, whatever the values of $\gamma \in \mathbb{R}$ and $R \in \operatorname{Proj} \mathcal{A}$. This shows, according to Theorem 4.9, that $A + \gamma R$ is not a linear combination of two projections with positive coefficients. The case of arbitrary real coefficients is then obtained easily in Theorem 4.20.

Note that the operator $A := B + C + 4E_1 + 21E_2$ constructed in Lemma 4.19 is a linear combination of 9 projections, with B (constructed in Lemma 4.15 and Corollary 4.16) a linear combination of 4 mutually orthogonal projections, say $E_{B0}, \ldots, E_{B3}$, then C (built in Lemma 4.17 and Corollary 4.18) of 3 mutually orthogonal projections, say E_{C1}, E_{C2}, E_{C3}, and finally with E_1 and E_2 mutually orthogonal. The constructions of B, C and the pair E_1, E_2 are independent of each other. Specific values of τ on the projections will be used in proving Corollary 4.22, dealing with a factor of type I_n.

Only the values of the trace on the 9 projections (and, of course, the values of the coefficients of the linear combinations) matter for the validity of the lemmas mentioned above. The choice of allowed values, used in Lemmas 4.15 and 4.17, was made to avoid complicated fractions. The assumption $\tau(1) \geq 1701$ is needed to make room for the construction of these projections.

Lemma 4.12. *Let $B, C \in \mathcal{A}_+$, $\|B\| < 1$, $\tau(\operatorname{supp} C) < 1$, and let $E_1, E_2, R \in \operatorname{Proj} \mathcal{A}$ with $E_1 \perp E_2$. Denote $\Lambda_i := \tau(E_i)$, $c := \tau(R)$ and assume that $c > 0$, $\Lambda_1 > \Lambda_2$ and that*

$$A = B + C + \lambda_1 E_1 + \lambda_2 E_2 + \gamma R$$

for some $0 < \lambda_1 < \lambda_2 < \gamma$. *Then the distribution function* F *of* A *satisfies:*

1. $F(t) + F^+(\gamma + \lambda_1 - t) \le 2\tau(1) - c - \Lambda_1 - \Lambda_2$ *for* $t \in]0, \lambda_1]$;
2. $F(t) + F^+(\gamma + \lambda_2 - t) \le 2\tau(1) - c - \Lambda_2$ *for* $t \in]0, \lambda_2]$;
3. $F(\gamma) \le \tau(1) - c$;
4. $F(1) > \tau(1) - (c + 1 + \Lambda_1 + \Lambda_2)$;
5. $F(\gamma + 1) > \tau(1) - (1 + \Lambda_1 + \Lambda_2)$;
6. $F(\lambda_1 + 1) > \tau(1) - (c + 1 + \Lambda_2)$;
7. $F(\gamma + \lambda_1 + 1) > \tau(1) - (1 + \Lambda_2)$;
8. $F(\lambda_2 + 1) > \tau(1) - (c + 1)$;
9. $F(\gamma + \lambda_2 + 1) > \tau(1) - 1$.

Proof. For 3., use $A \ge \gamma R$ and Lemma 4.6. For 1. and 2., use respectively $A \ge \lambda_1(E_1 + E_2) + \gamma R$ and $A \ge \lambda_2 E_2 + \gamma R$, Lemma 4.6 and (17) in Theorem 4.9. The others result from Lemma 4.10, and representations: 4. $A = B + (C + \lambda_1 E_1 + \lambda_2 E_2 + \gamma R)$; 5. $A = (B + \gamma R) + (C + \lambda_1 E_1 + \lambda_2 E_2)$; 6. $A = (B + \lambda_1 E_1) + (C + \lambda_2 E_2 + \gamma R)$; 7. $A = (B + \lambda_1 E_1 + \gamma R) + (C + \lambda_2 E_2)$; 8. $A = (B + \lambda_1 E_1 + \lambda_2 E_2) + (C + \gamma R)$; 9. $A = (B + \lambda_1 E_1 + \lambda_2 E_2 + \gamma R) + C$. □

Corollary 4.13. *Let* $B, C \in \mathcal{A}_+$, $\|B\| < 1$, $\tau(\operatorname{supp} C) < 1$, *and let* $E_1, E_2, R \in \operatorname{Proj} \mathcal{A}$, *with* $E_1 \perp E_2$, $\tau(E_1) = 16$, $\tau(E_2) = 4$ *and* $c := \tau(R) > 0$. *Let*

$$A = B + C + 4E_1 + 21E_2 + \gamma R \quad \textit{with} \quad \gamma > 21.$$

Then the distribution function F *of* A *satisfies:*

1. $F(t) + F^+(\gamma + 4 - t) \le 2\tau(1) - c - 20$ *for* $t \in]0, 4]$; *in particular* $F(t) \le \tau(1) - c - 10$ *or* $F^+(\gamma + 4 - t) \le \tau(1) - 10$ *for any* $t \in]0, 4]$;
2. $F(t) + F^+(\gamma + 21 - t) \le 2\tau(1) - c - 4$ *for* $t \in]0, 21]$; *in particular* $F(t) \le \tau(1) - c - 2$ *or* $F^+(\gamma + 21 - t) \le \tau(1) - 2$ *for any* $t \in]0, 21]$;
3. $F(\gamma) \le \tau(1) - c$;
4. $F(1) > \tau(1) - c - 21$;
5. $F(\gamma + 1) > \tau(1) - 21$;
6. $F(5) > \tau(1) - c - 5$;
7. $F(\gamma + 5) > \tau(1) - 5$;
8. $F(22) > \tau(1) - c - 1$;
9. $F(\gamma + 22) > \tau(1) - 1$.

Proof. Put $\lambda_1 = 4$, $\lambda_2 = 21$, $\Lambda_1 = 16$, $\Lambda_2 = 4$ and $\gamma > 21$ in Lemma 4.12. □

Lemma 4.14. *If* F *is a distribution function satisfying* $1. - 9.$ *of Corollary 4.13 for* $\gamma > 44$ *and* $c > 42$, *then* F *is not symmetric.*

Proof. Assume F is symmetric. We shall consider four cases, one of which would necessarily take place if the function F were symmetric with respect to some (α, β).

I $\beta \in [\gamma, \gamma + 1[$). By 4.13.8,

$$F(\beta) \ge F(\gamma) \ge F(22) > \tau(1) - c - 1.$$

Hence, by 4.13.2 with $t = 11 + \gamma - \beta$, (5) and (6), either

$$F^+(10) \le F(11+\gamma-\beta) \le \tau(1) - c - 2 < F(\beta) = F^+(\alpha)$$

or

$$F^+(\beta+10) \le \tau(1) - 2 < \tau(1) = F^+(\beta+\alpha).$$

Thus $\alpha \ge 10$, and by 4.13.6 and 4.13.7,

$$F(5) + F^+(\alpha+\beta-5) \ge F(5) + F(\gamma+5) > 2\tau(1) - c - 10. \tag{18}$$

Note that $\alpha+\beta-\gamma-2 \le 2\beta-\gamma-2 < 2(\gamma+1)-\gamma-2 = \gamma$. Hence, by 4.13.1 with $t=2$, (5) and 4.13.3, we have either

$$F(2) + F^+(\alpha+\beta-2) \le (\tau(1) - c - 10) + \tau(1) = 2\tau(1) - c - 10 \tag{19}$$

or

$$F^+(\gamma+2) + F(\alpha+\beta-\gamma-2) \le (\tau(1) - 10) + (\tau(1) - c) = 2\tau(1) - c - 10. \tag{20}$$

By Definition 4.2, the left hand sides of (18), (19) and (20) are all equal to the value of $F^{(\alpha+\beta)}$ on $]0,\alpha]\cup]\beta,\alpha+\beta]$, which yields a contradiction.

II $\beta \ge \gamma+1$. By (6) and 4.13.5, $F^+(\alpha) = F(\beta) > \tau(1) - 21 > \tau(1) - c$. By 4.13.3, $\alpha \ge \gamma$.

By 4.13.5, (5) and the definition of symmetry 4.2, we have as well $21 > \tau(1) - F(\gamma+1) \ge F(\alpha+\beta) - F^+(\beta) = F(\alpha) - F^+(0)$; by 4.13.3 and 4.13.5, $F(\gamma+1) - F^+(0) \ge F(\gamma+1) - F(\gamma) \ge 21$. This implies $\alpha < \gamma+1$.

Since $\alpha \ge \gamma$, we have $F(\alpha+\beta-22) \ge F(2\gamma-22) \ge F(\gamma+22) > \tau(1)-1$, by 4.13.9. This implies, again by symmetry of F and 4.13.8, $F(2)+F^+(\alpha+\beta-2) = F(22)+F^+(\alpha+\beta-22) > (\tau(1)-c-1)+(\tau(1)-1) = 2\tau(1)-c-2$, hence

$$F^+(2) > \tau(1) - c - 2. \tag{21}$$

In particular, $F(3) > \tau(1) - c - 10$, and by (6) with 4.13.1, $F(\beta) = F^+(\alpha) \le F^+(\gamma+1) \le \tau(1) - 10$, so that, by 4.13.7, $\beta < \gamma+5$.

The inequality (21) yields also $F(11) > \tau(1) - c - 2$, so that, by 4.13.2, $F^+(\gamma+10) \le \tau(1) - 2$. Since $\gamma \le \alpha < \gamma+1$ and $\gamma+1 \le \beta < \gamma+5$, we have $\gamma+10 = \alpha+\beta-t$ for some $t \in]0,\alpha]$ with $t < \gamma$. Consequently, $F(t) + F^+(\gamma+10) = F(22) + F^+(\alpha+\beta-22)$, so that $F^+(\gamma+10) = F(22) + F^+(\alpha+\beta-22) - F(t) > (\tau(1)-c-1) + (\tau(1)-1) - (\tau(1)-c) = \tau(1)-2$, from 4.13.8 and 4.13.3. We obtained a contradiction.

III $\beta \in [0,\gamma/2[$ We have $\alpha \le \beta < \gamma/2$, so that $\gamma > \alpha+\beta$ and $F(\gamma) = \tau(1)$, which contradicts 4.13.3.

IV $\beta \in [\gamma/2,\gamma[$ We have $\alpha+\beta \ge \gamma$, since by (5) $F^+(\alpha+\beta) = \tau(1)$ and $F(\gamma) \le \tau(1) - c$, by 4.13.3. By 4.13.4, there is a $0 < \epsilon < 1$ satisfying $F(\epsilon) > \tau(1) - c - 21$. Suppose $\alpha+\beta > \gamma+1+\epsilon$. Choose γ' with $\beta < \gamma' < \gamma$. We have $\gamma+1 = \alpha+\beta-t_1, \gamma' = \alpha+\beta-t_2$ for some $\epsilon < t_i < \alpha$, $i=1,2$. Using $\alpha \le \beta < \gamma$, 4.13.3 and the choice of ϵ, we have $F(t_i) \in]\tau(1)-c-21, \tau(1)-c]$ for $i=1,2$. On the other hand, by 4.13.3, $F^+(\alpha+\beta-t_2) = F^+(\gamma') \le \tau(1)-c$, and $F^+(\alpha+\beta-t_1) = F^+(\gamma+1) > \tau(1)-21$, by 4.13.5. This contradicts $c > 42$ and the equality $F(t_1) + F^+(\alpha+\beta-t_1) = F(t_2) + F^+(\alpha+\beta-t_2)$, satisfied for $t_1, t_2 \in]0,\alpha]$. We showed that $\alpha+\beta \le \gamma+1+\epsilon < \gamma+2$, in particular, by (5), $F(\gamma+2) = \tau(1)$.

Since $F^+(\gamma+10) = \tau(1)$, by 4.13.2 with $t=11$ we have $F(11) \le \tau(1)-c-2$. By 4.13.8 and (6), we have $F^+(\alpha) = F(\beta) \ge F(\gamma/2) \ge F(22) \ge \tau(1)-c-1$, so that $\alpha \ge 11$. Consequently, $F(2) + F^+(\alpha+\beta-2) = F(5) + F^+(\alpha+\beta-5)$. Since $\gamma \le \alpha+\beta < \gamma+2$, we have $\alpha+\beta-2, \alpha+\beta-5 \in [\gamma-5,\gamma[\subseteq]22,\gamma[$, and it follows from 4.13.3 and 4.13.8 that $F^+(\alpha+\beta-2) - F^+(\alpha+\beta-5) < 1$ and $F(5) - F(2) < 1$. By 4.13.6 this implies $F(2) > \tau(1) - c - 6$, hence $F^+(\gamma + 2) \leq \tau(1) - 10$ by 4.13.1 with $t = 2$. We got contradiction with the equality $F(\gamma + 2) = \tau(1)$, which ends the proof. □

Lemma 4.15. *Let τ' be a multiple of τ_0 such that $\tau'(1) > 17$. There is an operator $B \in \mathcal{A}_+$ satisfying $\|B\| < 1$ and the following condition: for any operator $C \in \mathcal{A}_h$ with $\tau'(\operatorname{supp} C) \leq 1$, the distribution function F_{B+C} (w.r.t. τ') is not symmetric.*

Proof. If the distribution function F of an operator (w.r.t. τ') satisfies, for some $0 = \lambda_0 < \lambda_1 < \lambda_2 < \lambda_3/2$, the inequalities $F^+(\lambda_i) - F(\lambda_i) \geq 4$ for $i = 0, 1, 2$, $F^+(\lambda_2) \geq \tau'(1) - 3$, $F(\lambda_3) < \tau'(1)$, then F cannot be symmetric. In fact, suppose that F is $(\alpha, \beta)-$symmetric for some $0 \leq \alpha \leq \beta$, and put $\delta := \alpha + \beta$. For $\delta < \lambda_3$ one has $F^+(\delta) \leq F(\lambda_3) < \tau'(1)$, contradicting (5). On the other hand, if $\delta \geq \lambda_3$, then

$$\begin{aligned} F^{(\delta)+}(\lambda_i) - F^{(\delta)+}(\lambda_{i-1}) &= \big(F^+(\lambda_i) - F^+(\lambda_{i-1})\big) + \big(F(\delta - \lambda_i) - F(\delta - \lambda_{i-1})\big) \\ &\geq \big(F^+(\lambda_i) - F(\lambda_i)\big) + \big(F(\delta - \lambda_i) - F(\delta - \lambda_{i-1})\big) \\ &\geq 4 + F(\delta - \lambda_2) - \tau'(1) \\ &\geq 4 + F^+(\lambda_2) - \tau'(1) \geq 1, \end{aligned}$$

for $i = 0, 1, 2$ and any $\lambda_{-1} < 0$. Consequently, $F^{(\delta)}$ has at least three jumps, so it takes on more than three values, contradicting (7). Thus F is not (α, β)-symmetric.

Put now $B := \sum_{i=0}^{3} \lambda_i E_i$, where E_i are mutually orthogonal projections from $\mathcal{A}$ with sum $1_{\mathcal{A}}$ such that $\tau'(E_0), \tau'(E_1), \tau'(E_2) \geq 5$ and $\tau'(E_3) = 2$, with λ_i as above, and assume additionally that $\lambda_3 < 1$, so that $\|B\| < 1$. The operator $B + C$ has eigenspaces $E_i'(\mathbb{H})$ for eigenvalues λ_i, with $E_i' \geq E_i \wedge (\operatorname{supp} C)^\perp$, for $i = 0, 1, 2, 3$. Hence $\tau'(E_0'), \tau'(E_1'), \tau'(E_2') \geq 4, \tau'(E_3') \geq 1$ and $F^+_{B+C}(\lambda_i) - F_{B+C}(\lambda_i) \geq 4$ for $i = 0, 1, 2$, $F^+_{B+C}(\lambda_3) - F_{B+C}(\lambda_3) > 0$. Moreover, by Lemma 4.10 applied to τ', $F^+_{B+C}(\lambda_2) \geq \tau'(1) - (\tau'(E_3) + \tau'(\operatorname{supp} C)) \geq \tau'(1) - 3$, so that the first part of the proof applies and, consequently, B has the desired properties. □

Corollary 4.16. *There exists an operator $B \in \mathcal{A}_+$ satisfying $\|B\| < 1$ and the following condition: for any operator $C \in \mathcal{A}_h$ with $\tau(\operatorname{supp} C) \leq 63$, the distribution function of $B + C$ (w.r.t. τ) is not symmetric.*

Proof. Put $\tau' = (1/63)\tau$ and build B for τ' as in Lemma 4.15. For any C with $\tau(\operatorname{supp} C) \leq 63$, we have $\tau'(\operatorname{supp} C) \leq 1$, so that F_{B+C} is not symmetric. □

Lemma 4.17. *There is $C' \in \mathcal{A}_+$ satisfying $\tau(\operatorname{supp} C') < 1$ and the following condition: for any operator $B' \in \mathcal{A}_h$ with $-\epsilon \cdot 1_{\mathcal{A}} \leq B' \leq (1 - \epsilon) \cdot 1_{\mathcal{A}}$ for some $\epsilon \in [0, 1]$, the distribution function F' of $B' + C'$ (w.r.t. τ) is not symmetric.*

Proof. Define $C' := \sum_{i=0}^{3} \lambda_i E_i$, where $\lambda_0 = 0$, $\lambda_1 = 2$, $\lambda_2 = 4$ and $\lambda_3 = 11$, and where E_i are mutually orthogonal projections from $\mathcal{A}$ with sum $1_{\mathcal{A}}$ satisfying $\tau(E_1 + E_2 + E_3) < 1$ and $\tau(E_i) > \tau(E_3) > 0$ for $i = 0, 1, 2$. By Lemma 4.10, for any $B \in \mathcal{A}_+$ with $\|B\| \leq 1$, the distribution function $F := F_{B+C'}$ of the operator $B + C'$ satisfies $F^+(\lambda_i + 1) \geq \tau(1) - \tau(E_{i+1} + \cdots + E_3) = \tau(E_0 + \cdots + E_i)$ for $i = 0, 1, 2, 3$ (an empty sum is the zero operator). By Lemma 4.6, it satisfies $F(\lambda_i) \leq \tau(E_0 + \cdots + E_{i-1})$ for $i = 0, 1, 2, 3$. Hence, for an operator B' as in the statement of our lemma, we can take $B := B' + \epsilon \cdot 1_{\mathcal{A}}$, and then the distribution function F' satisfies $F'(t) = F(t + \epsilon)$, so that ${F'}^+(\lambda_i + 1 - \epsilon) \geq \tau(E_0 + \cdots + E_i)$, $F'(\lambda_i - \epsilon) \leq \tau(E_0 + \cdots + E_{i-1})$ for $i = 0, 1, 2, 3$. Thus we have

$$F'(t) = \begin{cases} 0 & \text{for } t \in]-\infty, -\epsilon] \\ \tau(E_0) & \text{for } t \in]1-\epsilon, 2-\epsilon], \\ \tau(E_0+E_1) & \text{for } t \in]3-\epsilon, 4-\epsilon], \\ \tau(1)-\tau(E_3) & \text{for } t \in]5-\epsilon, 11-\epsilon], \\ \tau(1) & \text{for } t \in]12-\epsilon, \infty[. \end{cases} \tag{22}$$

Suppose that F' is (α,β)-symmetric for some $0 \le \alpha \le \beta$, and let $\delta := \alpha+\beta$. If $\delta < 11-\epsilon$, then $F'^{+}(\delta) \le F'(11-\epsilon) < \tau(1)$. If $\delta \ge 11-\epsilon$, then, for any $t \le 6$, $F'^{(\delta)}(t) = F'(t)+F'^{+}(\delta-t) \in [F'(t)+F'^{+}(5-\epsilon), F'(t)+\tau(1)] \subseteq [F'(t)+\tau(1)-\tau(E_3), F'(t)+\tau(1)]$. We have got $F'^{(\delta)}(t_0) < F'^{(\delta)}(t_1) < F'^{(\delta)}(t_2) < F'^{(\delta)}(t_3)$ for arbitrarily chosen $t_0 \le -\epsilon, 1-\epsilon < t_1 \le 2-\epsilon, 3-\epsilon < t_2 \le 4-\epsilon, 5-\epsilon < t_3 \le 6$, using $\tau(E_3) < \tau(E_i)$ for $i = 0, 1, 2$. Thus, for any $\delta \in \mathbb{R}$, either $F'^{+}(\delta) < \tau(1)$ or $F'^{(\delta)}$ takes on more than 3 values, which contradicts (7). □

Corollary 4.18. *There is an operator $C \in \mathcal{A}_+$ satisfying $\tau(\operatorname{supp} C) < 1$ and the following condition: for any operator $B \in \mathcal{A}_h$ with $-1 \cdot 1_{\mathcal{A}} \le B \le 66 \cdot 1_{\mathcal{A}}$, the distribution function of $B + C$ is not symmetric.*

Proof. Take $C := 67C'$, where C' is the operator from Lemma 4.17, used with $\epsilon = 1/67$. Then, for any $B \in \mathcal{A}_h$ with $-1\cdot 1_{\mathcal{A}} \le B \le 66\cdot 1_{\mathcal{A}}$, we have for $B' := (1/67)B$ that $F_{B'+C'}$ is not symmetric, so that F_{B+C} is not symmetric, either. □

Lemma 4.19. *Put $A := B + C + 4E_1 + 21E_2$, where $E_1, E_2 \in \operatorname{Proj} \mathcal{A}$ with $E_1 \perp E_2, \tau(E_1) = 16, \tau(E_2) = 4$, and where B and C are operators from Corollaries 4.16 and 4.18, respectively. Whatever the choice of projection $R \in \mathcal{A}$ and $\gamma \in \mathbb{R}$, the distribution function of $A + \gamma R$ is not symmetric.*

Proof. I For R such that $\tau(R) \le 42$ (and any $\gamma \in \mathbb{R}$) we have $\tau(\operatorname{supp}(C+4E_1+21E_2+\gamma R)) \le \tau(\operatorname{supp} C) + \tau(E_1) + \tau(E_2) + \tau(R) < 1 + 16 + 4 + 42 = 63$, and Corollary 4.16 together with the definition of B imply that the distribution function of $A + \gamma R$ is not symmetric.

II For $-1 < \gamma \le 44$, we have $-1_{\mathcal{A}} \le B + 4E_1 + 21E_2 + \gamma R \le 66 \cdot 1_{\mathcal{A}}$. Corollary 4.18 implies that the distribution function of $A + \gamma R = C + (B + 4E_1 + 21E_2 + \gamma R)$ is not symmetric.

III For $\gamma \le -1$ and $\tau(R) > 42$ the operator $A + \gamma R$ is not positive. In fact, $\tau(\operatorname{supp}(C + 4E_1 + 21E_2)) \le \tau(\operatorname{supp} C) + \tau(E_1) + \tau(E_2) = 1 + 4 + 16 < \tau(R)$. Hence, for $0 \ne \xi \in (R \wedge (\operatorname{supp}(C + 4E_1 + 21E_2))^{\perp})(\mathbb{H})$, we have $\langle (A+\gamma R)\xi, \xi\rangle = \langle B\xi, \xi\rangle + \gamma\|\xi\|^2 \le (\|B\| + \gamma)\|\xi\|^2 < (1+\gamma)\|\xi\|^2 < 0$.

Thus $F_{A+\gamma R}(0) > 0$, which by Remark 4.3(4) means that $A + \gamma R$ is not symmetric.

IV For $\gamma > 44$ and $\tau(R) > 42$, the distribution function of the operator $A+\gamma R$ is not symmetric. In fact, it satisfies the assumptions of Corollary 4.13, so we can use Lemma 4.14. □

It should be noted that if a self-adjoint operator A cannot be written as a real linear combination of three projections, it cannot be written as a complex linear combination of three projections, either. In fact, if $A = \alpha P + \beta Q + \gamma R$ with $\alpha, \beta, \gamma \in \mathbb{C}$, then $A = (1/2)(A + A^*) = (\Re\alpha)P + (\Re\beta)Q + (\Re\gamma)R$. Hence, Theorem 4.20, Corollaries 4.21 and 4.22, as well as Theorem 5.3 of the next section, hold for arbitrary combinations of projections.

Theorem 4.20. *Let $\mathcal{A}$ be a σ-finite type II_1 algebra with a faithful normal tracial state τ_0. For any $0 < \delta \le 1$ there exists an operator $A \in \mathcal{A}_+$ with $\tau_0(\operatorname{supp} A) \le \delta$ such that A is not a real linear combination of three projections from $\mathcal{A}$.*

Proof. Put $\tau := (1071/\delta)\tau_0$. We will apply Lemma 4.19 with the trace τ, and show that the operator A defined there cannot be of the form $\alpha P + \beta Q + \gamma R$, whatever $\alpha, \beta, \gamma \in \mathbb{R}$ and choice of projections $P, Q, R \in \mathcal{A}$. In fact:

I If at least two of the numbers α, β and γ are non-negative, the result follows immediately from Theorem 4.9 and Lemma 4.19.

II If at least two of them, say β and γ, are negative, we have $0 \le A - \gamma R = \alpha P + \beta Q$. Hence, if $0 \ne \xi \in P^{\perp}(\mathbb{H})$, then $Q\xi = 0$, so that $Q \le P$. If $Q = 0$, we can replace β with any $\beta' \ge 0$, and we have again case I. Otherwise, we have $0 \le A - \gamma R = \alpha(P - Q) + (\alpha + \beta)Q$, and the RHS consists of two operators with disjoint supports, both necessarily positive. Hence $\alpha, \alpha + \beta \ge 0$, and we are back to the case I of at least two non-negative coefficients. □

Corollary 4.21. *Let $\mathcal{A}$ be a type II_1 algebra. There exists an operator $A \in \mathcal{A}_+$ such that A is not a real linear combination of three projections from $\mathcal{A}$.*

Proof. Any type II_1 algebra is a direct sum of σ-finite type II_1 algebras (see e.g. Takesaki [18], Corollary V.2.9). It is enough to build the operator A as in Theorem 4.20 in one of those direct summands. □

Corollary 4.22. *For any type II_1 algebra $\mathcal{A}$ there exists a natural number n, a factor $\mathcal{A}_n \subset \mathcal{A}$ of type I_n and a self-adjoint operator $A \in \mathcal{A}_n$ which cannot be written as a real linear combination of three projections from $\mathcal{A}$ (and not just in $\mathcal{A}_n$).*

Proof. We can first restrict our attention to a σ-finite algebra, which allows us to use a finite trace. By the halving lemma (see e.g. Takesaki [18]) there exists in $\mathcal{A}$ an orthogonal family $\{F_i\}$ of $n := 2^{13}$ equivalent projections with sum $1_{\mathcal{A}}$. Let $\{U_{ij}\}$ be a matrix unit in $\mathcal{A}$ such that $U_{ii} = F_i$ for each i (see Takesaki [18], IV.1.7), and let $\mathcal{A}_n$ be the factor of type I_n generated by the matrix unit. Take a trace tr on $\mathcal{A}$ such that $\mathrm{tr}(1_{\mathcal{A}}) = 2^{13}$. Now construct all the 9 projections mentioned in Remark 4.11 as sums of minimal projections from $\mathcal{A}_n$ so that $\mathrm{tr}(E_1) = 6 \cdot 16$, $\mathrm{tr}(E_2) = 6 \cdot 4$, $\mathrm{tr}(E_{B0}) = \mathrm{tr}(E_{B1}) = \mathrm{tr}(E_{B2}) = 6 \cdot 5 \cdot 63$, $\mathrm{tr}(E_{B3}) = 6 \cdot 2 \cdot 63$, $\mathrm{tr}(E_{C1}) = \mathrm{tr}(E_{C2}) = 2$ and $\mathrm{tr}(E_{C3}) = 1$. We easily check that if we take $\tau = (1/6)\mathrm{tr}$, then all the assumptions of Lemma 4.19 are satisfied. This ends the proof of the corollary. □

Rabanovich showed in [17] an example of an operator that cannot be written as a real linear combination of three projections in a factor of type I_n, for $n = 76$. Although the above corollary is slightly more general than his Proposition 4.6 (even using projections from a surrounding algebra does not help in writing the operator as a linear combination of three projections), the n obtained here is much larger.

5. Type II_∞ factors

The construction of an operator in an algebra (or just a factor) of type II_1 that cannot be written as a linear combination of three projections leads to a fairly easy construction of such an operator in a II_∞ factor $\mathcal{A}$. It is constructed in a reduced factor $\mathcal{A}_E$ of type II_1, where E is a finite projection in $\mathcal{A}$. It is shown that if this operator were a linear combination of three projections from the factor $\mathcal{A}$, we could also find such a decomposition inside $\mathcal{A}_E$.

We start with a lemma:

Lemma 5.1. *Let $\mathcal{A}$ be a factor of type II_∞ with a faithful normal semifinite trace τ. Let further $P, Q, S \in \mathrm{Proj}\,\mathcal{A}$, and let α, β and γ be real numbers with $0 < \alpha \le \beta$. If $B \in \mathcal{A}_h$ is such that $\tau(\mathrm{supp}\,B) < \tau(S)$ (so, in particular, $\tau(\mathrm{supp}\,B) < \infty$), $\mathrm{supp}\,B \perp S$ and $B + \gamma S = \alpha P + \beta Q$, then $B = \alpha P' + \beta Q'$ for some $P', Q' \in \mathrm{Proj}\,\mathcal{A}$.*

Proof. Let $e(\cdot) := e_{B+\gamma S}(\cdot) = e_{\alpha P + \beta Q}(\cdot)$. If $\gamma = 0$, the result is obvious. If $\gamma \ne 0$, then necessarily $\gamma > 0$, since $\alpha P + \beta Q$ is positive, and $\mathrm{supp}\,B \perp S$. If $\gamma \notin]0, \alpha[$, then $e(]0, \alpha[) \perp S$. Since $e(]0, \alpha[) \le \mathrm{supp}(B + \gamma S) = \mathrm{supp}\,B + S$, we have $e(]0, \alpha[) \le \mathrm{supp}\,B$. Similarly, if $\gamma \notin]\beta, \alpha + \beta[$, then $e(]\beta, \alpha + \beta[) \le \mathrm{supp}\,B$. One of the two possibilities has to occur, hence, by (8) in Corollary 4.5, $\tau(e(]0,\alpha[)) = \tau(e(]\beta,\alpha+\beta[)) \le \tau(\operatorname{supp} B) < \tau(S)$. On the other hand, if $\gamma \in]0,\alpha[\cup]\beta,\alpha+\beta[$, then $S \le e(]0,\alpha[)$ or $S \le e(]\beta,\alpha+\beta[)$, so that $\tau(S) \le \tau(e(]0,\alpha[)) = \tau(e(]\beta,\alpha+\beta[)) \le \tau(\operatorname{supp} B)$, which contradicts one of the assumptions. Again by (8), we also have $e(]\alpha,\beta[) = 0$, so that $\gamma \in \{\alpha,\beta,\alpha+\beta\}$ and one of the following cases must hold (cf. Corollary 4.5):

(1) $\gamma = \alpha+\beta$. Then $S \le e(\{\alpha+\beta\}) = P \wedge Q$ and $B = \alpha(P-S) + \beta(Q-S)$.
(2) $\gamma = \alpha < \beta$. Then $S \le e(\{\alpha\}) = P \wedge Q^\perp$ and $B = \alpha(P-S) + \beta Q$.
(3) $\gamma = \beta > \alpha$. Then $S \le e(\{\beta\}) = P^\perp \wedge Q$ and $B = \alpha P + \beta(Q-S)$.
(4) $\gamma = \alpha = \beta$. Then $S \le e(\{\alpha\}) = P \wedge Q^\perp + P^\perp \wedge Q$ and $B = \alpha(P - P\wedge Q^\perp) + \alpha(Q + P\wedge Q^\perp - S)$.

In each case, we have the desired representation of B as a linear combination of two projections. □

Lemma 5.2. *Let $\mathcal{A}$ be a factor of type II_∞ with a faithful normal semifinite trace τ, and let E be a non-zero finite projection from $\mathcal{A}$ with trace 1. Suppose $A \in \mathcal{A}_{E,+} \subseteq \mathcal{A}_+$ is not a real linear combination of three projections in the reduced algebra $\mathcal{A}_E$. Then there are no $\alpha,\beta,\gamma \in \mathbb{R}$ and $P,Q,R \in \operatorname{Proj}\mathcal{A}$ such that $\tau(\operatorname{supp} A \vee P \vee Q \vee R) \le 1$ and $A = \alpha P + \beta Q + \gamma R$.*

Proof. Assume that for some $\alpha,\beta,\gamma \in \mathbb{R}$ and $P,Q,R \in \operatorname{Proj}\mathcal{A}$ with $\tau(\operatorname{supp} A \vee P \vee Q \vee R) \le 1$, $A = \alpha P + \beta Q + \gamma R$. Let $F \in \operatorname{Proj}\mathcal{A}$ be such that $\operatorname{supp} A \vee P \vee Q \vee R \le F$ and $\tau(F) = 1$. Then $E - \operatorname{supp} A \sim F - \operatorname{supp} A$, so there exists a partial isometry $U \in \mathcal{A}$ such that $U^*U = E, UU^* = F$ and U acts as identity on $(\operatorname{supp} A)\mathbb{H}$. Thus, we would have $A = U^*AU = \alpha U^*PU + \beta U^*QU + \gamma U^*RU$, with the decomposition taking place in $\mathcal{A}_E$, which contradicts the assumption. □

Theorem 5.3. *Let $\mathcal{A}$ be a factor of type II_∞. There exists an operator $A \in \mathcal{A}_h$ such that A is not a real linear combination of three projections from $\mathcal{A}$.*

Proof. Let $E \in \operatorname{Proj}\mathcal{A}$ be a non-zero finite projection with trace 1. By Theorem 4.20, there exists an operator $A \in \mathcal{A}_{E,+}$ with $\tau(\operatorname{supp} A) \le 1/4$, satisfying the assumptions of Lemma 5.2. Suppose that for some $\alpha,\beta,\gamma \in \mathbb{R}$, we have $A = \alpha P + \beta Q + \gamma R$ with $P,Q,R \in \operatorname{Proj}\mathcal{A}$. We shall show that then either $\tau(\operatorname{supp} A \vee P \vee Q \vee R) \le 1$, contradicting Lemma 5.2, or there are other projections in $\mathcal{A}$, say P', Q' and R', such that $A = \alpha P' + \beta Q' + \gamma R'$ and $\tau(\operatorname{supp} A \vee P' \vee Q' \vee R') \le 1$, again contradicting Lemma 5.2.

Clearly, A is non-zero, and if $A = \alpha P$ with $P \in \operatorname{Proj}\mathcal{A}$ and non-zero $\alpha \in \mathbb{R}$, then $P = \operatorname{supp} A$ and $\tau(\operatorname{supp} A \vee P) \le 1$, contradicting Lemma 5.2.

Assume $A = \alpha P + \beta Q$ for some non-zero $\alpha,\beta \in \mathbb{R}$ with $\alpha \le \beta$. If $\alpha > 0$, then $P, Q \le \operatorname{supp} A$, so that $\tau(\operatorname{supp} A \vee P \vee Q) \le \tau(\operatorname{supp} A) \le 1$. If $\alpha < 0$, then $0 \ne \xi \in Q^\perp(\mathbb{H})$ implies $\langle A\xi,\xi\rangle = \alpha\langle P\xi,\xi\rangle \ge 0$, so that $\xi \in P^\perp(\mathbb{H})$. Hence, $P \le Q$ and $A = (\alpha+\beta)P + \beta(Q-P)$, which yields $\operatorname{supp} A \ge Q$ and again $\tau(\operatorname{supp} A \vee P \vee Q) = \tau(\operatorname{supp} A) \le 1$, contradicting Lemma 5.2.

The last case to consider is that of a $A = \alpha P + \beta Q + \gamma R$ for some non-zero $\alpha,\beta,\gamma \in \mathbb{R}$ and $P,Q,R \in \operatorname{Proj} A$. As in part I of the proof of Theorem 4.20, we can assume that at least two of the coefficients, say α and β, are positive, with $\alpha \le \beta$. Changing γ to $-\gamma$ we get $A + \gamma R = \alpha P + \beta Q$.

Let $S := R \wedge (\operatorname{supp} A)^\perp$. If $\tau(S) \le 1/2$, then $\tau(R) \le 3/4$, $\tau(\operatorname{supp} A \vee R) \le 1$, and $P, Q \le \operatorname{supp} A \vee R$, which leads to $\tau(\operatorname{supp} A \vee P \vee Q \vee R) \le 1$, contradicting Lemma 5.2.

Assume now $\tau(S) > 1/2$ and put $R' := R - S$, $B := A + \gamma R'$. Note that $S \perp \operatorname{supp} B$. Since $R^\perp + R - S = 1 - R \wedge (\operatorname{supp} A)^\perp = R^\perp \vee \operatorname{supp} A$, we have, by Kaplansky's formula ([18, Proposition V.1.6]), $R' = R^\perp \vee \operatorname{supp} A - R^\perp \sim \operatorname{supp} A - R^\perp \wedge \operatorname{supp} A \precsim \operatorname{supp} A$. Hence $\tau(R') \le 1/4$ and $\tau(\operatorname{supp} B) \le 1/2$, so that $\tau(\operatorname{supp} B) < \tau(S)$. By Lemma 5.1, $B = \alpha P' + \beta Q'$ for some $P', Q' \in \operatorname{Proj}\mathcal{A}$ and $B = A + \gamma R' = \alpha P' + \beta Q'$. Since α and β are positive, $P', Q' \le \operatorname{supp} B \le \operatorname{supp} A \vee R'$ so that $\tau(\operatorname{supp} A \vee P' \vee Q' \vee R') \le 1$, contradicting Lemma 5.2. □

6. Open problems

There are many interesting open problems connected with linear combinations of projections with complex (cf. [17]), positive (cf. [9]) and integral (cf. [5]) coefficients. Here we present only three problems strictly connected with the subject of our paper.

(1) In [17], Rabanovich proved that there exists a 76 by 76 Hermitian matrix that cannot be written as a linear combination of 3 projections. As shown by Nakamura [12], any Hermitian n by n matrix with $n \leq 7$ can be written as a linear combination of 3 projections. What is the largest number n such that each Hermitian n by n matrix is a linear span of 3 projections?

(2) As shown in [5], each self-adjoint operator in a factor (or von Neumann algebra) of type II_1 is a linear combination of 12 projections. Is Theorem 4.20 sharp, that is, is any self-adjoint operator in a type II_1 factor a linear span of 4 projections?

(3) We know from 3.5 (and, for a separable factor of type I_∞, from [17]) that any self-adjoint operator in an infinite factor can be written as a linear combination of 4 projections. We also know from [17] that if K is a positive compact operator of infinite rank, then neither $1 + K$ nor $1 - K$ can be written as a linear combination of 3 projections. Can we find self-adjoint operators in type III factors that are not linear combinations of 3 projections?

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