A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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GEOMETRICAL PRESENTATION OF PREFERENCES BY USING UNFOLDING MODELS

Abstract: Unfolding is a special case of multidimensional scaling. It assumes that different individuals perceive various objects of choice in the same space, but differ with respect to what they consider an ideal combination of the objects' attributes. In unfolding, the data are usually preference scores of different individuals for a set of choice objects. These data can be conceived as proximities between the elements of two sets, individuals and choice objects.

In the article, some special unfolding models are presented. First, internal and external unfolding are distinguished. Than, the vector model for unfolding is introduced as a special case of the ideal-point model and weighted unfolding models are discussed.

Key words: Multidimensional scaling, Unfolding, Preference models.

I. INTRODUCTION

Unfolding attempts to produce a configuration **Y** of points in the space with each point $\mathbf{y}_k$ ($k = 1, \dots, m$) representing one of m judges, together with another configuration **X** of points $\mathbf{x}_i$ ($i = 1, \dots, n$) in the same space, these points representing choice objects. Individuals are represented as „ideal" points in the multidimensional space, so that the distances from each ideal point to the object points correspond to the preference scores. The ideal point model is used to find a point in a stimulus space which is most like an attribute. If the attribute is a subject's preference for the stimuli, then this point is interpreted as a subject's ideal stimulus. It is the hypothetical stimulus which, if it existed, the subject would prefer most.

The article presents two groups of unfolding models. There are internal and external models. In internal unfolding configurations **X** and **Y** are derived directly from preference matrix, whereas in the external case it is assumed that object point configuration is done.

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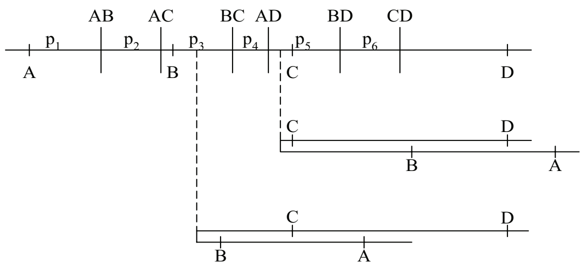
II. FULLY NONMETRIC UNFOLDING

Unfolding models can be categorized into unidimensional or multidimensional models. In the simplest nonmetric unidimensional case (see: Coombs 1950) respondents and objects could be represented by points on a straight line, where for each respondent, the rank order of the distances from his point to the points representing objects is the same as his rank ordering of the objects. Generating the joint for respondents and objects scale is started by finding the positions for objects and the midpoints between them. Midpoints split the scale into intervals. Any respondent represented by a point in a particular interval has the same preference order, and crossing the midpoint of the pair of points representing objects corresponds to interchanging the preference order between that pair (see: Fig. 1a).

a)



Interval:	p_1	p_2	p_3	p_4	p_5	p_6
Ordering:	ABCD	BACD	BCAD	CBAD	CBDA	CDBA

b)

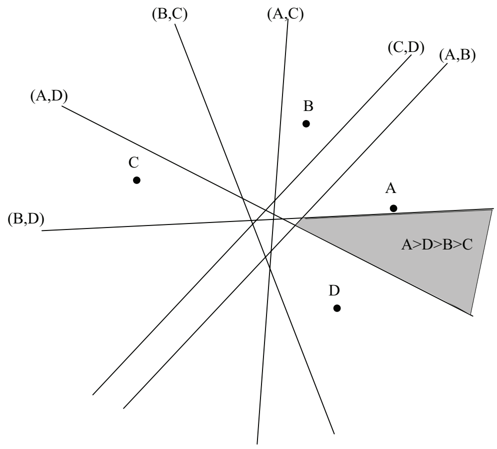


Figure 1. Fully nonmetric unfolding analysis: a) unidimensional; b) two dimensional

Source: own research based on: Cox and Cox 2001.

The unidimensional model can be generalized to several dimensions (see: Bennett, Hays 1960). In the multidimensional case the space can be divided up into isotonic regions by the hyperplanes defined by each pair of objects (see: Fig. 1b). The isotonic regions are labeled according to the preferred order for the points in a particular isotonic region. For example, all points in the shaded region in Figure 1b have the preferred ordering $A \succ D \succ B \succ C$.

The main problem with these unfolding techniques is that, certain preferred orderings cannot occur. This will happen when there are more than $r+1$ objects in *r*-dimensional space.

III. INTERNAL UNFOLDING

In internal unfolding, both the object configuration and the ideal points are derived only from preference matrix. We can conceive preference matrix as a submatrix of dissimilarity matrix in which the dissimilarity between objects and between respondents are treated as missing values (see: Fig. 2).

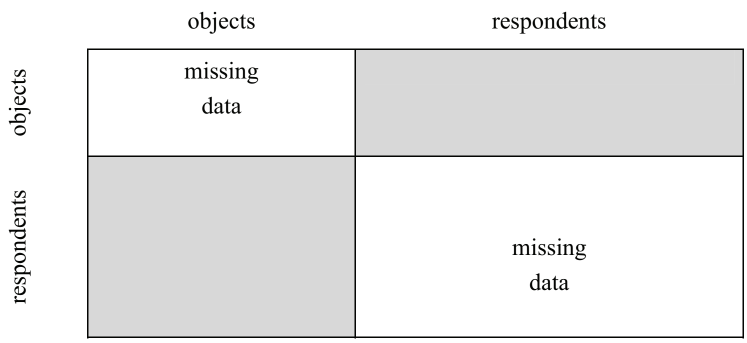


Figure 2. Preference matrix as a submatrix dissimilarity matrix
Source: own research based on: Borg and Groenen 2005.

For preference judgements δ_{ik} internal unfolding attempts to find configurations **X** and **Y** that minimize STRESS function:

$$S = \sqrt{\sum_{i<k}\left(d_{ik} - \hat{d}_{ik}\right)^2 \Big/ \sum_{i<k} d_{ik}^2}\ , \qquad (1)$$

where:

$d_{ik} = \sqrt{\sum_{a=1}^{r}\left(x_{ia} - y_{ka}\right)^2}$ – distance between $\mathbf{x}_i$ and $\mathbf{y}_k$,

$\hat{d}_{ik} = f(d_{ik})$ – monotonic regression of d_{ik} on δ_{ik}.

For nonmetric preference judgements disparities $\hat{d}_{ik}$ must satisfy the monotonic restriction:

$$\delta_{ik} \prec \delta_{i'k'} \Rightarrow \hat{d}_{ik} \le \hat{d}_{i'k'}$$

Unfolding solution can be computed by the majorization algorithm, where STRESS is reduced by iteratively taking Guttman transform. After K step of iteration the updates of **X** and **Y** becomes (see: Borg, Groenen 2005, p. 297–298)

$$\begin{aligned} \mathbf{X}^K &= \mathbf{V}_{11}^{+}\left[\mathbf{B}_{11}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})\mathbf{X}^{K-1} + \mathbf{B}_{12}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})\mathbf{Y}^{K-1}\right] \\ \mathbf{Y}^K &= \mathbf{V}_{22}^{+}\left[\mathbf{B}_{12}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})^T\mathbf{X}^{K-1} + \mathbf{B}_{22}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})\mathbf{Y}^{K-1}\right] \end{aligned} \tag{2}$$

where:

$\left[\mathbf{V}_{11}^{+}\right]_{n\times n} = m^{-1}(\mathbf{I} - (n+m)^{-1}\mathbf{1}\mathbf{1}^T)$,

$\left[\mathbf{V}_{22}^{+}\right]_{m\times m} = n^{-1}(\mathbf{I} - (n+m)^{-1}\mathbf{1}\mathbf{1}^T)$,

1 – a column vector of ones,

$\mathbf{B}_{12}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})$ – matrix with elements

$$b_{ik} = \begin{cases} \dfrac{-\delta_{ik}}{d_{ik}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})} & \text{for } d_{ik}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1}) \ne 0 \\ 0 & \text{for } d_{ik}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1}) = 0 \end{cases},$$

$\mathbf{B}_{11}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})$ – diagonal matrix with elements $b_{ii} = -\sum_k b_{ik}$,

$\mathbf{B}_{22}(\mathbf{X}^{K-1}, \mathbf{Y}^{K-1})$ – diagonal matrix with elements $b_{kk} = -\sum_i b_{ik}$.

An example

38 respondents made their judgements of preferences with regards to nine light entertainments on television (for details see: Zaborski 2002). The judgements forms preference matrix. Based on it, a multidimensional internal unfolding procedure was made using PREFSCAL algorithm. Configurations of ideal points and TV program points presents Fig. 3.

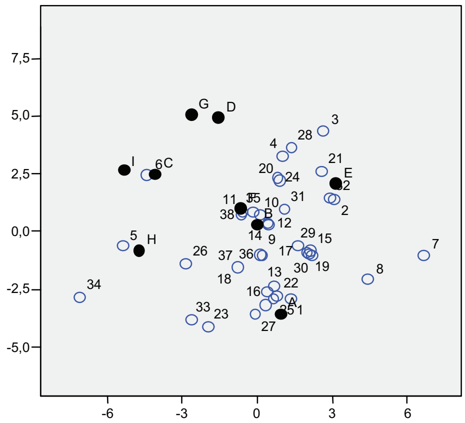


Figure 3. Configuration of points representing respondents (○) and programs (●)
Source: own research.

The positions of the points on the perceptual map show that the most popular programs are A, B, F and E. The layout of points indicates also that there are two biggest groups of respondents. One of them gathers round program A, and the second one most prefers programs B and F.

IV. EXTERNAL UNFOLDING MODELS

In external unfolding we assume that a similarity configuration of choice object is given. If we have preference data on these objects than external unfolding puts the ideal point for each subject in the space so that the closer this point lies to a point that represents an object, the more this object is preferred by individual. PREFMAP (PREFerence MAPping), which provides external analysis of the preference data, consist of four phases. The phases correspond to four preference-property models, as follows:

– vector model (phase IV),
– simple unfolding model (phase III),
– weighted unfolding model (phase II),
– general unfolding model (phase I).

The phases are in fact four distinct models, which are nested in the sense that each phase is a special case of the one preceding it.

The simplest **vector model** assumes that the subjects collapse the multidimensional stimulus space into one dimension representing the order of preference. We can use this model when a subject's liking for a stimulus is presumed to increase or decrease linearly along each dimension. The model assumes that the preferences would have the following form (Davison, 1984, p. 162):

$$\delta_{ki} = \sum_{a=1}^{r} w_{ka} x_{ia} + e_k, \qquad (3)$$

where:

δ_{ki} – the strength of person *k*' preference[1] ($k = 1, 2, \ldots, m$) for stimuli *i* ($i = 1, 2, \ldots, n$),

w_{ka} – linear regression weight,

x_{ia} – location of stimulus *i* along attribute a ($a = 1, 2, \ldots,$ r),

e_k – additive constant unique to subject *k*.

The vector model represents each subject's preference as a vector directed towards his region of maximum preference. It is a special case of the ideal point model whose ideal points are all infinitely far away from the points representing the choice objects. The projections of the stimulus points onto the vector reproduce the subject's preference values, and a preference ranking is interpreted as the order of the projections of the stimuli points on this line. Moreover, the angle which the vector makes with each dimension can be thought of as representing the salience of that dimension in the preference judgment. Individual differences in preference are expressed by the differing directions which the vectors have in the common space.

The simple unfolding model is the first of three models which assume nonlinear liking function. In this model, preferences have the following form (Davison, 1984, p. 163):

$$\delta_{ki} = \sum_{a=1}^{r} (y_{ka} - x_{ia})^2 + e_k, \qquad (4)$$

[1] it will be assumed that higher values of δ_{ki} designate smaller amounts of preference

where y_{ka} is the level along dimension *a* that the subject *k* considers ideal (ideal point coordinate).

The model assumes that subjects share the same set of reference dimensions but they differ in terms of where their ideal points are located in the space. Each subject has one most preferred point in the space (ideal point) which serves as a reference point to preference objects' scores by comparing their distances from ideal point. It is the circular ideal point model in which preference decreases with the square of the distance from the ideal point.

In **the weighted unfolding model** subjects are also assumed to have an ideal point and to share the same set of reference dimension, but they differ considerably in the value they attach to the dimensions of the space. According to the weighted model, preferences have the following form (Davison, 1984, p. 165):

$$\delta_{ki} = \sum_{a=1}^{r} w_{ka}^2 \left(y_{ka} - x_{ia}\right)^2 + e_k . \tag{5}$$

This is an elliptical ideal point model. The longest axis corresponds with the least important dimension, since it takes greater variation in this dimension to produce a given change in attribute strength. The shortest axis is the most important, since a small change in this dimension yield large changes in attribute strength.

The general unfolding model drops the assumption that subjects share the same fixed set of reference dimension. It allows them to structure the space as they wish by providing their own reference axes space. Each subject is viewed as having a specific, most preferred ideal point in the space, rotating the axes to his own reference dimensions, and then attaching an evaluative weight to each of them. Consequently this is the elliptical ideal point model with rotation. According to this model, preferences have the following form (Davison, 1984, p. 166):

$$\delta_{ki} = \sum_{a=1}^{r} w_{ka}^2 \left(y_{ka}^* - x_{ia}^*\right)^2 + e_k , \tag{6}$$

where: $\mathbf{X}^* = \mathbf{X}\mathbf{T}_k$, $\mathbf{Y}^* = \mathbf{Y}\mathbf{T}_k$, $\mathbf{T}_k$ – orthogonal rotation matrix idiosyncratic to subject *k*.

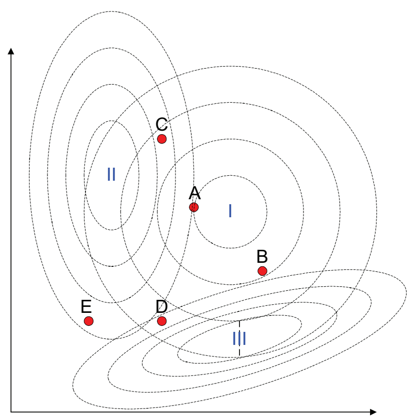


Figure 4. Iso-preference contours for ideal point models
Source: own research.

Iso-preference contours for three variants of ideal point model presents Fig. 4. Ideal point I has circular contours (simple unfolding model), ideal point II has elliptical contours whose axes parallel the dimensions (weighted unfolding model), and point III has elliptical contours whose axes have been rotated.

The applications of the external unfolding models for preference research presents among others: Borg and Groenen (2005), Zaborski (2008).

CONCLUSIONS

Unfolding models are a special case of multidimensional scaling methods. They try to determine the preferred mix of characteristic for a set of stimuli. Since individual preferences vary widely, the analysis of average preferences is rarely informative. Therefore, it is important to perform preference analysis at an individual level. Unfolding models do not explain the process by which a person makes a choice. Instead, they fit preference ratings to a stimulus space.

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GEOMETRYCZNA PREZENTACJA PREFERENCJI Z WYKORZYSTANIEM MODELI UNFOLDING

Analiza unfolding jest szczególnym przypadkiem skalowania wielowymiarowego. Zakłada ona, że respondenci postrzegają wybrane obiekty w tej samej przestrzeni, ale różnią się w ocenie ważności ich poszczególnych atrybutów. W analizie unfolding danymi są zazwyczaj oceny preferencji (np. rangowe uporządkowanie preferencji) respondentów względem badanych obiektów. Dane te można traktować jako podobieństwa między dwoma zbiorami: respondentów i obiektów. W wyniku analizy unfolding otrzymuje się w tej samej przestrzeni konfiguracje punktów reprezentujących oceniających oraz badane obiekty. Punkty reprezentujące respondentów to punkty „idealne", a odległości punktów reprezentujących obiekty od punktów idealnych odpowiadają rangowemu uporządkowaniu preferencji.

W artykule zaprezentowano wybrane modele analizy unfolding. Wskazano na różnicę między wewnętrzną i zewnętrzną analizą unfolding. Omówiono również model wektorowy będący szczególnym przypadkiem modelu punktu idealnego oraz ważone modele unfolding.