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THE MULTIDIMENSIONAL WEIGHTED ULTRASTRUCTURAL MODEL IN THE CROSS-SECTION OF EXPECTED STOCK RETURNS

Abstract. In the testing of CAPM and ICAPM models the problem of errors-in-variable is appears. To omit this problem some assumption about the relation between the random error variances of the model from which we estimate market beta and of the model from which we estimate risk premium was done. In this paper, we present weighted ultrastructural model, where the consistency of unknown parameters are discussed. The model where the replications dependent and independent variables allow us to calculate consistent estimators is considered.

The unknown variances are dependent on portfolio numbers i and independent on time t.

Moreover, for each factor, the possibility of the risk disturbance with variances dependent on i is permitted. We accept the assumption of variances of errors of the cross sectional model dependent on portfolio index, but we introduce homoscedastic errors indexed by time.

In this linear functional relationship using the maximum likelihood method the consistent estimators of unknown parameters and significance tests are calculated. This model can be use to testing of ICAPM parameter signification.

Key words: CPM and ICAPM model.

I. INTRODUCTION

The cross-sectional differences in asset expected returns have received high attention in finance literature. Sharpe (1964) and Lintner (1965) demonstrated that, in equilibrium, an expected return of financial asset must be linearly related to "beta", a measure of systematic risk with the market portfolio return $E_i = \gamma_0 + \beta_i \gamma_1$ $i = 1...N$, where E_i is the excess of expected return on asset i, β_i is asset i-th market beta, γ_0 is the measure disequilibria, and γ_1 is the market risk premium. However, since the empirical study of some markets implies that the risk premium should have got a multidimensional form.

In the literature there are many discussed models explaining cross-sectional differences. In many empirical studies the two-stage cross-sectional regression method is still preferred. In this approach the risk-return relation was estimated at two stages. At the first one beta estimates were obtained from the separate

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time-series regressions (TSR) for each asset and then these estimators were put into the second-pass cross-sectional regression (CSR). Since the independent variable in the CRS was measured with error, the second-pass estimator was subject to an errors-in-variables (EIV) problem where ordinary least square estimators are inconsistent. Some statistical problems connected with an errors-in-variables model were presented in the monographs of Kendall and Stuart (1979), Fuller (1987), Bunke and Bunke (1989). It is known that the model is nonidentifiable when the variables disturbances errors have a common normal distribution with unknown parameters (Reiersol, 1950). The discussion about assumptions and methods since consistent estimators of unknown parameters could be obtain is among others in Chan, Mak (1984). In the case when the variables are replicated the ultrastructural models can be adapted to estimate unknown parameters. The multidimensional case of model with replicated observations was discussed by Shalabh (2003). The multivariate models were also investigated by Fuller and Amemiya (1984), Gleser (1981), Isogawa (1985), Morton (1993), Jinadasa, Tracy (1990) and others. The historical overview of linear regression with errors in both variables was presented by J.W. Gillard (2006).

In the financial literature one of the most widely used methodologies of CRS parameters estimation was the two-pass regression approach, known as the Fama-MacBeth (FM) procedure developed by Fama and MacBeth (1973). Although early applications of the two-pass methodology focused on pricing with respect to a single market index, the procedure has been used more recently in the estimation of multifactor pricing relations. However, this approach neglected errors in variables.

Shanken (1992) discussed the traditional two-pass procedure and derives an asymptotic distribution of the CSR estimators when the special assumptions were introduced. He obtained the correction connected with EIV problem in this model. Kim (1995) proposed a new correction for the EIV problem in FM approach. In these approaches some assumption about variances of errors in CRS have been done.

In addition to the two-pass approach there were alternative procedures explored in empirical studies, for example, the maximum likelihood method or the generalized method of moments. The discussion of properties of some methods was in Chen and Kam (2004) paper. The simulation studies comparing some approaches were in the works of Shanken and Zhou (2007). As a result of using the most discussed methods was the T-consistent estimators of gamma, where T is the length of $\hat{\beta}$ estimation in TSR. The Kim approach involved N - consistent estimators of gamma, where N is the number of assets. The relevance of T-consistent or N-consistent estimators for the searching of the Polish market is questionable. In the emerging markets such approaches do not give satisfactory consistent results. The number of stocks has grown significantly lately. The range of listed firms is from 156 in the year 2002 to 354 in the year 2007. Big

firms have great weight in market capitalization (about 50 percent of capitalization mass is concentrated in twenty firms). It is very difficult to make empirical searching on such unstable market.

In this paper the author concentrate on finding such a method of testing the influence of β on the stock returns which does not require a long period of beta estimation and accepts moderate N. The idea of this approach is derived from the method of the parameters estimation in the ultrastructural model in paper of Dolby (1976) where one dimensional case was presented.

In this paper the author present the particular case of multidimensional model with replicated observations. In the case of special assumption the maximum likelihood method gives the *T*-consistent estimators where the *T* denotes the number of replications beta estimators. Using the maximum likelihood method the author obtained forms of estimators of unknown parameters in the case when *W* factors is considered.

II. THEORY AND ECONOMETRIC TESTS

2.1. Overview of some methodology

The standard cross-sectional model has the following form.

Let $X_t = \left[f_t^1, \ldots, f_t^W, R_{1t}, \ldots R_{Nt} \right]'$ be a $N + W$ - dimensional vector where $f_t^1, \ldots, f_t^W$ are the realizations of W factors at time t and $R_t^1, \ldots R_t^N$ are the returns on N assets at time *t*. Denote the mean and variance of X_t as $E(X_t) = [\mu_1, \mu_2]$ and $V = [V_{ij}]$ $i, j = 1, 2$. If the *W*- factor asset pricing model holds the expected returns of the *N* assets are given by

$$\mu_2 = 1_N \gamma_0 + \gamma_1 \beta_1 + \ldots + \gamma_W \beta_W \text{ and } \beta = [\beta_1, \ldots, \beta_W] = V_{21} V_{11}^{-1} \tag{1}$$

where 1_N is a vector of ones. The parameter γ_0 is called the zero-beta rate and $\gamma_1, \ldots \gamma_W$ are called the risk premia associated with *w* factor.

Suppose we have **T** observations of vector X_t. The popular two-pass CSR approach estimates the parameters of this relation by first estimating β using an OLS regression of R_t^i ($i = 1, \ldots, N$) on a constant and $f_t^1, \ldots, f_t^W$ as

$$R_t^i = \alpha_i + \beta_1^i f_t^1 + \ldots + \beta_W^i f_t^W + \varepsilon_t^i \quad t = 1, \ldots, T \tag{2}$$

thus we got estimator $\hat{\beta} = \left[\hat{\beta}_1, \ldots, \hat{\beta}_W \right]$, the second pass runs a CSR of some estimator of $\hat{\mu}_2$ on the matrix $[1_N, \hat{\beta}]$. It can be run in various ways, the most popular are the OLS (or GLS or WLS). In these approaches for each period t from running a cross-sectional regression of $R_t = \left(R_t^1, \ldots, R_t^N \right)'$ on $\hat{X} = [1_N, \hat{\beta}]$ we receive $\hat{\Gamma}_t = [\hat{\gamma}_0, \hat{\gamma}_1, \ldots, \hat{\gamma}_W] = \left(\hat{X}' \hat{X} \right)^{-1} \hat{X}' R_t$.

By repeating this CRS period by period we get a time series of $\hat{\Gamma}_t$ and the resulting estimate of $\hat{\Gamma}$ is the average of time series of $\hat{\Gamma}_t$.

The asymptotic analysis of this problem is presented in Shanken (1992) where he shows the T-consistency of $\hat{\Gamma}$ when the estimator error in beta goes to zero as T goes to infinity. He also calculated the adjustment to account for the estimation error in $\hat{\beta}$.

Another approach used in the two-pass methodology of estimating Γ is the maximum likelihood method discussed in Chen, Kan (2004) and Shanken (2007). To find the ML estimator the likelihood function of all the unknown parameters – the betas, covariance parameters and gammas need to maximize.

The other method is the Generalized Method of Moments (GMM), it was searched by Hansen (1982) and discussed by Shanken and Zhou (2007). An advantage of the GMM estimation is that it does not assume any hypothesis over the distribution of the series. However, this method has not been used to estimate the cross-sectional returns for the difficulty in finding out numerical solution to the problem.

In the methods discussed above the same $\hat{\beta}$ is used throughout the entire period. That is why using these methods to the emerging markets is questionable. In this situation it is better to allow $\hat{\beta}$ to change through the sample period and consider time-varying beta.

More precisely, let I_t denote the investors information at date t. Then, a conditional version of the standard linear beta pricing model is:

$$E\{R_{it} \mid I_t) = \gamma_0 + \gamma_1 \beta_{it}^1 + \ldots + \gamma_W \beta_{it}^W$$

where

$$(\beta_{t1}^i, \ldots \beta_{tW}^i)^T = Var^{-1}(f_1, \ldots, f_W \mid I_{t-1}) \left(Cov(R_t^i, f_1 \mid I_{t-1}), \ldots, Cov(R_t^i, f_W \mid I_{t-1})^T \right).$$

In this case the following relation is discussed:

$$R_{it} = \gamma_t^1 \beta_{it}^1 + \ldots + \gamma_t^W \beta_{it}^W + \gamma_{0t} + \eta_{it} \tag{3}$$

where

$$\hat{\beta}_{i,t-1}^w = \beta_{it}^w + \varepsilon_{it}^w$$

and $\hat{\beta}_{i,t-1} = \left(\hat{\beta}_{i,t-1}^1, \ldots \hat{\beta}_{i,t-1}^W \right)$ is the W-dimensional vector estimated from S time series data available up to t-1:

$$R_{is} = \alpha_i + \beta_{it}^1 f_s^1 + \ldots + \beta_{it}^W f_s^W + \sigma_s^i \quad s = t - S - 1, \ldots, t - 1 \quad i = 1, \ldots, N \tag{4}$$

The popular Fama-MacBeth methodology runs a CRS of returns $R_t = [R_{1t}, \ldots R_{Nt}]$ on $\hat{X} = [1_N, \hat{\beta}_{t-1}]$. The betas were estimated for several years of the data prior to each CSR. In this way, the time series of estimates were generated for each γ_j. Finally, as the estimate of gammas, the sample mean of the generated $\hat{\gamma}_t^1, \ldots, \hat{\gamma}_t^W, \hat{\gamma}_{0t}$ was taken. The use of the predictive beta $\hat{\beta}_{t-1}$ in the CRS estimation has some advantages. Firstly, the problem of a spurious cross-sectional relation arising from statistical correlation between returns and estimate market betas can be avoided. Secondly, the independence between the explanatory variable $\hat{\beta}_{t-1}$ and the regression error term in the CRS can be maintained. However, this approach does not take the errors in beta estimators into consideration.

The N-consistent estimator was discussed by Kim (1995) who develops an errors-in-variable problem of estimating Γ for the one-factor case. To omit the problem of the nonidentifiability it is assumed that matrix of the variance of η_t $\mathrm{var}(\eta_t) = \Sigma_\eta$ is the same as the matrix of the variance of σ_{t-1}, $\mathrm{var}(\sigma_{t-1}) = \Sigma_\sigma$. In this case the ratio between $\mathrm{var}(\eta_t) = \Sigma_\eta$ and $\mathrm{var}(\varepsilon_t^W) = \Sigma_\varepsilon^W$ ($W = 1$) could be estimated. It is known (Fuller (1987)) that in this situation the maximum likelihood method gives the consistent estimators. When the disturbances term of the market model is intertemporally homoscedastic with normal distribution the maximum likelihood methods gives analytical solution. The N-consistent $\hat{\Gamma}$ estimator can be

expressed as average of $\hat{\Gamma}_t$. The multidimensional case was discussed in Chen, Kan (2004) draft.

In the case when we have moderate number of portfolio such a kind of consistency is also irrelevant for the application.

2.2. The weighted ultrastructural model

In this section we discuss the methodology called in this paper "Weighted ultrastructual model" (WUM), where the beta estimator is allowed to change during the period and the error in beta estimator is under consideration.

To omit the EIV we apply the replications dependent and independent variables. Such an approach allows us to calculate T-consistent estimators, where T-denotes a number of replications. Another advantage of such an approach is the fact that we do not need a long history to estimate $\hat{\beta}$. It is particularly important for searching emerging markets.

Let us consider relations (3) and (4) where matrices of variance $\mathrm{var}(\eta_t) = \Sigma_\eta$ and $\mathrm{var}(\sigma_{t-1}) = \Sigma_\sigma$ are not equal. In this situation the matrix $\mathrm{var}(\varepsilon_t^w) = \Sigma_\varepsilon^w$ is unknown and the ratio between $\mathrm{var}(\eta_t) = \Sigma_\eta$ and $\mathrm{var}(\varepsilon_t^w) = \Sigma_t^w$ could not be estimated.

To obtain the consistent estimator we suggest that the variables $(\hat{\beta}_{t-1}^w, R_t)$, where $t = 1, \ldots, T$ will be replicated T times, and for each t, the $\hat{\beta}_{t-1}^w$ is estimated from data within range of S in (4).

Let us discuss the multidimensional case:

$$X_{it}^w = \hat{\beta}_{it}^w = \beta_{ii}^w + \varepsilon_{it}^w \quad \text{and} \quad Y_{it} = R_{it} = \gamma_0 + \sum_{w=1}^{W} \beta_{it}^w \gamma_w + \eta_{it}$$

$$w = 1, \ldots, W, \quad i = 1, \ldots N .$$

Moreover, for each w, we permit the possibility of the risk disturbance, so we assume that $E(\beta_t^w) = \beta^w$ and $\mathrm{var}(\beta_t^w) = \Sigma_s^w$.

Let us denote μ_t as $\mu_t = (x_t^1, x_t^2, \ldots, x_t^W, y_t)$ where $x_t^w = (X_{1t}^w, \ldots X_{Nt}^w)$ and $y_t = (Y_{1t}, \ldots Y_{Nt})$. The random variables μ_t are independent identically and dis-

tributed with the expected value: $\mu = \left(\beta^1, \ldots \beta^W, \sum_{w=1}^{W} \gamma_w \beta^w + \gamma_0 \right)$ and covariance

matrix V :

$$V = \begin{bmatrix} V_{11} & V_{12}^{'} \\ V_{12} & V_{22} \end{bmatrix},$$

where

$$V_{11} = [V_{11}^1 \ldots V_{11}^W] \times I_{NxW} \qquad V_{11}^w = \Sigma_{\varepsilon}^w + \Sigma_s^w$$

$$V_{22} = \Gamma^{'} \Sigma_s \Gamma + \Sigma_{\eta}$$

$$V_{12} = [V_{12}^1 \ldots V_{12}^W] \quad V_{12}^w = \gamma^w \Sigma_s^w$$

We consider the situation that factors $f_1, \ldots, f_W$ are uncorrelated so the matrix $\mathrm{var}(\varepsilon_{t-1}^w) = \Sigma_{\varepsilon}^w$ is diagonally. We assume that that matrices $\Sigma_{\eta}, \Sigma_{\varepsilon}^w, \Sigma_s^w$ are diagonal so $\Sigma_{\eta} = diag(\sigma_{\eta_1}^2, \ldots \sigma_{\eta_N}^2)$, $\Sigma_{\varepsilon}^w = diag(\sigma_{\varepsilon_1^w}^2, \ldots \sigma_{\varepsilon_N^w}^2)$, $\Sigma_s^w = diag(\sigma_{s_1^w}^2, \ldots \sigma_{s_N^w}^2)$. (This assumptions seems rather strong ones, but is necessary to calculate maximum likelihood estimators. It is interesting to consider the situation where this assumption can be relaxed.) .

The random variables μ_t are independent identically distributed with the expected value μ and covariance matrix V.

When we assume normal distribution of μ_t variables we could be able to obtain the T-consistent Γ estimators as the solution of likelihood equations (The discussion of consistency estimators in one dimensional case was in Dolby 1976). The consistency of estimators in W-dimensional case could be based on this argumentation).

The logarithm of the likelihood function was as follows:

$$\log L = C - \frac{T}{2} \log |V| - \frac{1}{2} \sum_{t=1}^{T} d_t V^{-1} d_t$$

where $d_t = \mu_t - \mu.$

Let V_ψ denote a matrix composed of the matrix V elements after derivation with respect to the ψ parameters (ψ being the arbitrary element of the unknown parameters set). The V matrix is the symmetric, thus:

$$\frac{\partial}{\partial\psi}\log L(\mu_j,\Psi) = m\left\{\frac{1}{2}tr(PV_\psi) - d_\psi^T V^{-1} d\right\}, \text{ with } P = V^{-1}(D-V)V^{-1}$$

and

$$D = \frac{1}{m}\sum_{j=1}^{m} d_j d_j^T.$$

It comes that

$$d_{\gamma_0}^T V^{-1} d = 0, \ d_{\beta_i^w}^T V^{-1} d = 0, \ \frac{1}{2}Tr(PV_{\gamma_w}) - d_{\gamma_w}^T V^{-1} d = 0, \ Tr(PV_{\sigma^2}) = 0,$$

where as σ^2 the parameters: $\sigma_{s_i^w}^2$, $\sigma_{\varepsilon_i^w}^2$ $\sigma_{\eta_i}^2$ is taken properly and $w = 1,\ldots,W$.

From the $Tr(PV_{\sigma^2}) = 0$ ($N(2W+1)$-relations), exploiting certain features of matrix algebra we obtained the following equations, for each i:

$$\sigma_{\varepsilon_i^w}^2 + \sigma_{s_i^w}^2 = \frac{1}{T}\sum_{t=1}^{T}(X_{it}^w - \beta_i^w)^2,$$

$$0 = \frac{1}{T}\sum_{t=1}^{T}(X_{it}^w - \beta_i^w)(X_{it}^v - \beta_i^v) \quad w \neq v \tag{5}$$

$$\sum_{w=1}^{W}\gamma_w^2 \sigma_{s_i^w}^2 + \sigma_{\eta_i}^2 = \frac{1}{T}\sum_{t=1}^{T}\left(Y_{ij} - \sum_{w=1}^{W}\gamma_w \beta^w - \gamma_0\right)^2$$

$$\gamma_w \sigma_{s_i^w}^2 = \frac{1}{T}\sum_{t=1}^{T}(X_{ij}^w - \beta_i^w)\left(Y_{ij} - \sum_{w=1}^{W}\gamma_w \beta^w - \gamma_0\right)$$

Furthermore, the matrix V^{-1} had a form:

$$V^{-1} = \begin{bmatrix} U_{11} & -U_{12}^T \\ -U_{12} & U_{22} \end{bmatrix},$$

where

$$U_{11} = [U]_{uv} \text{ and } U_{uv} = \begin{cases} \left(\gamma_w^2 \sigma_{s_i^w}^2 + \sigma_{\eta_i}^2 + \sum_{v \neq w} \frac{\gamma_v^2 \sigma_{s_i^v}^2 \sigma_{\varepsilon_i^v}^2}{\sigma_{s_i^v}^2 + \sigma_{\varepsilon_i^v}^2} \right) \frac{1}{k_i(\sigma_{s_i^w}^2 + \sigma_{\varepsilon_i^w}^2)} & \text{gdy } u = v \\ \\ \frac{\gamma_v \sigma_{s_i^v}^2}{\sigma_{s_i^v}^2 + \sigma_{\varepsilon_i^v}^2} \frac{\gamma_w \sigma_{s_i^v}^2}{\sigma_{s_i^{wv}}^2 + \sigma_{\varepsilon_i^w}^2} & \text{gdy } u \neq v \end{cases}$$

,

$$U_{12}^T = \begin{bmatrix} \left(\frac{\gamma_1 \sigma_{s_i^1}^2}{\sigma_{s_i^1}^2 + \sigma_{\varepsilon_i^1}^2} \frac{1}{k_i} \right)_{i=1}^N I_N \\ \vdots \\ \left(\frac{\gamma_W \sigma_{s_i^1}^2}{\sigma_{s_i^W}^2 + \sigma_{\varepsilon_i^W}^2} \frac{1}{k_i} \right)_{i=1}^N I_N \end{bmatrix} \quad U_{22} = \left(\frac{1}{k_i} \right)_{i=1}^N I_N \quad k_i = \sum_{w=1}^W \frac{\gamma_w^2 \sigma_{s_i^w}^2 \sigma_{\varepsilon_i^w}^2}{\sigma_{s_i^w}^2 + \sigma_{\varepsilon_i^w}^2} + \sigma_{\eta_i}^2.$$

Differentiating according to a appropriate parameter we have :

$$\sum_{1=1}^N \sum_{w=1}^W \frac{\gamma_w \sigma_{\varepsilon_i^w}^2}{(\sigma_{s_i^W}^2 + \sigma_{\varepsilon_i^W}^2)k_i}(X_{i.}^w - \beta_i^w) + \frac{1}{k_i}z_i = 0 \qquad (6)$$

$$\sum_{1=1}^N \sum_{w=1}^W \beta_i^w \left(\frac{\gamma_w \sigma_{\varepsilon_i^w}^2}{(\sigma_{s_i^W}^2 + \sigma_{\varepsilon_i^W}^2)k_i}(X_{i.}^w - \beta_i^w) + \frac{1}{k_i}z_i \right) = 0$$

To explicate unknown betas we should solve a linear relation:

$$
\begin{bmatrix}
\dfrac{\gamma_1^2\sigma_{\varepsilon_i^1}^2 + \sigma_{\eta_i}^2 + R_1}{\gamma_1\sigma_{\varepsilon_i^1}^2 k_i} & \cdots & \dfrac{\gamma_1\sigma_{\varepsilon_i^w}^2}{(\sigma_{s_i^w}^2 + \sigma_{\varepsilon_i^w}^2)k_i} \\
\vdots & \ddots & \vdots \\
\dfrac{\gamma_1\sigma_{\varepsilon_i^1}^2}{(\sigma_{s_i^1}^2 + \sigma_{\varepsilon_i^1}^2)k_i} & \cdots & \dfrac{\gamma_w^2\sigma_{\varepsilon_i^w}^2 + \sigma_{\eta_i}^2 + R_w}{\gamma_w\sigma_{\varepsilon_i^{w1}}^2 k_i}
\end{bmatrix}
\begin{bmatrix}
(X_{i.}^1 - \beta_i^1) \\
\vdots \\
(X_{i.}^w - \beta_i^w)
\end{bmatrix}
=
\begin{bmatrix}
-z \\
\vdots \\
-z
\end{bmatrix}
\tag{7}
$$

From (6) and (7) dependences we receive:

$$
X_{i.}^w - \beta_i^w = \frac{-\gamma_w\sigma_{\varepsilon_i^w}^2}{\gamma_1^2\sigma_{\varepsilon_i^1}^2 + \ldots + \gamma_W^2\sigma_{\varepsilon_i^W}^2}\,.
$$

Let us denote

$$
\alpha_i = \frac{1}{\gamma_1^2\sigma_{\varepsilon_i^1}^2 + \ldots + \gamma_W^2\sigma_{\varepsilon_i^W}^2}\,.
$$

From relations (5) we calculate that

$$
\sum_{w=1}^{W}\gamma_w^2\sigma_{\varepsilon_i^w}^2 = W_{yy}^i + \sum_w \gamma_w^2 W_{x_w x_w}^i - 2\sum_w \gamma_w W_{yx_w}^i + \sum_{\substack{w,v \\ w\neq v}} 2\gamma_w\gamma_v W_{x_v x_w}^i + z^2 =
$$

$$
= p(\gamma_1,\ldots,\gamma_W) + z_{i.}^2
$$

where

$$
z_{i.} = (Y_{i.} - \gamma_1 X_{i.}^1 - \ldots - \gamma_W X_{i.}^W - \gamma_0)
$$

and

$$
W_{\tau\omega}^i = \frac{1}{T}\sum (\tau_{it} - \tau_{i.})(\omega_{it} - \omega_{i.})\,.
$$

Finally, after tedious calculations we receive the equation:

$$\sum_{1=1}^{N} \alpha_i z_i = 0$$

and W equations:

$$\sum_{1=1}^{N} \alpha_i X_{i.}^{w} z_{i.} + \frac{\sum_{v=1}^{W} \gamma_v W_{x_w x_v}^{i} - W_{yx_w}^{i}}{p(\gamma_1, \ldots, \gamma_W)} \alpha_i z_{i.}^{2} = 0, \quad w = 1, \ldots, W$$

These equations may be determined numerically starting with the initial values equal to their estimators calculated when all variances are equal. In this case the forms of unknown parameters can be appointed contemporary.

In the one-dimensional case we have the formula:

$$\hat{\gamma}_0 = \ddot{R} - \hat{\gamma}_1 \ddot{\beta}$$

$$\sum_{i=1}^{N} \left(\gamma_1^2 (W_{\hat{\beta}R}^{i} s_{\hat{\beta}\hat{\beta}}^{i} - W_{\hat{\beta}\hat{\beta}}^{i} s_{\hat{\beta}R}^{i}) + \gamma_1 (W_{\hat{\beta}\hat{\beta}}^{i} s_{RR}^{i} - W_{RR}^{i} s_{\hat{\beta}\hat{\beta}}^{i}) + (W_{RR}^{i} s_{R\hat{\beta}}^{i} - W_{\hat{\beta}R}^{i} s_{RR}^{i}) \right) = 0$$

where

$$\ddot{R} = \frac{\sum_{i=1}^{n} \alpha_i R_{i.}}{\sum_{i=1}^{n} \alpha_i}, \quad \ddot{\beta} = \frac{\sum_{i=1}^{n} \alpha_i \hat{\beta}_{i.}}{\sum_{i=1}^{n} \alpha_i}$$

and

$$s_{XY}^{i} = \alpha_i (X_{i.} - \ddot{X})(Y_{i.} - \ddot{Y}), \quad \alpha_i = \frac{1}{\sigma_{\eta_i}^2 + \gamma_i^2 \sigma_{\varepsilon_i}^2}$$

where

$$\alpha_i = \left(p_i(\gamma_1) + (R_{i.} - \gamma_1 \hat{\beta}_{i.} - \gamma_0)^2 \right)^{-1}$$

and

$$p_i(\gamma_1) = W_{RR}^{i} - 2\hat{\gamma}_1 W_{\hat{\beta}R}^{i} + \hat{\gamma}_1^2 W_{\hat{\beta}\hat{\beta}}^{i}..$$

The variances have the forms:

$$\mathrm{var}(\gamma_1) = \frac{1}{m}\left(\sum_{i=1}^{n}\alpha_i\beta_i^2 - \frac{(\sum_{i=1}^{n}\alpha_i\beta_i)^2}{\sum_{i=1}^{n}\alpha_i}\right)^{-1}$$

$$\mathrm{var}(\gamma_0) = \frac{1}{m}\left(\sum_{i=1}^{n}\alpha_i - \frac{(\sum_{i=1}^{n}\alpha_i\beta_i)^2}{\sum_{i=1}^{n}\alpha_i\beta_i^2}\right)^{-1}.$$

For testing the significance of parameters $\gamma_1,\ldots,\gamma_W$ we can use a test based on the statistics $\dfrac{\gamma_s}{\sqrt{\mathrm{var}(\gamma_s)}}$ $s = 1,\ldots,W$. As the asymptotic variances of estimators γ_s we can take the appropriate diagonal elements of the matrix:

$$\left[-E\left(\frac{\partial^2}{\partial\psi\partial\omega}\log L(\mu_j,\Psi)\right)\right]^{-1}.$$

III. THE EMPIRICAL EXAMPLE

The empirical example cover only one stage: the influence of beta on portfolio returns. As comparison methods the author chose Fama MacBeth (1973) method (FM) both with and without Shanken (1992) correction because of its popularity, Kim method which has all the advantages of FM simultaneously taking into consideration beta error variables and WUM method presented in this paper.

There are two interpretations of the market portfolios used in the model, one as a value-weighed market return – WIG index (VW) and the other one as an equal-weighed market return (EW). The properties of the second index were discussed in Fisher's (1966) paper. The tests were not only made for different market portfolios, but also for the different periods in time-series regressions.

Firstly, the N portfolios was created after calculating $\beta_1,\ldots,\beta_n$ where n is a number of all assets. The N portfolios were constructed using the ranked values of the betas estimators. The estimators of beta were calculated in both cases

when as a market portfolio first the WIG (VW) and later the Fischer Index (EW) were taken into consideration. Later the N portfolios was created according to firm size and the book-to-market ratio (BV/MV).

Next, the returns of these portfolios were observed during T consecutive sub periods of a given period of time. Let us index t denoted sub periods from 1 to T.

The $\hat{\beta}_{it}$ is estimated from time series regression where the length of data taken to analysis was one year or two years. The data, used in the computations, were based on monthly periodization. Using the approaches discussed in previous section above for such grouped portfolios we obtain the estimators of γ_0 and γ_1 (see Table 1)

Below, for comparison we present the results when the length of time series of estimation $\hat{\beta}_i^t$ was one and two years.

Table 1. The estimation of parameters of $E\{R_{it} \mid I_t\} = \gamma_0 + \gamma_1\beta_{it}$ model

PANEL A: Sorting portfolios by beta VW							
TSR window:	2 years				1 year		
R^M:	VW		EW		VW		EW	
Method	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}
FM$_{WLS}$	0,2390	**3,4458**	0,2562	**3,4516**	0,2002	**3,0459**	0,2068	**3,1633**
SH$_{WLS}$	0,2390	**3,4458**	0,2562	**3,4516**	0,2002	**3,0458**	0,2068	**3,1633**
KIM	0,2116	**3,7864**	0,2136	**3,9463**	0,1743	**3,3766**	0,1881	**3,5758**
WUM	0,1876	**6,1970**	0,1712	**4,6037**	0,1432	**6,1379**	0,1235	**4,2940**
	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}
FM$_{WLS}$	−0,0014	−0,0334	−0,0103	−0,2145	0,0174	0,5774	0,0313	0,8185
SH$_{WLS}$	−0,0014	−0,0334	−0,0103	−0,2146	0,0174	0,5790	0,0313	0,8231
KIM	0,0076	0,2093	0,0123	0,3882	0,0185	0,7241	0,0049	0,2048
WUM	0,0219	0,6185	0,0444	0,9593	0,0367	1,5594	0,0653	**1,9200**
PANEL B: Sorting portfolios by beta EW							
TSR window:	2 years				1 year		
R^M:	VW		EW		VW		EW	
Method	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}
FM$_{WLS}$	0,1915	**3,2485**	0,2115	**3,4020**	0,1911	**3,1835**	0,2100	**3,5286**
SH$_{WLS}$	0,1915	**3,2484**	0,2115	**3,4020**	0,1911	**3,1835**	0,2100	**3,5286**
KIM	0,1867	**3,6533**	0,2016	**3,6622**	0,1665	**3,4698**	0,1796	**3,7520**
WUM	0,1586	**4,6220**	0,1730	**6,2585**	0,1504	**5,7113**	0,1685	**9,0013**
	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}
FM$_{WLS}$	0,0452	0,9574	0,0241	0,5484	0,0348	0,9376	0,0223	0,8333
SH$_{WLS}$	0,0452	0,9648	0,0241	0,5498	0,0348	0,9446	0,0223	0,8381
KIM	0,0415	0,9548	0,0222	0,5431	0,0344	1,0807	0,0127	0,6415
WUM	0,0603	1,3836	0,0431	1,2472	0,0344	1,1636	0,0317	1,1556

Table 1 (cont.)

PANEL C: Sorting portfolios by size								
TSR window:	2 years				1 year			
R^M :	VW		EW		VW		EW	
Method	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}
FM$_{WLS}$	0,2432	1,6587	0,2204	**3,4947**	0,0915	0,9957	0,1298	**2,6543**
SH$_{WLS}$	0,2432	1,6586	0,2204	**3,4947**	0,0915	0,9955	0,1298	**2,6542**
KIM	0,2869	**1,8966**	0,1889	**3,5822**	0,0593	0,7801	0,1354	**3,1524**
WUM	0,3439	**2,6501**	0,0200	0,2195	0,1041	1,1267	–0,5338	–1,4599
	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}
FM$_{WLS}$	0,0188	0,1373	0,0117	0,5071	0,1342	1,6276	0,0887	**2,4552**
SH$_{WLS}$	0,0188	0,1373	0,0117	0,5082	0,1342	1,6643	0,0887	**2,5885**
KIM	–0,0615	–0,4143	0,0205	0,9583	0,1520	**2,1676**	0,0624	**2,2936**
WUM	–0,2136	–1,3248	0,1742	**1,7110**	0,0610	0,5299	0,1851	**1,9302**
PANEL D: Sorting portfolios by BV/MV								
TSR window:	2 years				1 year			
R^M:	VW		EW		VW		EW	
Method	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}	γ_0	t_{stat}
FM$_{WLS}$	0,1562	**1,6989**	0,1936	**3,2530**	0,1071	1,4840	0,2053	**3,7857**
SH$_{WLS}$	0,1562	**1,6987**	0,1936	**3,2530**	0,1071	1,4837	0,2053	**3,7857**
KIM	0,2007	**1,9890**	0,1945	**3,5892**	0,1361	**2,2275**	0,2157	**4,2387**
WUM	0,3319	**1,8684**	0,2814	**2,6781**	0,0271	0,1405	0,1267	**1,8547**
	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}	γ_1	t_{stat}
FM$_{WLS}$	0,0940	0,9261	0,0203	0,7778	0,1113	1,0978	–0,0030	-0,1813
SH$_{WLS}$	0,0940	0,9327	0,0203	0,7817	0,1113	1,1088	–0,0030	–0,1814
KIM	0,0430	0,4065	0,0232	0,7728	0,0767	1,1100	–0,0182	–1,4898
WUM	–0,1649	–0,7515	–0,0877	–0,7989	0,1793	0,7577	0,0475	0,7090

Analyzing the results we can notice that one year window of estimation beta parameters leads to achieving results being in accordance with our pre-studies expectations. When we use one year window of estimation we obtain more remarkable estimators of unknown parameters.

Furthermore, comparing these methods we come to a conclusion that WUM is more stable and sensitive. In Panel A only this method shows the significance of the beta influence on the cross return in the case when EW was taken as a market portfolio and weak beta influence is applied for the case of VW. Analyzing the results of Panel C we conclude, that the beta influence on the case when EW was taken as a market portfolio, is remarkable, which is clearly visible only when the WUM method has been applied. This method indicated also the non-significant parameter γ_0. This example showed that the constructed tests presented in this paper (WUM) were quite sensitive and reliable and that is why it is worth further investigation.

The subsequent properties of this estimation will be researching using Monte-Carlo simulation.

IV. CONCLUSION

In this paper, we presented another approach for testing the significance of the cross regression parameters. This approach allows to receive consistent estimators of unknown parameters in the case when we have a short memory of beta parameter or a moderate number of portfolios. The properties of this approach are very important to test small markets with great fluctuations of asst price and number.

The numerical data from Warsaw Exchange Stock were used for the empirical example, with the portfolio defined either by WIG Index or Fischer Index.

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WIELOWYMIAROWY ULTRASTRUKTURALNY MODEL
W BADANIU PRZEKROJOWYCH STÓP ZWROTU

W badaniach empirycznych modelu CAPM (Capital Asset Pricing Model) lub jego wielowymiarowej wersji IACPM (Intemporal Capital Asset Pricing Model) testy sprawdzające poprawność modelu są dwuetapowe. Pierwszy etap to szacowanie regresji czasowych, z których estymatory wyznaczone metodą sumy najmniejszych kwadratów są zmiennymi niezależnymi dla etapu drugiego, gdzie bada się istotność parametrów regresji przekrojowej. W testowaniu istotności parametrów regresji przekrojowej pojawia się problem istnienia błędu w zmiennych objaśniających. Bez dodatkowych założeń o wariancjach tych błędów model taki jest nieidentyfikowalny.

W praktyce, najczęściej wprowadza się założenia o znajomości pewnych parametrów rozkładu zaburzeń błędów, o których zakłada się normalność lub stosuje się zabieg zminimalizowania błędu w zmiennych objaśniających. Podejścia takie doskonale spełniają swoją rolę w przypadku badania dużych rynków. W przypadku badania empirycznego Warszawskiej Giełdy Papierów Wartościowych

błąd w zmiennych objaśniających nie powinien być zaniedbywany. W pracy przedstawiono wielowymiarową wersję modelu ultrastrukturalnego, w którym w nieskorelowanych czynników jest obarczonych błędem obserwacji, które są niezależnymi zmiennymi losowymi o rozkładzie normalnym. Założono, że nieznane wariancje zależą od parametru i oraz nie zależą od parametru t. Dla rozwiązania problemu nieidentyfikowalności zastosowano replikację wszystkich zmiennych zależnych i niezależnej. Do wyznaczenia nieznanych wielkości zastosowano metodę największej wiarogodności oraz wykazano zgodność względem czasu T.

Prezentowane podejście może być używane do testowania istotności nieznanych parametrów modelu ICAPM.