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THE DEFINITION OF THE AVERAGE RATE OF RETURN OF OPEN PENSION FUNDS IN A MODEL WITH CONTINUOUS TIME¹

Abstract. In Polish law there exists a definition of the average rate of return of a group of pension funds which, as it was proved by Gajek and Kałuska (2000), does not satisfy some economic postulates. The authors presented other definitions of the average rate of return, both for discrete and continuous time. In this paper we propose a new definition which satisfies most of the postulates and is based on the model with continuous time.

Key words: average rate of return of a group of pension funds.

I. INTRODUCTION

Open Pension Funds are institutions which should invest their clients' money in the most effective way. There are lots of measures for the efficiency of these investments. The measures should be well constructed – it means that all changes of fund's assets, connected with any investment, should influence the given measure. It is very important to calculate the average rate of return of a group of pension funds. Firstly, having this result we can compare any fund with the group. The *good* fund should be more effective than, on average, the group. But, first of all, in the Polish law regulations (The Law on Organization and Operation of Pension Funds, Art. 173, Dziennik Ustaw Nr 139 poz. 934, Art 173; for the English translation see *Polish Pension...*, 1997) the definition of the average return of a group of funds determines a minimal rate for any fund. In case of deficit it is possible that this *weak* fund will have to cover it. It is always a very dangerous situation for funds. In the Polish law the following definition of the average return of a group of n pension funds can be found:

$$\bar{r}_0(T_1, T_2) = \sum_{i=1}^n \frac{1}{2} r_i(T_1, T_2) \cdot \left(\frac{A_i(T_1)}{\sum_{i=1}^n A_i(T_1)} + \frac{A_i(T_2)}{\sum_{i=1}^n A_i(T_2)} \right), \quad (1)$$

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where by $r_i(T_1, T_2)$ we denote the rate of the i -th fund during a given time period $[T_1, T_2]$ and by $A_i(t)$ we denote the value of i -th fund's assets at time t . After the year 2004 the results of funds for the last 36 months are verified once every half year, it means $[T_1, T_2] = [1, 36]$.

II. POSTULATES FOR THE MEASURE

In the paper of Gajek and Kałuszką (2000) the authors presented the following list of postulates for a correctly defined measure of the average rate of return $\bar{r}(T_1, T_2)$:

Postulate 1. In case when the group consists of one fund ($n = 1$) we get

$$\bar{r}(T_1, T_2) = r_1(T_1, T_2), \quad (2)$$

where

$$r_1(T_1, T_2) = \frac{w_1(T_2) - w_1(T_1)}{w_1(T_1)}. \quad (3)$$

Postulate 2. If all funds have the same values of their accounting units all the time, i.e.

$$\forall t \in [T_1, T_2] \quad w_1(t) = w_2(t) = \dots = w_n(t), \quad (4)$$

then

$$\bar{r}(T_1, T_2) = \frac{w_1(T_2) - w_1(T_1)}{w_1(T_1)}. \quad (5)$$

It means that if the unit's value changes in time in the same way in all funds, then it does not matter that the clients allocate from a fund to another fund. Their individual return rates will always be the same.

Postulate 3. If the number of units is constant at every fund during the time interval $[T_1, T_2]$, then

$$\bar{r}(T_1, T_2) = \frac{\sum_{i=1}^n A_i(T_2) - \sum_{i=1}^n A_i(T_1)}{\sum_{i=1}^n A_i(T_1)}. \quad (6)$$

When none of the clients change the fund or come into or out of the business, then any change of the assets A_i reflects only the investments results of the i -th fund.

Postulate 4. For each $t \in [T_1, T_2]$ we should get (multiplication rule):

$$1 + \bar{r}(T_1, T_2) = [1 + \bar{r}(T_1, t)][1 + \bar{r}(t, T_2)]. \quad (7)$$

Postulate 5. If the following condition holds

$$\exists k, l \in \{1, 2, \dots, n\} \quad \forall t \in \{T_1, \dots, T_2 - 1\} \quad \forall i \notin \{k, l\}$$

$$\frac{w_l(t+1)}{w_l(t)} \leq \frac{w_i(t+1)}{w_i(t)} \leq \frac{w_k(t+1)}{w_k(t)}, \quad (8)$$

then

$$r_l(T_1, T_2) \leq \bar{r}(T_1, T_2) \leq r_k(T_1, T_2).$$

Postulate 5 means that the average return rate is not greater than the return rate of the best fund, and is not smaller than the return rate of the worst fund.

Postulate 6. Let us assume that $n \geq 2$ and $k_i(t) = 0$ for $i = 1, 2, \dots, n$, $t \in [T_1, T_2 - \Delta t]$, where $\Delta t > 0$ is such that $T_2 - \Delta t > T_1$. Then

$$\lim_{\Delta t \rightarrow 0} \bar{r}(T_1, T_2) = r_1(T_1, T_2). \quad (9)$$

It means that if all the clients were members of one fund almost all the time then the average rate of return would be approximately equal to the rate of return of this fund.

Postulate 7. If for some $k \in \{1, 2, \dots, n\}$ the following condition holds

$$\max_{i \neq k} A_i(u) \leq \theta A_k(u) \quad \text{for } u \in [T_1, T_2], \quad (10)$$

then

$$\lim_{\theta \rightarrow 0} \bar{r}(T_1, T_2) = \frac{w_k(T_2) - w_k(T_1)}{w_k(T_1)}. \quad (11)$$

It means that the small funds have the limited influence on the rate of return of the group.

In the paper of Gajek and Kałuszką (2000) the authors show that the definition (1) does not satisfy several of these postulates. For example, it is easy to show that in case, when the number of units is constant at every fund during the time interval $[T_1, T_2]$, then

$$\bar{r}_0(T_1, T_2) \neq \frac{\sum_{i=1}^n A_i(T_2) - \sum_{i=1}^n A_i(T_1)}{\sum_{i=1}^n A_i(T_1)},$$

so the definition (1) does not satisfy the postulate 3. Moreover, considering an even number of funds, where half of them have the return rates equal to 50% and the rest of funds have the return rates equal to (−50%), we should get the real average return rate on the level 0%. But using formula (1) we get 12.5 %. The larger the differences between r_i , the stranger the values produced by \bar{r}_0 (see Białek (2005)). Finally, taking into consideration a group of Polish pension funds and the time period Jan 2002 – Dec 2004 (monthly observations) after the calculations we get $1 + \bar{r}_0(1,36) = 1.36407$ and (see also Białek (2005))

$$(1 + \bar{r}_0(1,12)) \cdot (1 + \bar{r}_0(12,36)) = 1.36384,$$

$$(1 + \bar{r}_0(1,24)) \cdot (1 + \bar{r}_0(24,36)) = 1.36381 \neq 1.36407.$$

So the definition (1) does not satisfy the postulate 4 neither. That is the reason for a construction an alternative definition of the average rate of return of a group of pension funds.

III. ALTERNATIVE MEASURE FOR THE AVERAGE RATE OF RETURN

In the papers of Gajek and Kałuszką (2001), and Białek (2005), (2008) the authors consider models with discrete time. It is justified because every fund is under an obligation to publish the results of investing every day – in particular the values of units must be known every day. But the insurance market seems to have continuous character, like a stock exchange. In the near future, when the

evaluation of units and assets will be made with a very high frequency, we could use the models with continuous time. These models can be used, provided we are able to approximate the considered processes by continuous functions. In this paper we propose a definition of the average rate of return of a group of funds based on the continuous time model.

Let us consider a group of n Open Pension Funds. We observe the following processes:

$w_i(t)$ – value (price) of unit of i -th fund at time t ,

$k_i(t)$ – number of unit of i -th fund at time t ,

$A_i(t)$ – the net assets of i -th fund at time t .

Hence we have

$$A_i(t) = w_i(t) \cdot k_i(t), \quad i = 1, 2, 3, \dots, n, \quad t \in [T_1, T_2], \quad (12)$$

where T is the time horizon for our observations.

Let us assume that functions $w_i(t)$, $k_i(t)$ and $A_i(t)$ are continuous on $[T_1, T_2]$ and $w_i(t)$ is differentiable on $[T_1, T_2]$. Let us signify

$$A(t) = \sum_{i=1}^n A_i(t). \quad (13)$$

The assets shares of the commodities at time t are defined by

$$A_i^*(t) = \frac{A_i(t)}{A(t)}, \quad \text{where } \sum_{i=1}^n A_i^*(t) = 1 \quad \text{for each } t \in [T_1, T_2], \quad (14)$$

Let us signify

$$\sigma_i(t) = \frac{w_i'(t)}{w_i(t)}. \quad (15)$$

Hence we have

$$\frac{dw_i(t)}{w_i(t)} = \sigma_i(t) dt. \quad (16)$$

Now, using the above significations and assumptions we can define the average rate of return of open pension funds as follows:

$$r(T_1, T_2) = \sum_{i=1}^n \frac{1}{w_i(T_1)} \int_{T_1}^{T_2} A_i^*(t) w_i(t) \sigma_i(t) dt. \quad (17)$$

Let us notice that in the case of $n = 1$ we have $A_1^*(t) = 1$ and (see postulate no. 1)

$$r(T_1, T_2) = \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} w_1(t) \sigma_1(t) dt = \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} dw_1(t) = \frac{w_1(T_2) - w_1(T_1)}{w_1(T_1)}.$$

The definition (17) satisfies most of the presented postulates, for example if we assumed that $w_1(t) = w_2(t) = \dots = w_n(t)$, we would get (see postulate no. 2):

$$\begin{aligned} r(T_1, T_2) &= \sum_{i=1}^n \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} A_i^*(t) w_1(t) \sigma_1(t) dt = \sum_{i=1}^n \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} \frac{k_i(t) w_1(t)}{\sum_{i=1}^n k_i(t) w_1(t)} w_1(t) \sigma_1(t) dt = \\ &= \sum_{i=1}^n \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} \frac{k_i(t)}{\sum_{i=1}^n k_i(t)} w_1'(t) dt = \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} \sum_{i=1}^n \frac{k_i(t)}{\sum_{i=1}^n k_i(t)} w_1'(t) dt = \\ &= \frac{1}{w_1(T_1)} \int_{T_1}^{T_2} w_1'(t) dt = \frac{w_1(T_2) - w_1(T_1)}{w_1(T_1)} = r_1(T_1, T_2). \end{aligned} \quad (18)$$

Example 1

Let us consider the group of $n = 5$ funds and let us assume that the values of units of all funds were identical on the time interval $[0, 3]$, it means: $w_1(t) = w_2(t) = \dots = w_5(t) = 100 + 20t$ PLN.

Let us assume that the net assets of funds have the following values:

$$A_1(t) = 2000; \quad A_2(t) = 1500 + 100t; \quad A_3(t) = 1700 + 50t^2;$$

$$A_4(t) = 2000 - 5t^3; \quad A_5(t) = 1300 + \sqrt{t} \quad (\text{the unit of money can be } 10^6 \text{ PLN}).$$

After calculations we get $r(0,3) = 60\%$ and $r_1(0,3) = r_2(0,3) = \dots = r_5(0,3) = 60\%$. The postulate no. 2 is satisfied.

Let us assume now that the values of units of funds differ in a fluctuation on the time interval:

$$w_1(t) = 100 + 20t; \quad w_2(t) = 80 + 20t^2; \quad w_3(t) = 100 - 10t;$$

$$w_4(t) = 120 + 15t; \quad w_5(t) = 100 + \sqrt{t};$$

and we correct the assets of the first of funds as follows:
 $\tilde{A}_1(t) = 10 \cdot A_1(t) = 20000$.

After calculations we get:

$r(0,3) = 59,7018\%$ and, as in the previous case, we have $r_1(0,3) = 60\%$.

We can notice that the postulate no. 7 is satisfied – we have $r(0,3) \approx r_1(0,3)$.

IV. PROPERTIES OF THE NEW DEFINITION

Property 1

If it holds $\forall i \in \{1, 2, \dots, n\} \quad w_i(t) = \text{const}$ on $[T_1, T_2]$, then we have $r(T_1, T_2) = 0$.

The property 1 is of axiomatic character. In the case of constant values of units of all funds, the average rate of return must equal zero.

Property 2

Let us assume that $\forall i \in \{1, 2, \dots, n\} \quad A_i^*(t) = A_i^* = \text{const}$. It means we assume that the relation between the given fund's assets and global assets from all funds is constant. Let us notice that under the above assumption we get

$$r(T_1, T_2) = \sum_{i=1}^n \frac{1}{w_i(T_1)} \int_{T_1}^{T_2} A_i^* w_i'(t) dt = \sum_{i=1}^n A_i^* \frac{1}{w_i(T_2)} \int_{T_1}^{T_2} dw_i(t) = \sum_{i=1}^n A_i^* r_i(T_1, T_2). \quad (19)$$

In the case when $A_i^* = A_j^*$ for $i \neq j$, we get from (19)

$$r(T_1, T_2) = \frac{\sum_{i=1}^n r_i(T_1, T_2)}{N}. \quad (20)$$

We come to the following conclusion: under the above assumption the average rate of return of funds defined in (17) is an arithmetic mean of returns of funds.

Certainly if $\forall i \in \{1, 2, \dots, n\} \quad r_i(T_1, T_2) = \tilde{r}(T_1, T_2)$, then we get from (19) that also $r(T_1, T_2) = \tilde{r}(T_1, T_2)$.

Property 3

If the numbers of units of all funds are the same on the time interval $[T_1, T_2]$, it means: $\forall i \in \{1, 2, \dots, n\} \quad k_i(t) = k(t)$,

then we can calculate the average rate of return of a group of funds as follows:

$$r(T_1, T_2) = \frac{1}{2} \sum_{i=1}^n \frac{1}{w_i(T_1)} \int_{T_1}^{T_2} \frac{dw_i^2(t)}{w(t)}, \quad (21)$$

where

$$w(t) = \sum_{i=1}^n w_i(t). \quad (22)$$

Property 4

Let us consider a function of t as follows:

$$r(T_1, t) = \sum_{i=1}^n \frac{1}{w_i(T_1)} \int_{T_1}^t A_i^*(s) w_i(s) \sigma_i(s) ds, \quad (23)$$

where $t = T_1 + \Delta t$.

Let us notice that we get from (23)

$$\frac{dr(T_1, t)}{dt} = \sum_{i=1}^n \frac{1}{w_i(T_1)} A_i^*(t) \frac{dw_i(t)}{dt}. \quad (24)$$

Hence, for small values of Δt , we get:

$$r(t, T_1) - r(T_1, T_1) = r(t, T_1) \approx \sum_{i=1}^n \frac{1}{w_i(T_1)} A_i^*(t) \cdot (w_i(t) - w_i(T_1)). \quad (25)$$

From the formula (25) we get that for small time intervals the average return can be written as

$$r(T_1, T_2) \approx \sum_{i=1}^n A_i^*(T_2) r_i(T_1, T_2). \quad (26)$$

The right side of the formula (26) is a weighted arithmetic mean of returns of all funds.

VI. CONCLUSIONS

The presented definition of the average rate of return of Open Pension Funds, satisfies most of the postulates proposed by Gajek and Kałuszka (2000). It is a natural alternative for the definition (1) and it is based on continuous time model. The proposed definition can be used provided we are able to approximate the considered processes by continuous functions. The presented properties of the new measure confirm its proper construction.

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*Jacek Białek***DEFINICJA PRZECIĘTNEJ STOPY ZWROTU OTWARTYCH FUNDUSZY
EMERYTALNYCH W MODELU Z CZASEM CIĄGŁYM**

W polskim prawie funkcjonuje definicja przeciętnej rentowności grupy funduszy emerytalnych, która – jak pokazali Gajek i Kałuska (2000) – nie spełnia pewnych ekonomicznie zasadnych postulatów. Autorzy ci zaproponowali inne definicje dla przeciętnego zwrotu grupy funduszy, zarówno dla czasu dyskretnego jak i ciągłego. W niniejszym artykule proponowana jest nowa definicja, która spełnia większość postawionych postulatów i jest oparta na modelu z czasem ciągłym.