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Preface

These notes are meant for graduate students and young researchers interested in the theory of noncommutative L^p and Orlicz spaces. We assume the reader has a basic knowledge of functional analysis, in particular that he or she is acquainted with the spectral theory and functional calculus of both bounded and unbounded self-adjoint operators. Knowledge of the theory of operator algebras is not strictly indispensable, but would be very helpful. In chapter 1 we have gathered results from that theory needed for the rest of the book. All this material is standard, and we highly recommend the prospective reader to have on her or his shelves at least one of the following excellent sources: "Operator algebras and quantum statistical mechanics 1" by Bratteli and Robinson, two volumes of "Fundamentals of the theory of operator algebras" by Kadison and Ringrose [**KR83, KR86**], "Lectures on von Neumann algebras" by Strătilă and Zsidó [**SZ79**] or "Theory of Operator Algebras I" by Takesaki [**Tak02**]. For more advanced material, Takesaki's "Theory of Operator Algebras II" [**Tak03a**] and Strătilă's "Modular Theory in Operator Algebras" [**Str81**] are among the best. Blackadar's encyclopedic "Operator algebras" [**Bla06**] is excellent for those who would like to find a piece of information quickly.

This is of course not the first set of notes to be written on noncommutative L^p-spaces. The mathematical community has for example long been served by the notes of Marianne Terp [**Ter81**]. One major difference in the present set of notes is the extent to which we have incorporated on the one hand the technologies of noncommutative decreasing rearrangements as developed by Fack and Kosaki [**FK86**], and on the other the fairly recent technology of Orlicz spaces for general von Neumann algebras [**Lab13**]. The theory of Orlicz spaces we present here stems from the research interests of the second-named author. As yet no exposition of this theory paralleling Terp's notes on L^p-spaces is in existence. Part

of the aim of the present set of notes is to remedy this shortcoming. The importance of Orlicz spaces is explained in the Introduction, and on the basis of that explanation we do feel that these spaces are worthy of serious study. In addition to the issues mentioned in the Introduction, the refinement brought about by the development of this technology have enabled us to come up with a much smoother, more streamlined path through the theory of Haagerup L^p spaces.

Chapter 1 revises essential background with chapters 2 to 4 presenting what may be regarded as the noncommutative theory of measures and measurable functions, and chapters 5 to 7 the noncommutative theory of spaces of measurable functions. Readers wishing to get to the tracial theory of noncommutative L^p-spaces as quickly as possible should at least master the material on traces and τ-measurable operators in chapters 2 and 3, and then read chapter 4 and section 5.1 of chapter 5. The theory of Haagerup L^p-spaces, which is valid for arbitrary von Neumann algebras, is ultimately presented in 7. For this theory to be comprehensible all of chapters 4 and 5 needs to be covered (including section 5.2) and a high degree of familiarity achieved with Theorems 6.50, 6.62, 6.65, 6.72, 6.74, and of Propositions 6.61, 6.67, and 6.70. The theory of Haagerup L^p-spaces is deeply intertwined with the theory of crossed products. Readers wishing to ultimately master the deeper subtleties of Haagerup L^p-spaces are therefore advised to at some stage take the time to master chapter 6 in its entirety.

Manuscripts which greatly assisted in galvanising our thoughts regarding these notes include the iconic notes of Terp [**Ter81**], the extremely useful paper of Fack and Kosaki [**FK86**], the more recent very elegant set of notes by Xu [**Xu07**], and the unpublished monograph of Dodds, de Pagter and Sukochev [**DdPS**]. (We are deeply grateful to the authors for sharing a draft copy with us.) There are of course many people who in different ways have directly or indirectly contributed to getting the notes to the point where they are now. People like Jurie Conradie, Pierre de Jager and Claud Steyn, who read large tracts of the preliminary draft of these notes. However, some individuals deserve special mention.

SG: First of all, I would like to thank Martin Lindsay and Jill Anderson for their long-standing friendship and extraordinary hospitality. The time I spent with Martin was also very important for my life as a mathematician. I am grateful to Oleg Tikhonov for making me acquainted with a series of excellent papers of Trunov [**Tru78, Tru82, Tru85**] and other

representatives of Kazan's group [**STS02**], and turning my attention to the papers [**Gar79**] [**HKZ91**]. Warm thanks to my colleagues and friends Andrzej Łuczak, Adam Paszkiewicz, Hanna Podsędkowska, Andrew Tomlinson and Rafał Wieczorek for their help in reducing the number of errors in this work.

LL: I would like to particularly thank Adam Majewski, whose insight into physics and ability to apply noncommutative L^p spaces to concrete problems in physics was a constant inspiration, and my wife who in so many ways supported and encouraged me all along. I also want to acknowledge the kindness and grace of God who carried me through all the dark days of physical challenges and gave me the strength to finish this work.

Both of us are grateful to Adam Skalski, for his very insightful review of the book.

Introduction

The theory of operator algebras originated from a paper of John von Neumann [**vN30**] from 1930, followed by a series of papers of Francis Murray and himself ([**MvN36**], [**MvN37**], [**vN40**], [**MvN43**], [**vN49**]) from 1936 to 1949 on "rings of operators". The measure theoretic or probabilistic aspects of those such rings equipped with trace-like functionals were clear to von Neumann from the very beginning. In [**MvN36**] the authors write about the trace value $T(A)$ of an operator A as an "*a priori* expectation value of the observable A". This is even more strongly pronounced in section 8 of the same paper, where the dimension function is used to measure projections, just as we measure sets in classical measure theory. The next major steps forward were made by Irving Segal [**Seg53**] and Jacques Dixmier [**Dix53**] in 1953. For a semifinite von Neumann algebra with a faithful normal semifinite trace, Segal introduced the algebra of measurable operators and introduced L^1, L^2 and L^∞, whilst Dixmier defined all the L^p-spaces. The term "von Neumann algebra" was coined by Dixmier in [**Dix57**], a first book on the subject of "rings of operators". (Now one can use either the second French edition [**Dix96a**] or the English one [**Dix81**].) The 1975 paper of F.J. Yeadon then provided a very complete discussion of L^p-spaces in the tracial case [**Yea75**]. The important notion of a τ-measurable operator was introduced by Edward Nelson in [**Nel74**]. The significance of this concept is that it enables one to realise the tracial L^p-spaces as concrete spaces of operators.

Further progress would've been impossible without the modular theory of Minoru Tomita and Masamichi Takesaki [**Tak70**]. Using this theory, the first constructions of non-commutative L^p-spaces for general von Neumann algebra appeared at the end of 70s. They were due to Uffe Haagerup [**Haa79a**] and Alain Connes [**Con80**]. The Haagerup construction was beautifully presented in the notes of Marianne Terp [**Ter81**], and the Connes construction by Michel Hilsum [**Hil81**]. These are far

from being the only known constructions; one may for example note the ingenious construction of Huzihiro Araki and Tetsuya Masuda [**AM82**] using only relative modular operators, but they are certainly the most successful ones. Especially Haagerup's L^p spaces are now 'standard' — if we speak about non-commutative L^p-spaces for general von Neumann algebras without giving any names, we certainly mean Haagerup's spaces. One more approach to non-commutative L^p-spaces needs to be mentioned here: they can all be viewed as interpolation spaces. Marianne Terp [**Ter82**] first proved this fact in the setting of Connes-Hilsum L^p-spaces with Hideki Kosaki then shortly afterward publishing a "Haagerup oriented" interpolative construction of L^p-spaces in the state case (σ-finite algebras) [**Kos84a**].

Non-commutative L^p spaces feature in a variety of applications, of which we only mention one of the first ones, by Ray Kunze [**Kun58**], to L^p-Fourier transforms on locally compact unimodular groups. This very early paper is interesting since it also combines the results of Segal and Dixmier on L^p-spaces, to, for the first time, realize these spaces as spaces of measurable operators. Among other results, Kunze proved a Hausdorff-Young inequality in this setting. One would expect a generalization of his results to Haagerup spaces and non-unimodular groups, and in fact such results were obtained by Terp [**Ter17**]. The application of noncommutative harmonic analysis of Hausdorff locally compact groups also clearly shows that these spaces occur naturally, and not as some very exotic pathological phenomenon. Specifically given a Hausdorff locally compact group G, one may form the group von Neumann algebra which is the von Neumann algebra generated by the left-shift operators on $L^2(G)$. If one is interested in quantum harmonic phenomena, it then makes sense to do Fourier analysis on the noncommutative L^p spaces associated with this algebra as Kunze did. The nature of the algebra one has to deal with depends on the nature of the group one starts with, with "wilder" groups leading to wilder group von Neumann algebras. There is in fact now renewed interest in noncommutative harmonic analysis with a lot of attention being given to Quantum Groups and Fourier multipliers. There are a large number of researchers currently working on this topic - too many to all mention here. We therefore content ourselves with mentioning a mere sampling of papers ([**Cas13, CdlS15, CFK14, DKSS12, DFSW16, FS09, JMP14, JNR09, CPPR15, NR11**]) involving Martijn Caspers, Matt Daws, Mikael de la Salle, Pierre Fima,

Uwe Franz, Marius Junge, Pawel Kasprzak, Tao Mei, Mathhias Neufang, Stefan Neuwirth, Javier Parcet, Mathilde Perrin, Éric Ricard, Zhong-Jin Ruan, Adam Skalski, Piotr Soltan, and Stuart White. At the same time we offer our sincere apologies to authors who may feel slighted by their exclusion from this incomplete list.

Noncommutative Orlicz spaces started with the papers of Wolfgang Kunze on the one hand and Peter Dodds, Theresa Dodds and Ben de Pagter on the other. The first introduces these spaces directly in a very algebraic way (see [**Kun90**], [**ARZ07**]), whereas the other introduces them as part of the category of Banach function spaces (see [**DDdP89**]). Ultimately these two approaches can be shown to be equivalent (see [**LM11**]). It is interesting to note that the papers [**Kun90, DDdP89**] were developed independently with each sparking a tradition which for some time developed independently of each other. This can be seen by looking at the citation profile of these two papers. It was only recently that this theory was extended by Labuschagne to even the type III context [**Lab13**].

At this point we should note that the paper of Dodds, Dodds and Pagter [**DDdP89**] in no small way helped synthesize ideas of several authors that had been brewing behind the scenes for some time and as such helped to kick-start a very successful and burgeoning theory of noncommutative rearrangement invariant Banach function spaces, which has attracted a very large number of adherents. Readers wishing to know more should consult the survey paper of Ben de Pagter [**dP07**] and the references therein. Yet despite the great success of this theory, it is at present only know to hold in the semifinite setting. The only known extension of this theory to the type III setting, is the theory of Orlicz spaces which we present in these notes. Though the type III theory of Orlicz spaces has received little attention to date, our hope is that a deeper understanding of this theory by the mathematical community, will help to pave the way for the ultimate extension of the theory of rearrangement invariant Banach function spaces to the type III setting. However as can be seen from the discussion below, there are justifications reaching beyond mathematics for studying these spaces.

Much of the current motivation for studying these spaces comes from Physics. Although we shall not cover any of these applications in these notes, it is nevertheless instructive to review them. The issue of return to equilibrium is for example still not fully settled in Quantum Statistical Mechanics (QSM). For this issue to be settled a better understanding of

entropy for QSM is required. At a naive level one may for semifinite algebras consider the formal quantity $\tau(f \log(f))$ as starting point for a quantum theory of entropy. The problem with the current QM formalism where the pair (L^1, L^∞) is used as "home" for states and observables, is that the L^1 topology is notoriously bad at distinguishing states with "good" entropy. In this topology one may have a sequence (f_n) of positive elements of L^1 converging to some f for which $\tau(f \log(f))$ is a well-defined finite quantity, but with $\tau(f_n \log(f_n))$ infinite for each n. So a better technology for studying entropy is required. In addition to the above log-Sobolev inequalities also play an important role in studying the "return to equilibrium" issue (see the concluding remarks in for example [**Zeg90**]). So such a technology should also be well suited to such inequalities. These two factors already strongly suggest the use of noncommutative Orlicz spaces as the appropriate technology. However the classical theory of entropy itself also suggests Orlicz spaces as the appropriate tool.

Let us quote from [**LM**]:*The origins of a quantity representing something like entropy may be found in the work of Ludwig Boltzmann. In his study of the dynamics of rarefied gases, Boltzmann formulated the so-called* spatially homogeneous Boltzmann equation *as far back as 1872, namely*

$$\frac{\partial f_1}{\partial t} = \int d\Omega \int d^3 v_2 I(g, \theta) |\mathbf{v}_2 - \mathbf{v}_1| (f_1' f_2' - f_1 f_2)$$

where $f_1 \equiv f(\mathbf{v}_1, t), f_2' \equiv f(\mathbf{v}_2', t), \ldots,$ are velocity distribution functions, $I(g, \theta)$ denotes the differential scattering cross section, $d\Omega$ the solid angle element, and $g = |\mathbf{v}|$.

The natural Lyapunov-type functional for this equation is the so-called Boltzmann H-function, which is

$$H_+(f) = \int f(x) \log f(x) dx,$$

*where f is a postulated solution of the Boltzmann equation. The connection to entropy may be seen in the fact that the classical description of continuous entropy S differs from the functional H only by sign. Hence Boltzmann's H-functional may be viewed as the first formalisation of the concept of entropy. Lions and DiPerna were the first to rigorously demonstrate the existence of solutions to Boltzmann's equation. (Lions later received the Fields medal for his work on nonlinear partial differential equations.) Their solution was for the density of colliding hard spheres, given general initial data (see for example [**DL88**, **DL89**] for a sampling of this*

*work). Villani subsequently announced, see [**Vil02**], Chapter 2, Theorem 9, that for particular cross sections (collision kernels in Villani's terminology) weak solutions of Boltzmann equation are actually in $L \log(L+1)$. So one consequence of the work by these authors was to give a strong indication that the Orlicz space $L \log(L+1)$ is the appropriate framework for studying entropy-like quantities like the Boltzmann H-functional.*

Physicists who on the basis of these facts strongly advocated the use of noncommutative Orlicz spaces for studying QSM include Ray Streater [**Str04**], Boguslaw Zegarlinski [**ARZ07**] and Adam Majewski [**Maj17**]. The 1995 paper of Giovanni Pistone and Carlo Sempi [**PS95**] added another strand of thought to this mix of ideas, namely the concept of regular observables. In [**PS95**] the authors introduce a moment generating class of random variables which they call *regular random variables*, and then go on to show that the weighted Orlicz space $L^{\cosh -1}(X, \Sigma, f \, d\nu)$ forms the natural home for these regular random variables. The significant fact regarding this concept, is that the space $L^{\cosh -1}$ is (up to isomorphism) the Köthe dual of $L \log(L+1)$. One may therefore expect that at the quantum level, noncommutative versions of $L^{\cosh -1}$ would similarly be home to *regular observables*. This was formalised in the paper [**LM11**]. So the picture that begins to emerge is that the (dual) pairing $(L \log(L+1), L^{\cosh -1})$ may be better suited to studying and refining QSM (and ultimately clarifying the issue of return to equilibrium) than the more classical pairing of (L^1, L^∞). Readers should note that such a paradigm shift will in no way impact the well-established paradigm for elementary QM pioneered by Paul Dirac et al., since in the case of $B(H)$ the two approaches agree (as was shown in [**LM11**]). The utility of this pairing for the noncommutative context was strongly demonstrated in [**ML14**].

Thus far all the theory we have presented was developed in the context of semifinite von Neumann algebras. That in itself is a problem since it is known that many of the most important von Neumann algebras in Quantum Physics are necessarily type III algebras (see [**Yng05**]). One may also note the work of Robert Powers. In [**Pow67**] Powers studied representations of uniformly hyperfinite algebras. The types of algebras Powers studied may in Physical terms be regarded as thermodynamic limits of an infinite number of sites, with the algebra $M_2(\mathbb{C})$ associated to each site. (See [**Maj17**] for details of the Physical interpretation of Powers' result.) Yet despite the simplicity of these "local" algebras, the algebra obtained in the limit turned out to be a type III algebra. To

appreciate the significance of this fact, readers should take note of the fact that type III algebras exhibit markedly different behaviour than their semifinite cousins. Consider for example the work of Stephen Summers and Reinhard Werner [**SW87**] who made the almost shocking discovery that in local algebras corresponding to wedge-shaped regions in QFT, Bell's inequalities are maximally violated in every single vector state! Thus for a theory of noncommutative Orlicz spaces to fully address the challenge emanating from Physics, one dare not ignore the type III setting. A theory of Orlicz spaces for type III algebras therefore had to be developed. This was eventually done in [**Lab13**], and then slightly refined in [**LM**]. However type III algebras do not admit f.n.s. traces. Hence in passing to type III algebras an alternative prescription for entropy to the naive one of $\tau(f \log(f))$ needed to be found. This was ultimately done in [**ML18**]. The contribution of the paper [**LM**] was to show that complete Markov dynamics canonically extends to even the most general noncommutative $L^{\cosh -1}$ spaces. The theory of noncommutative Orlicz spaces is therefore now well set for an onslaught on the challenge of further refining and developing QSM.

It is of interest to note that in a recent preprint [**LM17**], noncommutative Orlicz spaces were also shown to naturally occur in Algebraic Quantum Field Theory. The significance of these spaces for Physics therefore reaches beyond just QSM. Readers wishing to know more about these applications to physics and also about what still needs to be done should consult not just the references mentioned above, but pay careful heed to the paper [**Maj17**] and the references mentioned therein. This paper clearly outlines some of the remaining challenges and the development they require.

Preliminaries

In this chapter we gathered various facts from functional analysis and the theory of operator algebras that we will use freely in the sequel. There are many excellent books on functional analysis, so the reader will find the material we use, for example that on spectral theory, without any problems. One book that stands out for future operator-algebraists is Gert Pedersen's "Analysis Now" [**Ped89**].

Section 1.1 sets the stage for future material on operator algebras. In particular, it identifies various classes of elements of a C^*-algebra and introduces functional calculi that will be used in the sequel. In Section 1.2 we deal mainly with various topologies in $B(H)$ and with Borel functional calculus. Section 1.3 gathers most important notions and results on von Neumann algebras, together with a Structure Theorem 1.86. Sections 1.4 and 1.5 deal with general unbounded operators and with those unbounded operators that "almost belong" to a von Neumann algebra. Finally, Section 1.6 introduces a useful notion of generalized positive operators, corresponding to not-necessarily densely defined unbounded positive self-adjoint operators.

1.1 C^*-algebras

In this section we will give definition and the most basic properties of C^*-algebras. In addition to the monographs already mentioned in the Introduction, the reader interested in the theory of C^*-algebras could learn a lot from the classical book of Naimark [**Naĭ72**] and books of Sakai [**Sak71**], Dixmier [**Dix96b**], Pedersen [**Ped18**], Murphy [**Mur90**], Arveson [**Arv76**], Fillmore [**Fil96**] and Davidson [**Dav96**].

DEFINITION 1.1. An *algebra with involution* or a **-algebra* $\mathcal{A}$ is a (complex) algebra with a map $a \mapsto a^*$ from $\mathcal{A}$ into itself satisfying $(\lambda a)^* = \bar{\lambda}a^*$, $(a + b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $a^{**} = a$.

DEFINITION 1.2. A *Banach algebra* is a (complex) algebra that is also a Banach space, with a submultiplicative norm: $\|ab\| \leq \|a\|\|b\|$. A *Banach algebra with involution* is a Banach algebra with involution satisfying additionally $\|a^*\| = \|a\|$. A Banach algebra with involution is said to be an (abstract) C^*-*algebra* if the norm satisfies the C^*-condition: $\|a^*a\| = \|a\|^2$. If the C^*-algebra is *unital*, we denote its unit by $\mathbb{1}$; then $\|\mathbb{1}\| = 1$.

REMARK 1.3. The natural morphisms between two *-algebras $\mathcal{A}$ and $\mathcal{B}$ are algebraic homomorphisms $\mathscr{I}$ which preserve the involutive structure in that $\mathscr{I}(a^*) = \mathscr{I}(a)^*$ for all $a \in \mathcal{A}$. Such homomorphisms are called *-homomorphisms (*-isomorphisms if they are injective). It is well-known that any *-homomorphism from one C^*-algebra into another is automatically contractive (see, for example, [**Tak02**, Proposition I.5.2]). Hence the *-homomorphisms are also the natural morphisms for the category of C^*-algebras.

As seen from the above, C^*-algebras constitute a subclass of the class of Banach algebras. The importance of this particular subclass stems from the following:

EXAMPLE 1.4. There are two prototypical examples of C^*-algebras:
(1) $\mathcal{C}_0(X)$, where X is a locally compact Hausdorff space, and $\mathcal{C}_0(X)$ denote continuous functions on X vanishing at infinity. With the supremum norm and the natural involution $f \mapsto (\bar{f}\colon x \mapsto \overline{f(x)})$ it becomes a commutative C^*-algebra. For compact X we get a unital commutative C^*-algebra $\mathcal{C}(X)$, with unit $\mathbb{1} := \mathbb{1}_X$.
(2) $B(H)$, where: H is a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, linear in the first and antilinear in the second argument; the norm on H denotes the norm given by the inner product; $B(H)$ consists of bounded (or continuous) linear operators on H.
     A linear map $a\colon H \to H$ is bounded if

$$\|a\| := \sup\{\|a\xi\|\colon \|\xi\| \leq 1\} < \infty.$$

Endowed with the operator norm, $B(H)$ becomes a Banach space. With the usual algebraic operations and the adjoint map $a \mapsto a^*$, where a^* satisfies $\langle a\eta, \xi \rangle = \langle \eta, a^*\xi \rangle$, $B(H)$ becomes a *-algebra. The operator norm satisfies all the conditions of a C^*-norm, turning $B(H)$ into a C^*-algebra with unit $\mathbb{1} := \mathbb{1}_H$.

To show the importance of the first example, we will introduce the notion of a *spectrum* of a commutative C^*-algebra $\mathcal{A}$. Namely:

Definition 1.5. The *spectrum* of $\mathcal{A}$, denoted by $\mathrm{Sp}(\mathcal{A})$, is the set of *characters* of $\mathcal{A}$, i.e. non-zero homomorphisms of $\mathcal{A}$ into $\mathbb{C}$.

Since $\mathrm{Sp}(\mathcal{A})$ is contained in $\mathcal{A}^*$, we can endow it with the (restriction of) the weak*-topology:

Proposition 1.6. *The space* $\mathrm{Sp}(\mathcal{A})$ *endowed with the restriction of the* $\sigma(\mathcal{A}^*, \mathcal{A})$-*topology is locally compact Hausdorff. It is compact if* $\mathcal{A}$ *is unital.*

Definition 1.7. For any $a \in \mathcal{A}$, let $\hat{a}$ denote a map from $\mathcal{C}_0(\mathrm{Sp}(\mathcal{A}))$ into $\mathbb{C}$ given by $\hat{a}(\chi) := \chi(a)$. The map $a \mapsto \hat{a}$ from a into $\mathcal{C}_0(\mathrm{Sp}(\mathcal{A}))$ is called the *Gelfand transform*.

Theorem 1.8 (Gelfand-Naimark theorem for abelian C^*-algebras). *Every commutative C^* algebra $\mathcal{A}$ (resp. unital commutative C^* algebra) is isometrically isomorphic to $\mathcal{C}_0(\mathrm{Sp}(\mathcal{A}))$ (resp. $\mathcal{C}(\mathrm{Sp}(\mathcal{A}))$), with the isomorphism given by the Gelfand transform. Given two commutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, there is moreover a bijective correspondence between $*$-homomorphisms $\mathscr{I} : \mathcal{A} \to \mathcal{B}$ and continuous functions $\vartheta : \mathrm{Sp}(\mathcal{B}) \to \mathrm{Sp}(\mathcal{A})$ given by the formula $\widehat{\mathscr{I}(a)} = \hat{a} \circ \vartheta$ for all $a \in \mathcal{A}$.*

It is well known that topological properties of a locally compact or compact space X can be read from algebraic properties of $\mathcal{C}_0(X)$ or $\mathcal{C}(X)$. Thus one can treat 'commutative' topology (or at least a part of it) as the study of commutative C^*-algebras. That is why the general theory of C^*-algebras is often called 'noncommutative topology'.

We turn now to the second example. It is obvious that norm-closed *-subalgebras of $B(H)$ become themselves C^*-algebras. C^*-algebras obtained in this way are called *concrete* or *represented*. One prominent example is a (in general non-unital) algebra $K(H)$ of compact operators on H. To go from an abstract to a concrete C^*-algebra, we need a notion of a representation.

Definition 1.9. By a *representation* π of a *-algebra $\mathcal{A}$ on a Hilbert space H we understand a *-homomorphism from $\mathcal{A}$ into $B(H)$. A representation π is *faithful* if its kernel is $\{0\}$, and *non-degenerate* if $\pi(\mathcal{A})H$ is (norm) dense in H.

Theorem 1.10 (Gelfand-Naimark theorem for general C^*-algebras). *Every abstract C^*-algebra $\mathcal{A}$ is isometrically isomorphic to a represented one, i.e. there exists a (faithful) representation π of $\mathcal{A}$ on some Hilbert*

space H such that $\|\pi(a)\| = \|a\|$ for all $a \in \mathcal{A}$ and $\pi(\mathcal{A})$ is a C^*-subalgebra of $B(H)$.

The possibility of switching between abstract and represented pictures is of fundamental importance. We will often use the possibility of changing the representation so that it suits our needs.

Various classes of elements of C^*-algebras correspond to various classes of bounded operators:

DEFINITION 1.11. Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ is called *self-adjoint* or *hermitian* if $a = a^*$, *normal* if $a^*a = aa^*$, *unitary* if $aa^* = a^*a = \mathbb{1}$ and *positive* if $a = b^*b$ for some $b \in \mathcal{A}$. The self-adjoint elements of $\mathcal{A}$ are denoted by $\mathcal{A}_h$ and the positive elements by $\mathcal{A}_+$. We write $a \leq b$ for $a, b \in \mathcal{A}_h$ if $b - a \in \mathcal{A}_+$.

REMARK 1.12. The self-adjoint part $\mathcal{A}_h$ of a C^*-algebra $\mathcal{A}$ becomes an algebra when equipped with the so-called Jordan product $a \circ b = \frac{1}{2}(ab + ba)$. The morphisms on $\mathcal{A}$ which behave well with respect to this structure, are the so-called Jordan *-morphisms, namely linear maps $\mathscr{J}$ from one C^*-algebra $\mathcal{A}$ into another $\mathcal{B}$ which preserve both involution and the Jordan product. In other words $\mathscr{J}(a^*) = \mathscr{J}(a)^*$ and $\mathscr{J}(a \circ b) = \mathscr{J}(a) \circ \mathscr{J}(b)$. It is well-known that Jordan *-morphisms on C^*-algebras are also automatically contractive, and that they satisfy the following useful identities for all $a, b, c \in \mathcal{A}$:

(1) $\mathscr{J}(aba) = \mathscr{J}(a)\mathscr{J}(b)\mathscr{J}(a)$
(2) $\mathscr{J}(abc) + \mathscr{J}(cba) = \mathscr{J}(a)\mathscr{J}(b)\mathscr{J}(c) + \mathscr{J}(c)\mathscr{J}(b)\mathscr{J}(a)$
(3) $[\mathscr{J}(ab) - \mathscr{J}(a)\mathscr{J}(b)][\mathscr{J}(ab) - \mathscr{J}(b)\mathscr{J}(a)] = 0$

DEFINITION 1.13. Let $\mathcal{A}$ be a unital C^*-algebra. The *spectrum* $\mathrm{sp}(a)$ of an element $a \in \mathcal{A}$ is the set

$$\{\lambda \in \mathbb{C} : a - \lambda\mathbb{1} \text{ is not invertible in } \mathcal{A}\}.$$

PROPOSITION 1.14. *For any element $a \in \mathcal{A}$, $\mathrm{sp}(a)$ is a compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. If a is self-adjoint, then $\mathrm{sp}(a) \subseteq \mathbb{R}$, if a is positive, then $\mathrm{sp}(a) \subseteq \mathbb{R}_+$. If p is a self-adjoint idempotent, then $\mathrm{sp}(p) \subseteq \{0, 1\}$, and if u is unitary, then $\mathrm{sp}(p) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Moreover if $\mathcal{B}$ is a unital C^*-subalgebra of $\mathcal{A}$, then for any $a \in \mathcal{B}$, $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{\mathcal{B}}(a)$.*

We introduce two functional calculi valid for this context. When working with non-normal operators one may use the so-called holomorphic functional calculus which is based on Cauchy's integration formula.

THEOREM 1.15 (Holomorphic functional calculus). *Let $a \in \mathcal{A}$ be given, let $\mathcal{D}$ be a simply-connected domain containing $\mathrm{sp}(a)$ and Γ a simply closed positively oriented contour inside $\mathcal{D}$ encircling $\mathrm{sp}(a)$. Then, for any function f holomorphic on $\mathcal{D}$,*

$$f(a) = \int_\gamma f(z)(z\mathbb{1} - a)^{-1}\, dz$$

is a well-defined element of $\mathcal{A}$. In fact, the prescription $f \to f(a)$ yields an algebra homomorphism from the set of functions which are holomorphic on $\mathcal{D}$ into $\mathcal{A}$ which maps the function $\iota : z \mapsto z$ onto a.

When dealing with normal elements of a C^*-algebra, one has access to the more powerful continuous functional calculus.

THEOREM 1.16 (Continuous functional calculus). *Let $\mathcal{A}$ be unital and $a \in \mathcal{A}$ normal. There exists a unique $*$-isomorphism $f \mapsto f(a)$ from $C(\mathrm{sp}(a))$ onto the C^*-subalgebra of $\mathcal{A}$ generated by a and $\mathbb{1}$, mapping $\iota : t \mapsto t$ onto a and satisfying $\|f\|_\infty = \|f(a)\|$ for each $f \in C(\mathrm{sp}(a))$.*

COROLLARY 1.17. *Any $a \in \mathcal{A}$ can be written as a linear combination of four unitaries.*

COROLLARY 1.18. *If $a \in \mathcal{A}_+$, then there is a unique element $b \in \mathcal{A}_+$, such that $b^2 = a$.*

DEFINITION 1.19. We denote the element in $\mathcal{A}_+$ whose square is $a \in \mathcal{A}_+$ by $a^{1/2}$. For any $a \in \mathcal{A}$, we define $|a| := (a^*a)^{1/2}$, and call it the *absolute value* or *modulus* of a.

We pause to collate some basic properties of the cone $\mathcal{A}_+$.

PROPOSITION 1.20. *Let $\mathcal{A}$ be a C^*-algebra.*
  (1) *For any $a \in \mathcal{A}_h$ the elements $a_\pm = \frac{1}{2}(|a| \pm a)$ belong to $\mathcal{A}_+$, and are the unique elements of $\mathcal{A}_+$ satisfying $a = a_+ - a_-$ and $a_+ a_- = 0$. In the case where $\mathcal{A}$ is unital we have for any $a \in \mathcal{A}_+$ that $a \leq \|a\|\mathbb{1}$.*
  (2) *If $0 \leq a \leq b$ for some $a, b \in \mathcal{A}$, then*
     (a) $0 \leq a^r \leq b^r$ *for any $0 < r \leq 1$,*
     (b) $\|a\| \leq \|b\|$,
     (c) $0 \leq c^*ac \leq c^*bc$ *for any $c \in \mathcal{A}$,*
     (d) *in the case where $\mathcal{A}$ is unital we have that $0 \leq (b + \lambda\mathbb{1})^{-1} \leq (a + \lambda\mathbb{1})^{-1}$ for any $\lambda > 0$*

The above technology now enables one to introduce the important notion of approximate identity:

THEOREM 1.21. *Let $\mathcal{L}$ be a left ideal of a C^*-algebra $\mathcal{A}$. Then there is a net (f_λ) of positive contractive elements of $\mathcal{L}$ such that f_λ increases as λ increases, and $\lim_\lambda \|af_\lambda - a\| = 0$ for all $a \in \mathcal{A}$. A similar claim holds for right-ideals.*

DEFINITION 1.22. Let $\mathfrak{m}$ be a left (resp. right) ideal of a C^*-algebra $\mathcal{A}$. A net (f_λ) of positive contractive elements of $\mathfrak{m}$ is called a *right* (resp. *left*) *approximate identity* of $\mathfrak{m}$, if f_λ increases as λ increases, and $\lim_\lambda \|af_\lambda - a\| = 0$ (resp. $\lim_\lambda \|f_\lambda a - a\| = 0$) for all $a \in \mathcal{A}$.

We are now moving to linear forms on a C^*-algebra $\mathcal{A}$.

DEFINITION 1.23. A linear functional ω on $\mathcal{A}$ is said to be *real* or *hermitian* if $\omega(a) \in \mathbb{R}$ for all $a \in \mathcal{A}_h$, and *positive* if $\omega(a) \geq 0$ for any $a \in \mathcal{A}_+$. A positive functional of norm 1 is called a *state*. A positive functional ω on $\mathcal{A}$ is called *faithful* if $\omega(a) = 0$ for $a \in \mathcal{A}_+$ implies $a = 0$.

PROPOSITION 1.24. *A positive linear functional on a C^*-algebra $\mathcal{A}$ is automatically bounded, i.e. $\omega \in \mathcal{A}_+^*$. If the algebra $\mathcal{A}$ is unital (with unit $\mathbb{1}$), then $\|\omega\| = \omega(\mathbb{1})$.*

DEFINITION 1.25. If $\mathcal{A}$ is a C^*-subalgebra of $B(H)$ and $\xi \in H$, then $\omega_\xi : a \mapsto \langle a\xi, \xi \rangle$ is a positive linear functional on $\mathcal{A}$. If $\|\xi\| = 1$, then ω_ξ is a state, called a *vector state*.

The following definition introduces an important class of linear forms:

DEFINITION 1.26. A functional $\omega \in \mathcal{A}^*$ is called *tracial* if $\omega(ab) = \omega(ba)$ for all $a, b \in \mathcal{A}$.

NOTATION 1.27. For $\omega \in \mathcal{A}_+^*$ we write $N_\omega := \{a \in \mathcal{A} \colon \omega(a^*a) = 0\}$.

It is easy to see that N_ω is a left ideal in $\mathcal{A}$. If ω is tracial, then the ideal is two-sided.

NOTATION 1.28. For $\omega \in \mathcal{A}_+^*$ we denote by η_ω the quotient map $\mathcal{A} \ni a \mapsto a + N_\omega \in \mathcal{A}/N_\omega$. We denote by $\langle \cdot, \cdot \rangle_\omega$ the inner product on $\mathcal{A}/N_\omega$ defined by $\langle \eta_\omega(a), \eta_\omega(b) \rangle_\omega := \omega(b^*a)$.

The Cauchy-Bunyakovsky-Schwarz (*CBS* for short) inequality for the above inner product leads to an important inequality for elements of $\mathcal{A}_+^*$:

$$|\omega(b^*a)|^2 \leq \omega(a^*a)\omega(b^*b). \tag{1.1}$$

To construct a faithful representation of an abstract C^*-algebra on a Hilbert space we use an ingenious *Gelfand-Naimark-Segal construction* (*GNS* for short).

DEFINITION 1.29 (GNS representation). Let H_ω be the Hilbert space completion of $\mathcal{A}/N_\omega$ with the inner product from Notation 1.28. Then $\pi_\omega(a) : \eta_\omega(b) \mapsto \eta_\omega(ab)$ extends to a bounded operator on H_ω. The representation π_ω of A on H_ω obtained in this manner is called the *GNS representation of $\mathcal{A}$ associated with ω.*

1.2   **Bounded operators**

In this section we gather important information on bounded operators on a Hilbert space H. For functional calculi for self-adjoint (or normal) operators, we recommend first volume of Kadison and Ringrose [**KR83**], Strătilă and Zsidó [**SZ79**] and Arveson [**Arv02**].

We have special notation for the most important classes of bounded operators on H:

DEFINITION 1.30. The operators satisfying $\langle a\eta, \xi \rangle = \langle \eta, a\xi \rangle$ for all $\xi, \eta \in H$ are called *self-adjoint* or *hermitian* and the real subspace of self-adjoint operators is denoted by $B(H)_h$. A bounded operator a is *positive* if $\langle a\xi, \xi \rangle \geq 0$ for all $\xi \in H$ (or, equivalently, if a is positive as an element of the C^*-algebra $B(H)$), and the pointed cone of positive operators is denoted by $B(H)_+$. For $a, b \in \mathcal{M}_h$, we say that $a \leq b$ if $\langle a\xi, \xi \rangle \leq \langle b\xi, \xi \rangle$ for any $\xi \in H$.

It is obvious that $a \in B(H)$ is self-adjoint (resp. positive) in the above sense if and only if it is self-adjoint (resp. positive) as an element of the C^*-algebra $B(H)$. Similarly, it is clear that $a \leq b$ means the same for a, b treated as bounded operators on H and elements of the C^*-algebra $B(H)$.

DEFINITION 1.31. An (*orthogonal*) *projection p* is a bounded operator on H satisfying $p = p^* = p^2$, and the complete lattice of projections is denoted by $\mathbb{P}(B(H))$. We write $p^\perp$ for an (*orthogonal*) *complement* of p. Projections p, q are *orthogonal*, which is written as $p \perp q$, if $pq = 0$. An *orthogonal* family is a family of mutually orthogonal projections.

DEFINITION 1.32. An operator $u \in B(H)$ is *unitary* if $u^*u = uu^* = \mathbb{1}$, an *isometry* if $u^*u = \mathbb{1}$, and a *partial isometry* if $p := u^*u$ is a projection. Then $q := uu^*$ is also a projection, and p and q are called, respectively, the *initial* and *final* projection of u.

Besides the *norm* (or *uniform*) topology on $B(H)$, there are several other topologies that are constantly used in the theory of operator algebras. Here are a few of the most important ones:

DEFINITION 1.33.

(1) *weak (operator)* topology is given by the family of seminorms $a \mapsto |\langle a\eta, \xi \rangle|$ for $\xi, \eta \in H$;

(2) *strong (operator)* topology is given by the family of seminorms $a \mapsto \|a\xi\|$ for $\xi \in H$;

(3) *strong* (operator)* topology is given by the family of seminorms $a \mapsto (\|a\xi\|^2 + \|a^*\xi\|^2)^{1/2}$ for $\xi \in H$;

(4) σ-*weak* (or *ultraweak*) topology is given by the family of seminorms $a \mapsto |\sum_{n=1}^{\infty} \langle a\eta_n, \xi_n \rangle|$ indexed by all sequences $(\xi_n), (\eta_n)$ of vectors from H with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|\eta_n\|^2 < \infty$.

(5) σ-*strong* (or *ultrastrong*) topology is given by the family of seminorms $a \mapsto (\sum_{n=1}^{\infty} \|a\xi_n\|^2)^{1/2}$ indexed by all sequences (ξ_n) of vectors from H with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

(6) σ-*strong** (or *ultrastrong**) topology is given by the family of seminorms $a \mapsto (\sum_{n=1}^{\infty}(\|a\xi_n\|^2 + \|a^*\xi_n\|^2))^{1/2}$ indexed by all sequences (ξ_n) of vectors from H with $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$.

We will put all the above topologies under one collective name of *non-uniform topologies*.

PROPOSITION 1.34. *The following diagram shows how the topologies relate to each other:*

$$
\begin{array}{ccccc}
weak & \subseteq & strong & \subseteq & strong^* \\
\text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} \\
\sigma\text{-}weak & \subseteq & \sigma\text{-}strong & \subseteq \sigma\text{-}strong^* \subseteq & uniform
\end{array}
\tag{1.2}
$$

On bounded subsets of $B(H)$, weak topology coincides with σ-weak (resp. strong with σ-strong, strong with σ-strong*) topology. For a convex subset of $B(H)$ each of the σ-weak, σ-strong and σ-strong* topologies yield the same closure.*

PROPOSITION 1.35. *A net (a_i) converges to a weakly (resp. strongly, strongly*) if for each $\xi, \eta \in H$ (resp. for each $\xi \in H$) we have $\langle a_i\eta, \xi \rangle \to \langle a\eta, \xi \rangle$) (resp. $a_i\xi \to a\xi$ in norm, $a_i\xi \to a\xi$ and $a_i^*\xi \to a^*\xi$ in norm).*

PROPOSITION 1.36.(1) *The adjoint operation $*$ is continuous in the weak, σ-weak, strong* and σ-strong* topologies, but in general not in the strong or σ-strong topology.*

(2) *With* $\mathrm{ball}(B(H))$ *denoting the unit ball of* $B(H)$*, multiplication* $(a, b) \to ab$ *is*

    (a) *continuous from* $\mathrm{ball}(B(H)) \times B(H)$ *to* $B(H)$ *for each of the σ-strong and strong topologies,*

    (b) *continuous from* $\mathrm{ball}(B(H)) \times \mathrm{ball}(B(H))$ *to* $B(H)$ *for each of the σ-strong* and strong* topologies,*

    (c) *separately but not jointly continuous from* $B(H) \times B(H)$ *to* $B(H)$ *for each of the σ-weak and weak operator topology.*

PROPOSITION 1.37. *If $(a_i)_{i \in I}$ is an increasing net from $B(H)_+$ bounded above by an operator $b \in B(H)_+$, then $a_i \nearrow a$ for some $a \in B(H)$ (i.e. $\langle a_i \xi, \xi \rangle \nearrow \langle a \xi, \xi \rangle$ for all $\xi \in H$), a is the supremum of a_i's and $(a_i)_{i \in I}$ converges to a strongly (and σ-strongly).*

COROLLARY 1.38. *Any family of projections $\{p_i\}_{i \in I}$ possesses both a supremum $\bigvee_{i \in I} p_i$ and an infimum $\bigwedge_{i \in I} p_i$. Moreover, any increasing net of projections $(p_i)_{i \in I}$ is strongly convergent to $\bigvee_{i \in I} p_i$, and any decreasing net of projections $(p_i)_{i \in I}$ is strongly convergent to $\bigwedge_{i \in I} p_i$. Finally, the sum $\sum_{i \in I} p_i$ of a family of projections $\{p_i\}_{i \in I}$ exists in strong topology and is a projection if and only if the elements of the family are mutually orthogonal.*

DEFINITION 1.39. We write $\mathrm{n}(a)$ for the *null projection* of a, that is the projections onto the *null space* or *kernel* $\{\xi : a\xi = 0\}$ of a. The *right support* of a is $\mathrm{s}_r(a) := \mathbb{1} - \mathrm{n}(a)$, and the *left support* or the *range projection* $\mathrm{s}_l(x)$ is the projection onto the closure (in H) of $a(H)$. If $a \in B(H)_h$, then $\mathrm{s}(a) := \mathrm{s}_l(a) = \mathrm{s}_r(a)$ is called the *support* of a.

PROPOSITION 1.40. *The right support is the smallest projection p satisfying $ap = a$, and the left support is the smallest projection p satisfying $pa = a$.*

DEFINITION 1.41. A family of projections $(e_\lambda)_{\lambda \in \mathbb{R}}$ that is increasing: $e_\lambda \leq e'_\lambda$ for $\lambda \leq \lambda'$, continuous from the right in the sense of strong convergence: $e_\lambda = \bigwedge_{\lambda' > \lambda} e_{\lambda'}$ for each $\lambda \in \mathbb{R}$ and satisfies both $\bigwedge_{\lambda \in \mathbb{R}} e_\lambda = 0$ and $\bigvee_{\lambda \in \mathbb{R}} e_\lambda = \mathbb{1}$ is called a *resolution of the identity*. A resolution of the identity is *bounded* if there is a $\lambda_0 > 0$ such that $e_\lambda = 0$ for $\lambda < -\lambda_0$ and $e_\lambda = \mathbb{1}$ for $\lambda > \lambda_0$, otherwise it is called *unbounded*.

THEOREM 1.42 (Spectral decomposition). *Each $a \in B(H)_h$ has a unique spectral decomposition*

$$a = \int_{-\|a\|}^{\|a\|} \lambda de_\lambda, \tag{1.3}$$

where $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is a bounded resolution of the identity satisfying $e_\lambda = 0$ for $\lambda < -\|a\|$ and $e_\lambda = \mathbb{1}$ for $\lambda \geq \|a\|$ and

$$ae_\lambda \leq \lambda e_\lambda, \ ae_\lambda^\perp \geq \lambda e_\lambda^\perp \ \text{for all } \lambda \in \mathbb{R},$$

and the integral is understood as a norm limit of approximating Riemann sums. The sums can be chosen as finite linear combinations of projections $e_{\lambda'} - e_\lambda$ with coefficients in $\mathrm{sp}(a)$. We call $(e_\lambda)_{\lambda \in \mathbb{R}}$ the spectral resolution of the operator a and the formula (1.3) the spectral decomposition of a.

We use the Borel functional calculus for bounded operators in the following form:

THEOREM 1.43 (Borel functional calculus for bounded operators). *Let $a \in B(H)_h$. There exists a unique injective *-homomorphism $f \mapsto f(a)$ from the *-algebra $\mathcal{B}_b(\mathrm{sp}(a))$ of bounded Borel functions on the spectrum of a into the *-algebra $B(H)$, mapping the identity function $\lambda \mapsto \lambda$ to a and satisfying the following continuity condition:*

if $f, f_n \in \mathcal{B}_b(\mathrm{sp}(a))$, $\sup \|f_n\| < \infty$, and $f_n \to f$ pointwise,
then $f_n(a) \to f(a)$ strongly.

One can write a spectral decomposition of $f(a)$:

$$f(a) = \int_{-\infty}^{\infty} f(\lambda) de_\lambda,$$

to be understood in a weak sense: for any $\xi, \eta \in H$,

$$\langle f(x)\xi, \eta \rangle = \int_{-\infty}^{\infty} f(\lambda) d\langle e_\lambda \xi, \eta \rangle.$$

We have

$$\|f(x)\xi\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d\langle e_\lambda \xi, \xi \rangle.$$

It should be noted that for $a \in B(H)_+$ and function $\lambda \mapsto \lambda^{1/2}$ the operator $f(a)$ is exactly the element $a^{1/2}$, as defined in Definition 1.19.

PROPOSITION 1.44. *For any $a \in B(H)$, we have $\mathrm{s}_l(a) = \mathrm{s}_r(a^*)$ and $\mathrm{s}_r(a) = \mathrm{s}_l(a^*)$. Moreover, $\mathrm{s}_l(a) = \mathrm{s}(aa^*)$ and $\mathrm{s}_r(a) = \mathrm{s}(a^*a)$. For positive a, $\mathrm{s}(a) = \mathrm{s}(a^{1/2})$, so that $\mathrm{s}(|a|) = \mathrm{s}_r(a)$ and $\mathrm{s}(|a^*|) = \mathrm{s}_l(a)$.*

THEOREM 1.45 (Polar decomposition). *Let $a \in B(H)$. There exists a partial isometry with initial projection $\mathrm{s}_r(a)$ and final projection $\mathrm{s}_l(a)$ such that $a = u|a| = |a^*|u$. If $a = vb$ with $b \in B(H)_+$ and v a partial isometry with initial projection $\mathrm{s}(b)$, then $v = u$ and $b = |a|$. If both a and a^* are injective, then $u \in \mathbb{U}(B(H))$, the unitary group of $B(H)$.*

DEFINITION 1.46. The unique representation of a in the form $a = u|a|$ is called *the polar decomposition of a*.

1.3 Von Neumann algebras

The theory of von Neumann algebras is very rich, and we are dealing here only with its most basic aspects. The reader interested in the theory, in addition to texts and monographs mentioned in the Preface and in Section 1.1 could consult books of Kaplansky [**Kap68**], Sunder [**Sun87**], Zhu [**Zhu93**] and volume 3 of Takesaki [**Tak03b**].

The presentation in this section is strongly influenced by [**Tak02**].

DEFINITION 1.47. Let $\mathcal{A}$ be any subset of $B(H)$. The *commutant* $\mathcal{A}'$ of $\mathcal{A}$ is the set $\{a' \in B(H) \colon aa' = a'a$ for all $a \in \mathcal{A}\}$. The *centre* $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ is defined as $\mathcal{A} \cap \mathcal{A}'$.

DEFINITION 1.48. A *-subalgebra $\mathcal{M}$ of $B(H)$ is called a (concrete) *von Neumann algebra* if $\mathcal{M} = \mathcal{M}''$. Note that $\mathbb{1} = \mathbb{1}_H \in \mathcal{M}$. We say that $\mathcal{M}$ *acts* on H. A von Neumann algebra is called a *factor* if the centre of the algebra is trivial, i.e. $\mathcal{Z}(\mathcal{M}) = \mathbb{C}\mathbb{1}_H$.

It is clear that $B(H)$ is a factor von Neumann algebra. In fact, $B(H)' = \mathbb{C}\mathbb{1}_H$ and $(\mathbb{C}\mathbb{1}_H)' = B(H)$.

DEFINITION 1.49. We say that von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$, acting respectively on H_1 and H_2, are *isomorphic* if there exists a *-preserving algebra isomorphism Φ of $\mathcal{M}_1$ onto $\mathcal{M}_2$. It is then automatically norm-preserving and σ-weakly bicontinuous. We denote the fact by $\mathcal{M}_1 \cong \mathcal{M}_2$. If, for some unitary $u : H_1 \to H_2$ (i.e. $u^*u = 1_{H_1}$, $uu^* = 1_{H_2}$) we have $\Phi(a) = uau^*$, we say that Φ is a *spatial isomorphism*, and the algebras are *spatially isomorphic*.

There is also an abstract counterpart to the notion of a von Neumann algebra, introduced by Shôichirô Sakai in [**Sak56**].

DEFINITION 1.50. An (abstract) C^* algebra $\mathcal{M}$ is called a W^*-*algebra* or an *abstract von Neumann algebra* if, as a Banach space, it is the dual of another Banach space.

In this case the *predual* Banach space is unique (see [**Sak71**, 1.13.3]) and we denote it by $\mathcal{M}_*$. Thus $(\mathcal{M}_*)^* \simeq \mathcal{M}$, where $\simeq$ denotes the Banach space isometric isomorphism. From the duality theory for locally convex spaces we know that $\mathcal{M}_*$ can be identified with the set of $\sigma(\mathcal{M}, \mathcal{M}_*)$-continuous functionals on $\mathcal{M}$, with duality given by $\langle a, \omega \rangle := \omega(a)$.

The following theorem corresponds to the Gelfand-Naimark theorem for general C^*-algebras:

THEOREM 1.51. *Every W^*-algebra $\mathcal{M}$ is isometrically isomorphic to a represented one, i.e. there exists a (faithful) representation π of $\mathcal{M}$ on some Hilbert space H such that $\|\pi(a)\| = \|a\|$ for all $a \in \mathcal{M}$ and $\pi(\mathcal{M})$ is a von Neumann algebra acting on H.*

PROPOSITION 1.52. *If $\mathcal{M}$ is a von Neumann algebra acting on a Hilbert space H, then the $\sigma(\mathcal{M}, \mathcal{M}_*)$-topology on $\mathcal{M}$ is exactly the σ-weak topology. In other words, the predual $\mathcal{M}_*$ of $\mathcal{M}$ consists of σ-weakly continuous functionals on $\mathcal{M}$.*

PROPOSITION 1.53. *For a functional $\omega \in \mathcal{M}^*$ the following conditions are equivalent:*

(1) *ω is weakly continuous;*
(2) *ω is strongly continuous;*
(3) *ω is strongly* continuous.*

For a functional $\omega \in \mathcal{M}^*$ the following conditions are equivalent:

(4) *ω is σ-weakly continuous;*
(5) *ω is σ-strongly continuous;*
(6) *ω is σ-strongly* continuous.*

DEFINITION 1.54. A functional $\omega \in \mathcal{M}_*^+$ is called

(1) *normal* if $a_i \nearrow a$ implies $\omega(a_i) \nearrow \omega(a)$ for any increasing net $(a_i)_{i \in I}$ from $\mathcal{M}_+$ with supremum $a \in \mathcal{M}$;
(2) *completely additive* if $\omega(\sum_{i \in I} a_i) = \sum_{i \in I} \omega(a_i)$ for any family $\{a_i\}_{i \in I}$ of positive operators from $\mathcal{M}$ with

$$\sup_{J \subseteq I, J \text{ finite}} \| \sum_{i \in J} a_i \| < \infty.$$

(3) *completely additive on projections* if $\omega(\sum_i p_i) = \sum_i \omega(p_i)$ for any orthogonal family of projections from $\mathcal{M}$.

THEOREM 1.55. *For a state $\omega \in \mathcal{M}_+^*$ the following conditions are equivalent:*

(1) $\omega \in \mathcal{M}_*$;

(2) ω is σ-weakly (or $\sigma(\mathcal{M}, \mathcal{M}_*)$, σ-strongly, σ-strongly*) continuous;

(3) ω is normal;

(4) ω is completely additive;

(5) ω is completely additive on projections;

(6) $\omega = \sum_{n \in \mathbb{N}} \omega_{\xi_n}$ for some $\xi_n \in H$ with $\sum_{n \in \mathbb{N}} \|\xi_n\|^2 = 1$.

Any element of $\mathcal{M}_*$ may be written as a linear combination of four such states.

The above theorem now easily yields the following conclusion:

PROPOSITION 1.56. *Any *-isomorphism $\mathscr{I}$ from one von Neumann algebra $\mathcal{M}_1$ onto another $\mathcal{M}_2$ is automatically a σ-weak-σ-weak homeomorphism.*

PROOF. Since both $\mathscr{I}$ and $\mathscr{I}^{-1}$ are contractive, $\mathscr{I}$ is an isometric isomorphism, and hence so is the Banach adjoint $\mathscr{I}^*$ of $\mathscr{I}$. For any $a \in \mathcal{M}_1$ we have that $\mathscr{I}(a^*a) = \mathscr{I}(a)^*\mathscr{I}(a)$ since any $b \in \mathcal{M}_1^+$ may be written in the form $b = a^*a$, this shows that $\mathscr{I}$ is an order-isomorphism. But if that is the case we will for any net $(a_i) \subseteq \mathcal{M}_1$ and $a \in \mathcal{M}_1$ have that $a_i \nearrow a$ if and only if $\mathscr{I}(a_i) \nearrow \mathscr{I}(a)$. But then the preceding theorem will ensure that for any state ω on $\mathcal{M}_2$ we have that $\omega \in (\mathcal{M}_2)_*$ if and only $\omega \circ \mathscr{I} \in (\mathcal{M}_1)_*$. Thus $\mathscr{I}^*$ restricts to an isometric isomorphism from $(\mathcal{M}_2)_*$ onto $(\mathcal{M}_1)_*$. It is now an exercise to see that $\mathscr{I}$ is the Banach adjoint of this restriction, which proves the claim. $\square$

Jordan *-morphisms on von Neumann algebras also exhibit somewhat more elegant behaviour than their C^*-algebraic counterparts:

PROPOSITION 1.57. [**BR87a**, Proposition 3.2.2] *Let $\mathscr{J}$ be a Jordan *-homomorphism from one von Neumann algebra $\mathcal{M}_1$ into another $\mathcal{M}_2$, and let $\mathcal{B}$ be the σ-weakly closed *-subalgebra of $\mathcal{M}_2$ generated by $\mathscr{J}(\mathcal{M}_1)$. Then there exists a projection $e \in \mathcal{B} \cap \mathcal{B}'$ such that $a \to e \mathscr{J}(a)$ is a *-homomorphism, and $a \to (\mathbb{1} - e) \mathscr{J}(a)$ a *-antihomomorphism.*

NOTATION 1.58. The following notation is 'inherited' from $B(H)$:

$$\mathcal{M}_h = \mathcal{M} \cap B(H)_h, \qquad \mathcal{M}_+ = \mathcal{M} \cap B(H)_+,$$
$$\mathbb{P}(\mathcal{M}) = \mathcal{M} \cap \mathbb{P}(B(H)), \qquad \mathbb{U}(\mathcal{M}) = \mathcal{M} \cap \mathbb{U}(B(H)).$$

The following two theorems belong to the most basic set of principles in the whole theory of operator algebras:

THEOREM 1.59 (Von Neumann double commutant theorem). *A* *-*subalgebra* $\mathcal{M}$ *of* $B(H)$ *containing* $\mathbb{1}_H$ *is a von Neumann algebra if and only if it is closed in any of the non-uniform topologies. In particular, if a unital* *-*subalgebra* $\mathcal{A}$ *of a von Neumann algebra* $\mathcal{M}$ *is dense in* $\mathcal{M}$ *in one of the non-uniform topologies, then it is dense in* $\mathcal{M}$ *in all non-uniform topologies.*

THEOREM 1.60 (Kaplansky's density theorem). *Let* $\mathcal{M}$ *be a von Neumann algebra, and* $\mathcal{A}$ *its* *-*subalgebra, dense in one of the non-uniform topologies. Then the unit ball of* $\mathcal{A}$ *(resp.* $\mathcal{A}_h$, $\mathcal{A}_+$*) is dense in any of the non-uniform topologies in the unit ball of* $\mathcal{M}$ *(resp.* $\mathcal{M}_h$, $\mathcal{M}_+$*).*

Below we list facts that follow easily from von Neumann's double commutant theorem.

PROPOSITION 1.61.(1) *For any family of projections* $\{p_i\}_{i \in I}$ *from* $\mathcal{M}$, *both* $\bigvee_{i \in I} p_i$ *and* $\bigwedge_{i \in I} p_i$ *belong to* $\mathcal{M}$.
(2) *The sum of an orthogonal family* $\{p_i\}_{i \in I}$ *from* $\mathcal{M}$ *belongs to* $\mathcal{M}$.
(3) *The initial and final projections of a partial isometry* u *from* $\mathcal{M}$ *belong to* $\mathcal{M}$.
(4) *If* $(a_i)_{i \in I}$ *is an increasing net of positive operators from* $\mathcal{M}$, *then its supremum belongs to* $\mathcal{M}$.
(5) *If* $\mathfrak{m}$ *is a* σ-*weakly closed left (resp. right) ideal in* $\mathcal{M}$, *then there exists a projection* $p \in \mathcal{M}$ *such that* $\mathfrak{m} = \mathcal{M}p$ *(resp.* $\mathfrak{m} = p\mathcal{M}$*). The projection* p *is the supremum of a right-approximate (resp. left-approximate) identity in* $\mathfrak{m}$. *If* $\mathfrak{m}$ *is a two-sided ideal, then* p *belongs to the centre* $\mathcal{Z}(\mathcal{M})$ *of* $\mathcal{M}$.
(6) *If* $a \in \mathcal{M}$, *then* $\mathbb{n}(a), \mathbb{s}_l(a), \mathbb{s}_r(a) \in \mathcal{M}$. *If* $a \in \mathcal{M}_h$, *then* $\mathbb{s}(a) \in \mathcal{M}$.
(7) *The spectral resolution* (e_λ) *of an operator* $a \in \mathcal{M}_h$ *consists of projections from* $\mathcal{M}$.
(8) *If* $a \in \mathcal{M}_h$ *and* $f \in \mathcal{B}_b(\mathrm{sp}(a))$, *then* $f(a) \in \mathcal{M}$. *In particular, if* $a \in \mathcal{M}_+$, *then* $a^{1/2} \in \mathcal{M}_+$.
(9) *If* $a \in \mathcal{M}$ *has polar decomposition* $a = u|a|$, *then* $u \in \mathcal{M}$ *and* $|a| \in \mathcal{M}_+$.

We can now introduce the important concept of a polar decomposition of functionals in $\mathcal{M}_*$. We start with characterization of N_ω for $\omega \in \mathcal{M}_*^+$:

LEMMA 1.62. *If* $\omega \in \mathcal{M}_*^+$, *then the left ideal* N_ω *is* σ-*weakly closed in* $\mathcal{M}$.

PROOF. Let $(a_i)_{i \in I}$ be a net in $\mathcal{N}_\omega$ which converges σ-weakly to some $a \in \mathcal{M}$. By the CBS inequality (see (1.1)), $\omega(a^* a_i) = 0$ for each $i \in I$. Since by Proposition 1.36(2)(c) $a^* a_i \to a^* a$ σ-weakly, we have, by Proposition 1.52, that $\omega(a^* a) = \lim_{i \in I} \omega(a^* a_i) = 0$, and hence that $a \in N_\omega$. $\qquad \square$

DEFINITION 1.63. Let $\omega \in \mathcal{M}_*^+$ be given. Then, by proposition 1.61(5), there exists a projection $p \in \mathcal{M}$ such that $N_\omega = \mathcal{M}p$. We define the *support projection* of ω to be $\operatorname{supp} \omega = \mathbb{1} - p$.

It is easy to see that $(\operatorname{supp} \omega)^\perp$ is the largest projection p in $\mathcal{M}$ such that $\omega(p) = 0$.

PROPOSITION 1.64 (Polar decomposition of normal functionals). *For any $\omega \in \mathcal{M}_*$ there exists a partial isometry $u \in \mathcal{M}$ and a functional $\rho \in \mathcal{M}_*^+$ related by the equality $\omega(a) = \rho(au)$ for all $a \in \mathcal{M}$, with in addition $\|\omega\| = \|\rho\|$ and $\operatorname{s}(\rho) = uu^*$. Such a decomposition is unique and we write $|\omega|$ for ρ. For the partial isometry in the decomposition we have that $u^* u = \operatorname{s}(\omega^*)$, where the functional ω^* is defined by $\omega^*(a) = \overline{\omega(a^*)}$ for all $a \in \mathcal{M}$.*

DEFINITION 1.65. A projection $p \in \mathcal{M}$ is called σ-*finite* if for any orthogonal family $\{p_i\}_{i \in I}$ of non-zero projections from $\mathcal{M}$ such that $p = \sum_{i \in I} p_i$ we have $\#I \leq \aleph_0$ where $\#I$ denotes the cardinality of I. A von Neumann algebra $\mathcal{M}$ is σ-*finite* if $\mathbb{1}$ is σ-finite.

DEFINITION 1.66. For each $a \in \mathcal{M}$ there exists a smallest central projection z such that $a = az = za$. It is called the *central support* or *central cover* of a and it is denoted by $\operatorname{z}(a)$.

Let K be a closed linear subspace of H invariant under $\mathcal{M}$, i.e. such that $aK \subseteq K$ for all $a \in \mathcal{M}$. We denote by a_K the operator on K obtained by restricting a to K. Note that if $e' \in \mathbb{P}(\mathcal{M}')$, then $e'H$ is invariant under $\mathcal{M}$.

DEFINITION 1.67. Let $e \in \mathbb{P}(\mathcal{M})$ and $e' \in \mathbb{P}(\mathcal{M}')$. We put $\mathcal{M}_e := \{a_{eH} : a \in e\mathcal{M}e\}$ (resp. $\mathcal{M}_{e'} := \{a_{e'H} : a \in \mathcal{M}\}$). Then $\mathcal{M}_e$ (resp. $\mathcal{M}_{e'}$) is a von Neumann algebra called the *reduced von Neumann algebra of $\mathcal{M}$ on $K := eH$* (resp. the *induced von Neumann algebra of $\mathcal{M}$ on $K := e'H$*). In particular, for any $z \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$ we can form the algebra $\mathcal{M}_z$, often written simply as $\mathcal{M}z$.

PROPOSITION 1.68. *If $e \in \mathcal{M}$, then $(\mathcal{M}_e)' = (\mathcal{M}')_e$.*

DEFINITION 1.69. Let $\{\mathcal{M}_i\}_{i \in I}$ be a family of von Neumann algebras acting on Hilbert spaces H_i. Let $H := \sum_{i \in I}^{\oplus} H_i = \{\{\xi_i\}_{i \in I} \colon \xi_i \in H_i, \sum_{i \in I} \|\xi_i\|^2 < \infty\}$. For each bounded family $\{a_i\}_{i \in I}$ in $\prod_{i \in I} \mathcal{M}_i$, we define a bounded operator a on H by $a(\{\xi_i\}_{i \in I}) = \{a_i \xi_i\}_{i \in I}$, and denote it by $\sum_{i \in I}^{\oplus} a_i$. The set of such operators is denoted by $\sum_{i \in I}^{\oplus} \mathcal{M}_i$ and called the *direct sum* of $\{\mathcal{M}_i\}_{i \in I}$. If the index set is finite, we use $\oplus$ instead of $\sum^{\oplus}$.

PROPOSITION 1.70. *If $\{z_i\}_{i \in I}$ is any orthogonal family of central projections in $\mathcal{M}$ such that $\sum_{i \in I} z_i = \mathbb{1}$, then $\mathcal{M} \cong \sum_{i \in I}^{\oplus} \mathcal{M}_{z_i}$; if, on the other hand, $\mathcal{M} = \sum_{i \in I}^{\oplus} \mathcal{M}_i$ for a family of von Neumann algebras $\{\mathcal{M}_i\}_{i \in I}$, then there is an orthogonal family $\{z_i\}$ of central projections in $\mathcal{M}$ such that $\sum_{i \in I} z_i = \mathbb{1}$ and $\mathcal{M}_i \cong \mathcal{M}_{z_i}$.*

DEFINITION 1.71. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras acting on H_1 and H_2, respectively. Let $H := H_1 \otimes H_2$. For $a_1 \in \mathcal{M}_1$ and $a_2 \in \mathcal{M}_2$, there is a unique bounded operator a acting on H such that $a(\xi_1 \otimes \xi_2) = a_1(\xi_1) \otimes a_2(\xi_2)$. We denote this operator by $a_1 \otimes a_2$ and call it a *simple tensor*. The von Neumann subalgebra of $B(H)$ generated by simple tensors, i.e. the closure of the *-algebra of finite linear combinations of simple tensors in one of the non-uniform topologies, is called the *tensor product* of $\mathcal{M}_1$ and $\mathcal{M}_2$ and is denoted by $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$.

THEOREM 1.72. *One has $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)' = \mathcal{M}_1' \overline{\otimes} \mathcal{M}_2'$. Hence*

$$\mathcal{Z}(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2) = \mathcal{Z}(\mathcal{M}_1) \overline{\otimes} \mathcal{Z}(\mathcal{M}_2).$$

DEFINITION 1.73. We say that projections p, q from $\mathcal{M}$ are equivalent and write $p \sim q$, if they are initial and final projections of a partial isometry $u \in \mathcal{M}$, i.e. $p = u^*u$ and $q = uu^*$. If $p \sim r \leq q$ for some $p, q, r \in \mathbb{P}(\mathcal{M})$, then we write $p \precsim q$ and say that p is *dominated* by q. If $p \nsim q$ and $p \precsim q$, we write $p \prec q$ and say that q *strictly dominates* p.

PROPOSITION 1.74. *$\sim$ is an equivalence relation on $\mathbb{P}(\mathcal{M})$, and $\precsim$ is a partial order on $\mathbb{P}(\mathcal{M})$. In particular, $p \precsim q$ and $q \precsim p$ implies $p \sim q$ for $p, q \in \mathbb{P}(\mathcal{M})$.*

PROPOSITION 1.75. *If $\{p_i\}_{i \in I}$ and $\{q_i\}_{i \in I}$ are two orthogonal families of projections from $\mathcal{M}$ such that $p_i \sim q_i$ for each $i \in I$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$. If, on the other hand, $p_i \precsim q_i$ for each $i \in I$, then $p \precsim q$.*

THEOREM 1.76 (Comparability theorem). *For any $p, q \in \mathbb{P}(\mathcal{M})$, there exists a central projection $z \in \mathcal{M}$ such that $pz \precsim qz$ and $pz^{\perp} \succsim qz^{\perp}$.*

We know from polar decomposition of $a \in \mathcal{M}$ that $\mathrm{s}_l(a) \sim \mathrm{s}_r(a)$. This leads to the highly useful result of Kaplansky [**Kap68**, p. 81]:

PROPOSITION 1.77 (Kaplansky's parallelogram law). *For any $p, q \in \mathbb{P}(\mathcal{M})$,*

$$p \vee q - q \sim p - p \wedge q.$$

PROOF. Note that $\mathrm{n}(q^\perp p) = p^\perp + p \wedge q$ and $\mathrm{n}(pq^\perp) = q + p^\perp \wedge q^\perp$. Hence $\mathrm{s}_r(q^\perp p) = \mathrm{n}(q^\perp p)^\perp = p - p \wedge q$ and $\mathrm{s}_l(q^\perp p) = \mathrm{s}_r(pq^\perp) = \mathrm{n}(pq^\perp)^\perp = p \vee q - q$. Thus $\mathrm{s}_l(q^\perp p) \sim \mathrm{s}_r(q^\perp p)$ yields the result. $\square$

DEFINITION 1.78. A projection $p \in \mathcal{M}$ is called:
  (1) *minimal*, if $p \neq 0$ and $0 \neq q \leq p$ implies $q = p$;
  (2) *abelian*, if $p \neq 0$ and $\mathcal{M}_p$ is abelian;
  (3) *finite*, if $p \sim q \leq p$ implies $q = p$;
  (4) *infinite*, if it is not finite;
  (5) *properly infinite*, if $p \neq 0$ and pz is infinite for any central projection z such that $pz \neq 0$.
  (6) *purely infinite*, if $p \neq 0$ and there is no non-zero finite projection $q \in \mathcal{M}$ with $q \leq p$;

PROPOSITION 1.79.(1) *A non-zero subprojection of a minimal projection is equal to the projection;*
  (2) *a subprojection of an abelian projection is an abelian projection;*
  (3) *a subprojection of a finite projection is finite;*
  (4) *a non-zero subprojection of a purely infinite projection is purely infinite.*

PROPOSITION 1.80. *If $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ are two orthogonal families of projections from $\mathcal{M}$ such that $e_i \sim f_i$ for all $i \in I$, then $\sum_{i \in I} e_i \sim \sum_{i \in I} f_i$.*

PROPOSITION 1.81. *If $z \in \mathcal{Z}(\mathcal{M})$ and $\{e_i\}_{i \in I}$, $\{f_j\}_{j \in J}$ are two orthogonal families of abelian projections from $\mathcal{M}$ such that $\mathrm{z}(e_i) = \mathrm{z}(f_j) = z$ for all $i \in I$, $j \in J$, and $\sum_{i \in I} e_i = \sum_{j \in J} e_i = z$, then the cardinal numbers of I and J are equal. Similarly, if $z \in \mathcal{Z}(\mathcal{M})$ and $\{e_i\}_{i \in I}$, $\{f_j\}_{j \in J}$ are two orthogonal families of equivalent finite projections from $\mathcal{M}$ such that $\mathrm{z}(e_i) = \mathrm{z}(f_j) = z$ for all $i \in I$, $j \in J$, and $\sum_{i \in I} e_i = \sum_{j \in J} e_i = z$, then the cardinal numbers of I and J are equal.*

PROPOSITION 1.82. *Abelian projections are 'minimal' in the following sense: if $p, q \in \mathbb{P}(\mathcal{M})$ with p abelian and $p \leq \mathrm{z}(q)$, then $p \precsim q$. An abelian projection in a factor is a minimal projection.*

LEMMA 1.83 (Halving lemma). *Let $p \in \mathbb{P}(\mathcal{M})$.*

(1) *If p is properly infinite, then there is a $q \in \mathbb{P}(\mathcal{M})$ with $q \leq p$ such that $q \sim p - q \sim p$.*
(2) *If p does not have any abelian subprojection, then there is a $q \in \mathbb{P}(\mathcal{M})$ such that $q \sim p - q$.*

DEFINITION 1.84. A von Neumann algebra $\mathcal{M}$ is said to be:

(1) *discrete*, if every non-zero central projection majorizes a non-zero abelian projection;
(2) *continuous*, if there are no abelian projections in it;
(3) *finite*, if $\mathbb{1}$ is finite;
(4) *infinite*, if $\mathbb{1}$ is infinite;
(5) *purely infinite*, if $\mathbb{1}$ is purely infinite;
(6) *properly infinite*, if $\mathbb{1}$ is properly infinite;
(7) *semifinite*, if there are no purely infinite projections in its centre.

DEFINITION 1.85. A von Neumann algebra $\mathcal{M}$ is said to be:

(1) *of type I*, if it is discrete;
(2) *of type II*, if there are no non-zero abelian projections and no purely infinite projections in $\mathcal{M}$;
(3) *of type III*, if it is purely infinite;
(4) *of type I_α* with α a cardinal number $\leq \#\mathcal{M}$, if $\mathbb{1}$ is the sum of α abelian projections with central support $\mathbb{1}$;
(5) *of type I_∞*, if it is properly infinite and of type I;
(6) *of type II_1*, if it is finite and of type II;
(7) *of type II_α* with α a cardinal number such that $\aleph_0 \leq \alpha \leq \#\mathcal{M}$, if $\mathbb{1}$ is the sum of α equivalent finite projections with central support $\mathbb{1}$;
(8) *of type II_∞*, if it is properly infinite and of type II.

THEOREM 1.86 (Structure of von Neumann algebras).

(1) *Every von Neumann algebra is (isomorphic to) a direct sum of algebras of type I, II_1, II_∞, III (with some summands possibly missing). Each factor von Neumann algebra is of one of the types I, II_1, II_∞, III.*
(2) *Every von Neumann algebra of type I can be uniquely represented as a direct sum of algebras of type I_α, $\alpha \leq \#\mathcal{M}$. If $\mathcal{M}$ is finite, then all the α's in the direct sum are finite. An algebra of type*

I_α is isomorphic to $\mathcal{K}\overline{\otimes}B(H)$, where $\mathcal{K}$ is abelian and H is α-dimensional. A factor of type I_α is isomorphic to $B(H)$ with H α-dimensional.

(3) Every von Neumann algebra of type II_∞ can be represented in a unique way as a direct sum of algebras of type II_α, $\aleph_0 \leq \alpha \leq \#\mathcal{M}$. An algebra of type II_α is isomorphic to $\mathcal{N}\overline{\otimes}B(H)$, where $\mathcal{N}$ is of type II_1 and H is α-dimensional. If $\mathcal{M}$ is a factor, then $\mathcal{N}$ is also a factor.

(4) Every finite-dimensional von Neumann algebra is a direct sum of a finite number of type I_n factors, with $n \in \mathbb{N}$. A finite-dimensional factor is isomorphic, for some $n \in \mathbb{N}$, to the algebra of complex $n \times n$-matrices.

(5) Every finite von Neumann algebra is a direct sum of σ-finite ones.

(6) Every commutative von Neumann algebra is of type I_1.

We will make the above structure theorem for von Neumann algebras clearer by explaining points (2) and (3) of the theorem in more detail. To this aim, we introduce a useful notion of a matrix unit (we use Definition IV.1.7 of Takesaki [**Tak02**]).

DEFINITION 1.87. A family $\{u_{i,j}\}_{i,j\in I}$ of elements of a von Neumann algebra $\mathcal{M}$ is called a *matrix unit* in $\mathcal{M}$ if

(1) $u_{i,j}^* = u_{j,i}$;
(2) $u_{i,j}u_{k,\ell} = \delta_{j,k}u_{i,\ell}$;
(3) $\sum_{i\in I} u_{i,i} = \mathbb{1}$,

with summation in the strong topology (observe that by (1) and (2) $u_{i,i} \in \mathbb{P}(\mathcal{M})$).

The simplest example of a matrix unit can be found in the full von Neumann algebra $B(H)$.

PROPOSITION 1.88. Let $\{e_i\}_{i\in I}$ be a maximal family of minimal projections in $B(H)$. By Structure Theorem 1.86 and Proposition 1.82, $\sum_{i\in I} e_i = \mathbb{1}$ and all the projections are mutually equivalent. Choose one minimal projection e_{i_0} from the family. Let $\{u_i\}_{i\in I}$ be partial isometries such that $e_{i_0} = u_i^* u_i$ and $e_i = u_i u_i^*$ for all $i \in I$. Put $u_{i,j} := u_i u_j^*$ for $i,i \in I$. Then $\{u_{i,j}\}_{i,j\in I}$ is a matrix unit. Let $\mathcal{M} = \mathcal{N}\overline{\otimes}B(H)$. Each element $a \in \mathcal{M}$ can be written in the form $a = \sum_{i,j\in I} a_{i,j} \otimes u_{i,j}$, with all $a_{i,j}$ in $\mathcal{N}$, where the sum converges strongly.

Now we can write every element of $\mathcal{M} = \mathcal{N} \overline{\otimes} B(H)$ as an infinite matrix with entries in $\mathcal{N}$.

NOTATION 1.89. Let $a \in \mathcal{M} = \mathcal{N} \overline{\otimes} B(H)$. If $a_{i,j}$'s are as in a previous proposition, we write $a = (a_{i,j})_{i,j \in I}$.

PROPOSITION 1.90. *Let $a, b \in \mathcal{M} = \mathcal{N} \overline{\otimes} B(H)$. Then, with the above notation, $(ab)_{i,j} = \sum_{k \in I} a_{i,k} b_{k,j}$, with the sum σ-strong*-convergent.*

To find an example of a commutative W^*-algebra, start with a finite measure space (X, Σ, μ), let $L^\infty(X, \Sigma, \mu)$ denote the C^*-algebra of (equivalence classes of) all essentially bounded μ-measurable (complex) functions on X. By the Radon-Nikodym theorem, $L^\infty(X, \Sigma, \mu) = L^1(X, \Sigma, \mu)^*$, where $L^1(X, \Sigma, \mu)$ is the space of (equivalence classes of) μ-integrable functions on X. Hence $L^\infty(X, \Sigma, \mu)$ is a W^*-algebra.

Note that it is easy to faithfully represent $L^\infty(X, \Sigma, \mu)$ on a Hilbert space. Indeed, put $H := L^2(X, \Sigma, \mu)$, the Hilbert space of (equivalence classes of) square integrable functions on X, and for each $f \in L^\infty(X, \Sigma, \mu)$, the operator m_f on H is defined by $m_f(g) = fg$ (where any representative of the class f will do on the right side of the equality).

For a general theory of commutative von Neumann algebras, we need more general measure spaces. To this aim, we introduce the notion of a measure algebra. The main advantage of this approach is that we obtain the corresponding measure space in a most natural way. Also, the needed notion of 'localizability' turns out to be much more natural for measure algebras.

DEFINITION 1.91. A *Boolean algebra* is a commutative ring $(\mathscr{A}, +, \cdot)$ with a multiplicative identity $1 = 1_{\mathscr{A}}$ satisfying $a^2 = a$ for all $a \in \mathscr{A}$. For $a, b \in \mathscr{A}$, we say that $a \leq b$ if $ab = a$. We call a Boolean algebra *(Dedekind) complete* (resp. *σ-complete*) if every non-empty subset (resp. non-empty countable subset) of the algebra has a least upper bound. A set $\mathcal{F}$ of elements of a Boolean algebra $\mathscr{A}$ is called *disjoint* if $ab = 0$ for any $a, b \in \mathcal{F}$, $a \neq b$. A *measure algebra* is a pair $(\mathscr{A}, \mu_{\mathscr{A}})$ consisting of a σ-complete Boolean algebra $\mathscr{A}$ and a function (called a *measure*) $\mu_{\mathscr{A}} \colon \mathscr{A} \to [0, \infty]$ such that $\mu_{\mathscr{A}}(0) = 0$, $\mu_{\mathscr{A}}(a) > 0$ for $0 \neq a \in \mathscr{A}$, and $\mu_{\mathscr{A}}(\sup_{n \in \mathbb{N}} a_n) = \sum_{n \in \mathbb{N}} \mu_{\mathscr{A}}(a_n)$ for any disjoint countable family $\{a_n\}_{n \in \mathbb{N}}$. We call the measure $\mu_{\mathscr{A}}$ *semifinite* if for any $a \in \mathscr{A}$ with $\mu_{\mathscr{A}}(a) = \infty$ there is a non-zero $b \in \mathscr{A}$ such that $b \leq a$ and $\mu_{\mathscr{A}}(b) < \infty$. A measure algebra $(\mathscr{A}, \mu_{\mathscr{A}})$ is *localizable* if $\mathscr{A}$ is complete and $\mu_{\mathscr{A}}$ is semifinite.

PROPOSITION 1.92. *Let (X, Σ, μ) be a measure space. The σ-field Σ with the operations $\triangle, \cap$ becomes a Boolean algebra (here $\triangle$ denotes symmetric difference). Put $N_\mu = \{A \in \Sigma \colon \mu(A) = 0\}$. Then N_μ is an ideal in Σ. The quotient ring $\mathscr{A} = \Sigma / N_\mu$ is a σ-complete Boolean algebra. Let $\mu_\mathscr{A}$ be the function on $\mathscr{A}$ given by $\mu_\mathscr{A}(a) = \mu(A)$ whenever a is the image of A for the quotient map. Then $(\mathscr{A}, \mu_\mathscr{A})$ is a measure algebra.*

DEFINITION 1.93. *The algebra $(\mathscr{A}, \mu_\mathscr{A})$ from the above proposition is called the measure algebra of (X, Σ, μ).*

DEFINITION 1.94. *The Stone space $\mathcal{X}$ of a measure algebra $(\mathscr{A}, \mu_\mathscr{A})$ is the set $\mathcal{X}$ of (ring) homomorphisms of $\mathscr{A}$ onto $\mathbb{Z}_2$. For any $a \in \mathscr{A}$, define $\hat{a} := \{\chi \in \mathcal{X} \colon \chi(a) = 1\} \subseteq \mathcal{X}$. The map $\mathscr{A} \ni a \mapsto \hat{a} \in \mathcal{P}(\mathcal{X})$ is called the Stone representation of $\mathscr{A}$. The topology $\mathcal{T}$ of $\mathcal{X}$ is the set*

$$\{O \in \mathcal{P}(\mathcal{X}) \colon \text{for any } \chi \in O \text{ there is an } a \in \mathscr{A} \text{ such that } \chi \in \hat{a} \subseteq O\}.$$

Let $\mathfrak{S}$ consist of those $A \in \mathcal{T}$ for which $\mathcal{X} \setminus A \in \mathcal{T}$.

PROPOSITION 1.95. *$\mathcal{T}$ is a locally compact topology on $\mathcal{X}$, and $\mathfrak{S}$ is a σ-algebra on $\mathcal{X}$. We always consider $\mathcal{X}$ as a topological space and measurable space with this topology and this σ-algebra.*

THEOREM 1.96 (Loomis-Sikorski). *Let $\mathscr{A}$ be a σ-complete Boolean algebra, $\mathcal{X}$ its Stone space (with topology $\mathcal{T}$ and σ-algebra $\mathfrak{S}$). Let $N_\mathcal{X}$ denote the set of meagre (or: category I) subsets of $\mathcal{X}$. Then $\mathscr{A}$ is isomorphic, as a Boolean algebra, to $\mathfrak{S}/N_\mathcal{X}$.*

NOTATION 1.97. *We will write $\tilde{A}$ for the image in $\mathscr{A}$ of a set $A \in \mathfrak{S}$ under the composition $\pi \circ \theta$ of the isomorphism $\pi \colon \mathfrak{S}/N_\mathcal{X} \to \mathscr{A}$ and the quotient map $\theta \colon \mathfrak{S} \to \mathfrak{S}/N_\mathcal{X}$.*

THEOREM 1.98. *Let $(\mathscr{A}, \mu_\mathscr{A})$ be a measure algebra. Let $\mathcal{X}$ be its Stone space, and let ν be defined on $\mathfrak{S}$ by $\nu(A) := \mu_\mathscr{A}(\tilde{A})$. Then $(\mathscr{A}, \mu_\mathscr{A})$ is the measure algebra of the measure space $(\mathcal{X}, \mathfrak{S}, \nu)$.*

DEFINITION 1.99. *The measure space $(\mathcal{X}, \mathfrak{S}, \nu)$ constructed above is called the (canonical) measure space of the measure algebra $(\mathscr{A}, \mu_\mathscr{A})$.*

THEOREM 1.100. *Each commutative von Neumann algebra $\mathcal{M}$ admits an additive and positively homogenous functional $\tau \colon \mathcal{M}_+ \to [0, \infty]$ such that $(\mathbb{P}(\mathcal{M}), \mu)$, with $\mu = \tau \upharpoonright \mathbb{P}(M)$, is a localizable measure algebra. If $(\mathcal{X}, \mathfrak{S}, \nu)$ is the corresponding measure space, then $L^\infty(\mathcal{X}, \mathfrak{S}, \nu)$ is a W^*-algebra isomorphic to the von Neumann algebra $\mathcal{M}$.*

Note that the functional τ from the theorem can be easily obtained as a sum of a maximal family of states on $\mathcal{M}$ with mutually orthogonal supports. Since in a σ-finite algebra such a family is at most countable, say $\{\tau_n\}_{n \in \mathbb{N}}$, we can easily get a state τ on such an algebra by putting $\tau := \sum_{n=1}^{\infty} (1/2^n) \tau_n$.

COROLLARY 1.101. *Each σ-finite commutative von Neumann algebra $\mathcal{M}$ admits a state τ such that $(\mathbb{P}(\mathcal{M}), \mu)$, with $\mu = \tau \restriction \mathbb{P}(\mathcal{M})$, is a finite measure algebra.*

1.4 Unbounded operators

A good acquaintance with unbounded operators on a Hilbert space is indispensable for dealing with non-commutative L^p and Orlicz spaces. This material is not known as well as that on bounded operators, and we try to prove whatever possible. A lot of material in this and the following section has been adapted from [**SZ79**].

Let H be a (complex) Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, linear in the first argument and antilinear in the second one.

DEFINITION 1.102. By an (*unbounded*) *operator* on H we understand a linear map x from a linear subspace $\mathrm{dom}(x) \subseteq H$ into H. We call $\mathrm{dom}(x)$ the *domain* of x, and denote by $\mathcal{G}(x)$ the set $\{(\xi, x\xi) : \xi \in \mathrm{dom}(x)\} \subseteq H \oplus H$, the *graph* of x. 0 denotes an operator x with $\mathrm{dom}(x) := H$ and $x\xi = 0$ for all $\xi \in H$.

DEFINITION 1.103 (Operations on unbounded operators). We say that:

(1) x and y are *equal* and write $x = y$ if $\mathcal{G}(x) = \mathcal{G}(y)$. Then obviously $\mathrm{dom}(x) = \mathrm{dom}(y)$ and $x\xi = y\xi$ for all $\xi \in \mathrm{dom}(x)$.

(2) y is an *extension* of x and write $x \subseteq y$ or $y \supseteq x$ if $\mathcal{G}(x) \subseteq \mathcal{G}(y)$. Then obviously $\mathrm{dom}(x) \subseteq \mathrm{dom}(y)$ and $x\xi = y\xi$ for all $\xi \in \mathrm{dom}(x)$.

(3) x is *positive* if $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathrm{dom}(x)$.

(4) For any $\lambda \in \mathbb{C}$ and any operator x, we define operator λx with $\mathrm{dom}(\lambda x) := \mathrm{dom}(x)$ by $(\lambda x)\xi := \lambda(x\xi)$ for all $\xi \in \mathrm{dom}(x)$.

(5) For any operators x, y we define the *sum* of x and y as the operator $x + y$ with $\mathrm{dom}(x+y) := \mathrm{dom}(x) \cap \mathrm{dom}(y)$ by $(x+y)\xi := x\xi + y\xi$ for all $\xi \in \mathrm{dom}(x+y)$. We define the *difference* of x and y as the operator $x - y := x + (-1)y$.

(6) For any operators x, y we define the *product* or *composition* of x and y as the operator xy with $\operatorname{dom}(xy) := \{\xi \in \operatorname{dom}(y): y\xi \in \operatorname{dom}(x)\}$ by $(xy)\xi := x(y\xi)$ for all $\xi \in \operatorname{dom}(xy)$.

(7) For any *injective* operator x, we define the *inverse* operator x^{-1} with $\operatorname{dom}(x^{-1}) := x\operatorname{dom}(x)$ by $x^{-1}(\eta) := \xi$ whenever $\eta = x\xi$ for all $\eta \in \operatorname{dom}(x^{-1})$.

PROPOSITION 1.104. *Addition of operators is commutative and associative, multiplication of operators is associative. We also have, for any operators x_1, x_2, y,*

$$(x_1 + x_2)y = x_1 y + x_2 y$$
$$y(x_1 + x_2) \supseteq yx_1 + yx_2.$$

If x is injective, then $(x^{-1})^{-1} = x$. If additionally y is injective and $x \subseteq y$, then $x^{-1} \subseteq y^{-1}$.

PROOF. Obvious from definitions. □

DEFINITION 1.105. Operator x is:
(1) *densely defined* if its domain is dense in H;
(2) *closed* if its graph is closed in $H \oplus H$;
(3) *closable* of *preclosed* if the closure $\overline{\mathcal{G}(x)}$ of the graph of x is itself a graph of some operator y. We write then $[x] := y$ and call $[x]$ the *closure* of x. It is the smallest closed extension of x, in the sense that $x \subseteq [x]$ and if, for some closed z, we have $x \subseteq z$, then $[x] \subseteq z$. x is preclosed if for any sequence (ξ_n) from $\operatorname{dom}(x)$, whenever $\xi_n \to 0$ and $x\xi_n$ converges, then $x\xi_n \to 0$.
(4) *bounded* if it is everywhere defined and $\|x\| := \sup\{\|x\xi\|: \xi \in H, \|\xi\| \leq 1\} < \infty$. In this case $\|x\|$ is the *norm* of x. The set of all bounded operators on H is denoted by $B(H)$.

PROPOSITION 1.106.(1) *x is closed if whenever (ξ_n) is a sequence from $\operatorname{dom}(x)$ such that $\xi_n \to \xi \in H$ and $x\xi_n \to \eta \in H$, then $\xi \in \operatorname{dom}(x)$ and $\eta = x\xi$.*
(2) *x is preclosed if for any sequence (ξ_n) from $\operatorname{dom}(x)$, whenever $\xi_n \to 0$ and $x\xi_n$ converges, then $x\xi_n \to 0$.*
(3) *if x is densely defined and $\sup\{\|x\xi\|: \xi \in \operatorname{dom}(x), \|\xi\| \leq 1\} < \infty$, then x is closable and $[x]$ is bounded.*
(4) *If x is closed and $\operatorname{dom}(x) = H$, then x is bounded.*
(5) *If x is closed and injective, then x^{-1} is closed.*

(6) *If x is closed, then its kernel is closed.*

PROOF. (1) The condition guarantees that the graph $\mathcal{G}(x)$ of x is closed.

(2) The condition guarantees that the closure of $\mathcal{G}(x)$ is the graph of a function. We can then define $\mathrm{dom}([x]) := \{\xi \in H \colon$ there is an $\eta \in H$ such that $(\xi, \eta) \in \overline{\mathcal{G}(x)}\}$ and $[x]\xi$, for $\xi \in \mathrm{dom}([x])$, as the unique element $\eta \in H$ such that $(\xi, \eta) \in \overline{\mathcal{G}(x)}$.

(3) If $\xi \in H$, there is a sequence (ξ_n) in $\mathrm{dom}(x)$ such that $\xi_n \to \xi$. The boundedness condition guarantees that the image $(x\xi_n)$ of the Cauchy sequence (ξ_n) is itself a Cauchy sequence and $x\xi_n \to \xi$. Consequently, the condition of closability is satisfied and the closure of x is everywhere defined and satisfies the boundedness condition, hence it is bounded.

(4) This is the famous *closed graph theorem*.

(5) Immediate from $\mathcal{G}(x^{-1}) = \{(\eta, \xi) \colon (\xi, \eta) \in \mathcal{G}(x)\}$.

(6) Immediate from (1).   □

DEFINITION 1.107. For a closed densely defined operator x on H, we define:

(1) the *null projection* $\mathtt{n}(x)$ as the projection onto the null space $\{\xi \colon x\xi = 0\}$, i.e. the kernel of x;

(2) the *right support* $\mathtt{s}_r(x) := \mathbb{1} - \mathtt{n}(x)$;

(3) the *left support* $\mathtt{s}_l(x)$ as the projection onto the closure (in H) of $x(\mathrm{dom}(x))$.

LEMMA 1.108. *Let x be a densely defined operator on H. Let $f_\eta \colon \mathrm{dom}(x) \ni \xi \mapsto \langle x\xi, \eta \rangle \in \mathbb{C}$. Put $D := \{\eta \colon f_\eta$ is bounded $\}$. Then D is a linear subspace of H. If $\eta \in D$, then there exists a unique $\zeta \in H$ such that $\langle \xi, \zeta \rangle = \langle x\xi, \eta \rangle$.*

PROOF. The density of $\mathrm{dom}(x)$ implies that f_η extends to a bounded linear form on the whole of H. By Riesz theorem there exists a unique $\zeta \in H$ such that $\langle \xi, \zeta \rangle = \langle x\xi, \eta \rangle$.   □

DEFINITION 1.109. We define the *adjoint* of x to be an operator x^* with domain $\mathrm{dom}(x^*) := D$ from the previous lemma such that $x^*\eta := \zeta$. In other words, we have $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle$ for $\xi \in \mathrm{dom}(x)$ and $\eta \in \mathrm{dom}(x^*)$. We say that a densely defined operator x is *self-adjoint* if $x = x^*$.

PROPOSITION 1.110. *For any densely defined operator x on H:*

(1) x^* *is closed;*

(2) x *is preclosed (closable) if and only if* x^* *is densely defined, in which case* $[x] = x^{**}$;

(3) $\mathbb{s}_r(x^*) = \mathbb{s}_l(x)$;

PROOF. (1) Let $\eta_n \in \operatorname{dom}(x^*)$, $\eta_n \to \eta$ and $x^*\eta_n \to \zeta$. Then, for $\xi \in \operatorname{dom}(x)$,

$$\langle x\xi, \eta \rangle = \lim_{n\to\infty} \langle x\xi, \eta_n \rangle = \lim_{n\to\infty} \langle \xi, x^*\eta_n \rangle = \langle \xi, \zeta \rangle.$$

Hence $\eta \in \operatorname{dom}(x^*)$ and $x^*\eta = \zeta$, which shows that x^* is closed.

(2) Let u be an operator on $H \oplus H$ given by $(\xi, \eta) \mapsto (\eta, -\xi)$. It is easy to check that u is a unitary, and $u^* = u^{-1} = -u$. If $(\eta, x^*\eta) \in \mathcal{G}(x^*)$ and $(\xi, x\xi) \in \mathcal{G}(x)$, then

$$\langle u(\xi, x\xi), (\eta, x^*\eta) \rangle = \langle (x\xi, -\xi), (\eta, x^*\eta) \rangle = \langle x\xi, \eta \rangle + \langle -\xi, x^*\eta \rangle = 0.$$

Hence $\mathcal{G}(x^*) \subseteq (u\mathcal{G}(x))^\perp$. To see that equality holds, observe that if $(\eta, \zeta) \in (u\mathcal{G}(x))^\perp$, then

$$0 = \langle u(\xi, x\xi), (\eta, \zeta) \rangle = \langle (x\xi, -\xi), (\eta, \zeta) \rangle = \langle x\xi, \eta \rangle + \langle -\xi, \zeta \rangle.$$

Thus $\xi \to \langle x\xi, \eta \rangle$ then corresponds to the continuous mapping $\xi \to \langle \xi, \zeta \rangle$, and hence by definition $\eta \in \operatorname{dom}(x^*)$ with $\zeta = x^*\eta$. Consequently $\mathcal{G}(x^*) = (u\mathcal{G}(x))^\perp$, whence

$$\mathcal{G}(x^*)^\perp = (u\mathcal{G}(x))^{\perp\perp} = \overline{u\mathcal{G}(x)} = u\overline{\mathcal{G}(x)}.$$

It can now easily be verified that $\eta \perp \operatorname{dom}(x^*)$ if and only if $(\eta, 0) \in (\mathcal{G}(x^*))^\perp$ if and only if $(0, \eta) = u^*(\eta, 0) \in \overline{\mathcal{G}(x)}$. The only way that $(0, \eta)$ can belong to $\overline{\mathcal{G}(x)}$ is if there existed a sequence (ξ_n) in $\operatorname{dom}(x)$ such that $\xi_n \to 0$ whilst $x(\xi_n) \to \eta$. This clearly shows that x fulfils the criteria for closability if and only if $\operatorname{dom}(x^*)$ is dense in H. Finally,

$$\mathcal{G}(x^{**}) = (u^*\mathcal{G}(x^*))^\perp = (u^*(u\mathcal{G}(x))^\perp)^\perp = \mathcal{G}(x)^{\perp\perp} = \mathcal{G}([x]),$$

whence $x^{**} = [x]$.

(3) Since x^* is closed, its null space is closed, and $\mathbb{s}_r(x^*)^\perp(H) = \mathfrak{n}(x^*)(H) \subseteq \operatorname{dom}(x^*)$. If $\eta \in \mathfrak{n}(x^*)(H)$, then $x^*\eta = 0$ and $\langle x\xi, \eta \rangle = \langle \xi, x^*\eta \rangle = 0$ for all $\xi \in \operatorname{dom}(x)$. Hence $\eta \perp \mathbb{s}_l(x)(H)$ and $\mathbb{s}_l(x)(H) \subseteq \mathbb{s}_r(x^*)(H)$. If, on the other hand, $\eta \perp \mathbb{s}_l(x)(H)$, then $\langle x\xi, \eta \rangle = 0$ for all $\xi \in \operatorname{dom}(x)$ implies that $\eta \in \operatorname{dom}(x^*)$ and $x^*\eta = 0$, which means that $\eta \perp \mathbb{s}_r(x^*)$. Consequently, $\mathbb{s}_r(x^*)(H) \subseteq \mathbb{s}_l(x)(H)$, which ends the proof. □

PROPOSITION 1.111. *Let $\lambda \in \mathbb{C}$, and assume that $x, y, x + y$ and xy are densely defined operators on H, and that $a \in B(H)$. Then:*

(1) $(\lambda x)^* = \bar{\lambda} x^*$;

(2) *if* $x \subseteq y$, *then* $x^* \supseteq y^*$;

(3) $(x + y)^* \supseteq x^* + y^*$;

(4) $(xy)^* \supseteq y^* x^*$;

(5) *if x is injective and $x(\mathrm{dom}(x))$ dense in H, then* $(x^{-1})^* = (x^*)^{-1}$;

(6) $(x + a)^* = x^* + a^*$;

(7) $(ax)^* = x^* a^*$.

PROOF. (1)–(4) follow easily from the definitions. We will show (4) to indicate the way. Assume $\eta \in \mathrm{dom}(y^* x^*)$ and $\xi \in \mathrm{dom}(xy)$. Then

$$\langle xy\xi, \eta \rangle = \langle y\xi, x^*\eta \rangle = \langle \xi, y^* x^* \eta \rangle$$

From the density of $\mathrm{dom}(xy)$ and continuity of the map $\xi \mapsto \langle xy\xi, \eta \rangle$ we infer $\eta \in \mathrm{dom}((xy)^*$, hence $\mathrm{dom}(y^* x^*) \subseteq \mathrm{dom}((xy)^*)$ and $\langle \xi, y^* x^* \eta \rangle = \langle \xi, (xy)^* \eta \rangle$, so that $(xy)^* \supseteq y^* x^*$.

(5) If $\eta \in \mathrm{dom}((x^*)^{-1})$, then $\eta = x^* \zeta$ for some $\zeta \in \mathrm{dom}(x^*)$, so that, for $\xi \in \mathrm{dom}(x)$,

$$\langle x\xi, (x^{-1})^* \eta \rangle = \langle x\xi, (x^{-1})^* x^* \zeta \rangle = \langle \xi, x^* \zeta \rangle = \langle x\xi, \zeta \rangle$$
$$= \langle x\xi, (x^*)^{-1} x^* \zeta \rangle = \langle x\xi, (x^*)^{-1} \eta \rangle.$$

Hence $\eta \in \mathrm{dom}((x^{-1})^*)$, so that $(x^*)^{-1} \subseteq (x^{-1})^*$. Put now $y := x^{-1}$. Then, by what we have just proved, $(y^*)^{-1} \subseteq (y^{-1})^* = x^*$. By Proposition 1.104, $(x^{-1})^* \subseteq (x^*)^{-1}$.

(6) We have $\mathrm{dom}(x+a) = \mathrm{dom}(x)$. Since $\langle (x+a)\xi, \eta \rangle = \langle x\xi, \eta \rangle + \langle a\xi, \eta \rangle$, domains of x^* and $(x + a)^*$ coincide. Now if $\xi \in \mathrm{dom}(x + a) = \mathrm{dom}(x)$ and $\eta \in \mathrm{dom}((x + a)^*) = \mathrm{dom}(x^*)$, then

$$\langle \xi, (x + a)^* \eta \rangle = \langle (x + a)\xi, \eta \rangle = \langle x\xi, \eta \rangle + \langle a\xi, \eta \rangle$$
$$= \langle \xi, x^* \eta \rangle + \langle \xi, a^* \eta \rangle = \langle \xi, (x^* + a^*)\eta \rangle.$$

Hence $(x + a)^* = x^* + a^*$.

(7) Note that $\mathrm{dom}(ax) = \mathrm{dom}(x)$ and by (4), $(ax)^* \supseteq x^* a^*$. Take $\eta \in \mathrm{dom}((ax)^*)$ and $\xi \in \mathrm{dom}(x)$. We have

$$\langle \xi, (ax)^* \eta \rangle = \langle ax\xi, \eta \rangle = \langle x\xi, a^* \eta \rangle$$

Hence $a^* \eta \in \mathrm{dom}(x^*)$ and $\eta \in \mathrm{dom}(x^* a^*)$. Consequently, $(ax)^* \subseteq x^* a^*$ and $(ax)^* = x^* a^*$. $\square$

One can find the following useful result, among others, in [**KR83**, Theorem 2.7.8(v)].

PROPOSITION 1.112. *If an unbounded operator x is closed and densely defined, then x^*x is self-adjoint.*

THEOREM 1.113 (Spectral decomposition for unbounded operators). *Every self-adjoint x acting on a Hilbert space H has a unique spectral decomposition*

$$x = \int_{-\infty}^{\infty} \lambda de_\lambda, \tag{*}$$

where $\{e_\lambda\}$ is a resolution of identity, that is a family of projections satisfying $e_\lambda \leq e'_\lambda$ for $\lambda \leq \lambda'$ with strong convergence of $e_\lambda \to 0$ as $\lambda \to -\infty$, $e_\lambda \to \mathbb{1}$ as $\lambda \to \infty$ and with $e_{\lambda+} = e_\lambda$ for each $\lambda \in \mathbb{R}$ (continuity from the right in the sense of strong convergence). We call $e_\lambda = e_\lambda(x)$ the spectral resolution of x.

The integral (*) can be understood in a weak sense:

$$\langle x\xi, \xi \rangle = \int_{-\infty}^{\infty} \lambda d\langle e_\lambda \xi, \xi \rangle.$$

The domain of x is given by

$$\mathrm{dom}(x) = \{\xi \in H \colon \int_{-\infty}^{\infty} \lambda^2 d\langle e_\lambda \xi, \xi \rangle < \infty\}.$$

and

$$\|x\xi\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\langle e_\lambda \xi, \xi \rangle.$$

We shall use the functional calculus of unbounded operators only for positive self-adjoint ones.

THEOREM 1.114 (Borel functional calculus for unbounded operators). *Let x be a positive self-adjoint operator on H and $f \in \mathcal{B}([0,\infty))$, the set of complex Borel measurable functions on $[0,\infty)$ that are bounded on compact sets. The following equation defines a unique operator $f(x)$ by:*

$$\langle f(x)\xi, \xi \rangle = \int_0^{\infty} f(\lambda)d\langle e_\lambda \xi, \xi \rangle,$$

written further as

$$f(x) = \int_0^{\infty} f(\lambda)de_\lambda,$$

with domain

$$\text{dom}(f(x)) = \{\xi \in H : \int_0^\infty |f(\lambda)|^2 d\langle e_\lambda \xi, \xi \rangle < \infty\}.$$

Moreover, there is a dense subspace D of H contained in $\text{dom}(f(x))$ for any $f \in \mathcal{B}([0, \infty))$, and $\overline{f(x) \upharpoonright D} = f(x)$, i.e. D is a core for all $f(x)$. The subspace D can be obtained as a union of a countable number of ranges of spectral projections of x. We have

$$\|f(x)\xi\|^2 = \int_0^\infty |f(\lambda)|^2 d\langle e_\lambda \xi, \xi\rangle \text{ for } \xi \in D.$$

The above theorem yields immediately a square root an unbounded positive self-adjoint operator, thus extending the notion of a square root of a positive bounded operator (cf. Definition 1.19 and the comment after Theorem 1.43). Similarly, we can define absolute value of an unbounded closed densely defined operator.

DEFINITION 1.115. For any closed densely defined x on H we define $|x| := (x^*x)^{1/2}$, and call it the *absolute value* (or *modulus*) of x.

The following proposition extends the results of Proposition 1.44 to unbounded operators (cf. Proposition 1.110(3) for a part that is true with weaker assumptions).

PROPOSITION 1.116. *For any closed densely defined operator x on H, we have $s_l(x) = s_r(x^*)$ and $s_r(x) = s_l(x^*)$. Moreover, $s_l(x) = s(xx^*)$ and $s_r(x) = s(x^*x)$. For positive x, $s(x) = s(x^{1/2})$, in particular $s(|x|) = s_r(x)$ and $s(|x^*|) = s_l(x)$.*

One has also polar decomposition of unbouded operators (cf. Theorem 1.45).

THEOREM 1.117 (Polar decomposition). *Let x be a closed densely defined operator on H. There exists a partial isometry $u \in B(H)$ with initial projection $s_r(x)$ and final projection $s_l(x)$ such that $x = u|x| = |x^*|u$. If $x = vy$ with y positive and $v \in B(H)$ a partial isometry with initial projection $s(b)$, then $v = u$ and $y = |x|$. Moreover, $u(x^*x)u^* = xx^*$.*

DEFINITION 1.118. The unique representation of a closed operator x in the form $x = u|x|$ is called *the polar decomposition of x*.

The following easy technical result will be used in the sequel:

LEMMA 1.119. *Let $\{p_n\}_{n\in\mathbb{N}}$ be an orthogonal family of projections on H and let $\{\lambda_n\}$ be a family of positive numbers. Then the operator x defined on $D := \bigcup_{n\in\mathbb{N}} p_n H$ by $p_n\xi = \lambda_n\xi$ is closable, and its closure is a positive self-adjoint operator on H.*

NOTATION 1.120. We will denote by $\sum_{n\in\mathbb{N}} \lambda_n p_n$ the positive self-adjoint operator from the previous proposition.

1.5   Affiliated operators

Operators affiliated with a von Neumann algebra are those unbounded operators whose spectral projections belong to the algebra. They are home to all the classes of operators important for the non-commutative theory.

LEMMA 1.121. *Let $\mathcal{M}$ be a von Neumann algebra. For an unbounded operator x on H, the following conditions are equivalent:*

(1) *$u'x = xu'$ (or, equivalently, $u'^*xu' = x$) for any $u' \in \mathbb{U}(\mathcal{M}')$;*
(2) *$u'x \subseteq xu'$ for any $u' \in \mathbb{U}(\mathcal{M}')$;*
(3) *$a'x \subseteq xa'$ for any $a' \in \mathcal{M}'$.*

PROOF. (1)$\Rightarrow$(2): is obvious.

(2)$\Rightarrow$(3) We start with representing a' as a linear combination of unitaries: $a' = \sum_{n=1}^4 \lambda_n u'_n$. Then

$$\operatorname{dom}(xa') = \{\xi \in H : a'\xi \in \operatorname{dom}(x)\}$$

$$= \{\xi \in H : \sum_{n=1}^4 \lambda_n u'_n\xi \in \operatorname{dom}(x)\}$$

$$\supseteq \bigcap_{n=1}^4 \{\xi \in H : u'_n\xi \in \operatorname{dom}(x)\} = \bigcap_{n=1}^4 \operatorname{dom}(xu'_n)$$

$$\supseteq \bigcap_{n=1}^4 \operatorname{dom}(u'_n x) = \operatorname{dom}(x) = \operatorname{dom}(a'x).$$

For $\xi \in \operatorname{dom}(a'x)$ we have

$$a'x\xi = \sum_{n=1}^4 \lambda_n u'_n x\xi = \sum_{n=1}^4 \lambda_n x u'_n\xi = x\left(\sum_{n=1}^4 \lambda_n u'_n\xi\right) = xa'\xi,$$

which ends the proof.

(3)$\Rightarrow$(1): Take $u' \in \mathbb{U}(\mathcal{M}')$. By (3), $u'x \subseteq xu'$, so that $\operatorname{dom}(x) = \operatorname{dom}(u'x) \subseteq \operatorname{dom}(xu') = u'^*(\operatorname{dom}(x))$, which yields $u'(\operatorname{dom}(x)) \subseteq \operatorname{dom}(x)$.

Using u'^* instead of u' gives $u'^*(\mathrm{dom}(x)) \subseteq \mathrm{dom}(x)$. Hence $u'(\mathrm{dom}(x)) = \mathrm{dom}(x)$, which yields $\mathrm{dom}(u'x) = \mathrm{dom}(xu')$, so that $u'x = xu'$. $\qquad\square$

DEFINITION 1.122. A (not necessarily bounded, not necessarily densely defined) operator x on H is *affiliated* to the von Neumann algebra $\mathcal{M}$ if it satisfies one of the equivalent conditions of the previous lemma. The set of operators affiliated to $\mathcal{M}$ is denoted by $^\eta\mathcal{M}$. The set of closed densely defined operators affiliated with $\mathcal{M}$ is denoted by $\overline{\mathcal{M}}$, the real-linear subspace of self-adjoint operators by $\overline{\mathcal{M}}_h$, and the subset of $\overline{\mathcal{M}}_h$ consisting of positive operators by $\overline{\mathcal{M}}_+$.

LEMMA 1.123.(1) *If $x \in {}^\eta\mathcal{M}$ and $x \in B(H)$, then $x \in \mathcal{M}$.*
(2) *If $x, y \in {}^\eta\mathcal{M}$ and $\lambda \in \mathbb{C}$, then $x + y \in {}^\eta\mathcal{M}$, $xy \in {}^\eta\mathcal{M}$ and $\lambda x \in {}^\eta\mathcal{M}$.*
(3) *If $x \in {}^\eta\mathcal{M}$ is densely defined, then $x^* \in {}^\eta\mathcal{M}$.*
(4) *If $x \in {}^\eta\mathcal{M}$ is preclosed (closable), then $[x] \in {}^\eta\mathcal{M}$.*
(5) *If $x \in {}^\eta\mathcal{M}$ is self-adjoint and (e_λ) is the spectral resolution of x, then $e_\lambda \in \mathcal{M}$ for all $\lambda \in \mathbb{R}$. Moreover, $f(x) \in {}^\eta\mathcal{M}$ for each $f \in \mathcal{B}([0, \infty))$, and one can choose a core D for all $f(x)$ with $f \in \mathcal{B}([0, \infty))$ (see Theorem 1.114) to be affiliated with $\mathcal{M}$.*
(6) *If x is a closed, densely defined operator on H with polar decomposition $x = u|x|$, then $x \in \overline{\mathcal{M}}$ if and only if $u \in \mathcal{M}$ and $|x| \in \overline{\mathcal{M}}_+$.*
(7) *If $\{p_n\}_{n \in \mathbb{N}}$ is an orthogonal family of projections from $\mathcal{M}$ and $\{\lambda_n\}$ is a family of positive numbers, then $\sum_{n \in \mathbb{N}} \lambda_n p_n \in {}^\eta\mathcal{M}$.*

PROOF. Let $u' \in \mathbb{U}(\mathcal{M}')$.
(1) By definition and von Neumann's double commutant theorem.
(2) Suppose $x, y \in {}^\eta\mathcal{M}$. Since $\mathrm{dom}(u'(x + y)) = \mathrm{dom}(x + y)$ and $\mathrm{dom}(u'x + u'y) = \mathrm{dom}(u'x) \cap \mathrm{dom}(u'y) = \mathrm{dom}(x) \cap \mathrm{dom}(y) = \mathrm{dom}(x + y)$, by Proposition 1.104 we have $u'(x+y) = u'x + u'y = xu' + yu' = (x+y)u'$, so that $x + y \in {}^\eta\mathcal{M}$. Also, $u'xy = xu'y = xyu'$, so that $xy \in {}^\eta\mathcal{M}$. It is obvious that $\lambda x \in {}^\eta\mathcal{M}$ for any $\lambda \in \mathbb{C}$.
(3) Suppose $x \in {}^\eta\mathcal{M}$ be densely defined. Then, by Proposition 1.111(4) and (7),
$$u'x^* \subseteq (xu'^*)^* = (u'^*x)^* = x^*u',$$
which shows that $x^* \in {}^\eta\mathcal{M}$.
(4) Assume $x \in {}^\eta\mathcal{M}$ is preclosed. Then u'^*xu' is preclosed and $[x] = [u'^*xu'] = u'^*[x]u'$, so that $[x] \in {}^\eta\mathcal{M}$.

(5) If $x \in {}^{\eta}\mathcal{M}$ and $u' \in \mathbb{U}(\mathcal{M}')$, then

$$x = u'xu'^* = u' \int_{-\infty}^{\infty} \lambda de_\lambda u'^* = \int_{-\infty}^{\infty} \lambda d(u'e_\lambda u'^*).$$

By the uniqueness of the spectral decomposition, $e_\lambda = u'e_\lambda u'^*$ for all $\lambda \in \mathbb{R}$, so that $e_\lambda \in {}^{\eta}\mathcal{M}$ and by (1), $e_\lambda \in \mathcal{M}$. This implies $f(x) \in {}^{\eta}\mathcal{M}$ for each $f \in \mathcal{B}([0,\infty))$. It is clear that a countable union of ranges of spectral projections of x (see Theorem 1.114) is affiliated with $\mathcal{M}$.

(6) Suppose $x \in \overline{\mathcal{M}}$, with polar decomposition $x = u|x|$. We have, for any $u' \in \mathbb{U}(\mathcal{M}')$,

$$x = u'^*xu' = u'^*u|x|u' = u'^*uu'u'^*|x|u'$$

whence $u'^*uu' = u$ and $u'^*|x|u' = |x|$ by the uniqueness of the polar decomposition. Hence $u \in \mathcal{M}$ and $|x| \in {}^{\eta}\mathcal{M}$. The other direction is obvious.

(7) is obvious (cf. Notation 1.120). $\qquad\square$

DEFINITION 1.124. A closed subspace K of H is called *affiliated* to $\mathcal{M}$ if the orthogonal projection onto the subspace belongs to $\mathcal{M}$.

We finish the section by introducing order in the set $\overline{\mathcal{M}}_+$. The definition is chosen in such a way that all our definitions of the order relation agree on common domains.

DEFINITION 1.125. Let $x, y \in \overline{\mathcal{M}}_+$. We say that x is *less than or equal to* y and write $x \leq y$ if $\text{dom}(y^{1/2}) \subseteq \text{dom}(x^{1/2})$ and $\|x^{1/2}\xi\| \leq \|y^{1/2}\xi\|$ for each $\xi \in \text{dom}(y^{1/2})$.

Note that for all $x \in \overline{\mathcal{M}}_+$ we have $0 \leq x$, so there is no conflict of notation.

1.6 Generalized positive operators

The presentation of generalized positive operators in this section is based on Haagerup's seminal paper on operator valued weights [**Haa79b**].

DEFINITION 1.126. A *generalized positive operator affiliated to* $\mathcal{M}$ is a map $m : \mathcal{M}_*^+ \to [0,\infty]$ satisfying:

(1) $m(\omega + \omega') = m(\omega) + m(\omega')$ for all $\omega, \omega' \in \mathcal{M}_*^+$;
(2) $m(\lambda\omega) = \lambda m(\omega)$ for $\lambda \geq 0$, $\omega \in \mathcal{M}_*^+$;
(3) m is lower semicontinuous (in the norm topology on $\mathcal{M}_*^+$).

The collection of these is called the *extended positive part* of $\mathcal{M}$ and is denoted by $\widehat{\mathcal{M}}_+$.

For $m, n \in \widehat{\mathcal{M}}_+, a \in \mathcal{M}$ and $\lambda \geq 0$, we define $m + n, \lambda m$ and $a^* m a$ in $\widehat{\mathcal{M}}_+$ by

$$(m + n)(\omega) := m(\omega) + n(\omega) \text{ for } \omega \in \mathcal{M}_*^+;$$

$$(\lambda m)(\omega) := \lambda m(\omega) \text{ for } \omega \in \mathcal{M}_*^+;$$

$$(a^* m a)(\omega) := m(a\omega a^*) \text{ for } \omega \in \mathcal{M}_*^+,$$

where $a\omega a^* \in \mathcal{M}_*^+$ is defined by $(a\omega a^*)(b) := \omega(a^* b a)$ for each $b \in \mathcal{M}$. If $z \in \mathcal{Z}(\mathcal{M})_+$, we write mz or zm instead of $z^{1/2} m z^{1/2}$.

For $m_1, m_2 \in \widehat{\mathcal{M}}_+$, we write $m_1 \leq m_2$ if $m_1(\omega) \leq m_2(\omega)$ for all $\omega \in \mathcal{M}_*^+$. If $(m_i)_{i \in I}$ is an increasing net in $\widehat{\mathcal{M}}_+$, then we define $m = \sup m_i \in \widehat{\mathcal{M}}_+$ by $m(\omega) := \sup_{i \in I} m_i(\omega)$ for $\omega \in \mathcal{M}_*^+$ and write $m_i \nearrow m$. Similarly, for an arbitrary family $\{m_i\}_{i \in I}$ from $\widehat{\mathcal{M}}_+$, an element $m = \sum_{i \in I} m_i$ is defined by $m(\omega) = \sum_{i \in I} m_i(\omega)$ for $\omega \in \mathcal{M}_*^+$.

REMARK 1.127. It is clear that $m + n$, λm and $a^* m a$ are well-defined. This is also true of $\sup m_i$ and $\sum m_i$: the additivity of $\sup m_i$ is easy to show and the supremum of lower semicontinuous functions is lower semicontinuous.

EXAMPLE 1.128. For $a \in \mathcal{M}_+$ define $m_a(\omega) = \omega(a)$, $\omega \in \mathcal{M}_*^+$. It is clear that the map $a \mapsto m_a$ preserves order. Thus we can view $\mathcal{M}_+$ as a subset of $\widehat{\mathcal{M}}_+$.

NOTATION 1.129. For $x \in \overline{\mathcal{M}}_+$ with spectral representation $x = \int_0^\infty \lambda de_\lambda$, put $m_x(\omega) = \int_0^\infty \lambda d\omega(e_\lambda)$ for $\omega \in \mathcal{M}_*^+$.

PROPOSITION 1.130. *If $x \in \overline{\mathcal{M}}_+$, then $m_x \in \widehat{\mathcal{M}}_+$. The map $x \mapsto m_x$ is injective.*

PROOF. In fact, if $a_n = \int_0^n \lambda de_\lambda$, then $m_x(\omega) = \sup_n \omega(a_n)$, hence m_x is lower semicontinuous. For $\xi \in H$

$$m_x(\omega_\xi) = \int_0^\infty \lambda d\langle e_\lambda \xi, \xi \rangle = \begin{cases} \|x^{1/2}\xi\|^2 & \xi \in \text{dom}(x^{1/2}) \\ \infty & \xi \notin \text{dom}(x^{1/2}). \end{cases}$$

If $m_x = m_y$, then $\text{dom}(x^{1/2}) = \text{dom}(y^{1/2})$ and $\|x^{1/2}\xi\| = \|y^{1/2}\xi\|$ for $\xi \in \text{dom}(x^{1/2})$. By the uniqueness of the polar decomposition $x^{1/2} = y^{1/2}$, so that $x = y$. $\qquad\square$

Thus, positive self-adjoint operators affiliated to $\mathcal{M}$ can be viewed as elements of $\widehat{\mathcal{M}}_+$. Note that under this identification, the positive self-adjoint operator $\sum_{n\in\mathbb{N}} \lambda_n p_n$ defined by Notation 1.120 can be regarded as a sum of a series of generalized positive operators $\lambda_n p_n$.

DEFINITION 1.131. A *positive quadratic form on H* is a map $\mathfrak{s}: H \to [0, \infty]$ satisfying:

(1) $\mathfrak{s}(\lambda\xi) = |\lambda|^2 \mathfrak{s}(\xi)$ for all $\xi \in H$;
(2) $\mathfrak{s}(\xi + \eta) + \mathfrak{s}(\xi - \eta) = 2\mathfrak{s}(\xi) + 2\mathfrak{s}(\eta)$ for all $\xi, \eta \in H$;
(3) $\mathfrak{s}$ is lower semicontinuous on H.
    The form $\mathfrak{s}$ is *affiliated* to $\mathcal{M}$, if
(4) $\mathfrak{s}(u'\xi) = \mathfrak{s}(\xi)$ for any unitary $u' \in \mathcal{M}'$ and any $\xi \in H$.

We define the *domain* of the positive quadratic form $\mathfrak{s}$ by

$$\mathrm{dom}(\mathfrak{s}) := \{\xi \in H : \mathfrak{s}(\xi) < \infty\},$$

and the *null space* of the form by

$$N_{\mathfrak{s}} := \{\xi \in H : \mathfrak{s}(\xi) = 0\}.$$

The set of positive quadratic forms on H affiliated to $\mathcal{M}$ is denoted by $\mathfrak{S}_{\mathcal{M}}$.

THEOREM 1.132. *There is a one-to-one correspondence between:*

(a) *generalized positive operators m affiliated to $\mathcal{M}$;*
(b) *positive quadratic forms $\mathfrak{s}$ on H affiliated to $\mathcal{M}$;*
(c) *pairs (K, x), where K is a closed subspace of H affiliated to $\mathcal{M}$, and x is a positive self-adjoint operator with domain dense in K, affiliated to $\mathcal{M}$.*

PROOF. We will show the existence of three maps: $\widehat{\mathcal{M}}_+ \ni m \mapsto \mathfrak{s}_m \in \mathfrak{S}_{\mathcal{M}}$, $\mathfrak{S}_{\mathcal{M}} \ni \mathfrak{s} \mapsto (K_{\mathfrak{s}}, x_{\mathfrak{s}})$, with $K_{\mathfrak{s}}$ and $x_{\mathfrak{s}}$ as in (b) above, and $(K, x) \mapsto m_{(K,x)} \in \widehat{\mathcal{M}}_+$, with K and x as in (c) above. It will be clear from the constructions that each of the mappings is injective and that their composition is the identity mapping on $\widehat{\mathcal{M}}_+$.

(a) $\Rightarrow$ (b). Define $\widehat{\mathcal{M}}_+ \ni m \mapsto \mathfrak{s}_m \in \mathfrak{S}$, where $\mathfrak{s}_m : H \ni \xi \mapsto m(\omega_\xi)$. Let $m \in \widehat{\mathcal{M}}_+$. To show that $\mathfrak{s}_m$ is a positive quadratic form on H affiliated to $\mathcal{M}$, we have to check that it satisfies conditions (1) to (4) of Definition 1.131. We have $\mathfrak{s}_m(\lambda\xi) = m(\omega_{\lambda\xi}) = |\lambda|^2 m(\xi) = |\lambda|^2 \mathfrak{s}_m(\xi)$, hence (1). Similarly, (2) follows from the equality $\omega_{\xi+\eta} + \omega_{\xi-\eta} = 2(\omega_\xi + \omega_\eta)$. For (3),

note first that for any $a \in \mathcal{M}$,

$$(\omega_\xi - \omega_\eta)(a) = \frac{1}{2} \left(\langle a(\xi + \eta), \xi - \eta \rangle + \langle a(\xi - \eta), \xi + \eta \rangle \right),$$

which implies that

$$\|\omega_\xi - \omega_\eta\| \le \|\xi - \eta\| \|\xi + \eta\|.$$

Hence, if the sequence $(\xi_n)_{n \in \mathbb{N}}$ converges to ξ in the norm of H, then also the sequence $(\omega_{\xi_n})_{n \in \mathbb{N}}$ converges in the norm of $\mathcal{M}_*$ to ω_ξ. Thus $\mathfrak{s}_m(\xi) = m(\omega_\xi) \le \limsup m(\omega_{\xi_n}) = \limsup \mathfrak{s}_m(\xi_n)$, which yields (3). (4) follows immediately from the equality $\omega_{u'\xi} = \omega_\xi$ for all $u' \in \mathbb{U}(\mathcal{M}')$.

(b) $\Rightarrow$ (c). Let $\mathfrak{s}$ be a positive quadratic form on H affiliated to $\mathcal{M}$. Put $K_\mathfrak{s} = \overline{\mathrm{dom}(\mathfrak{s})}$. Note that by Definition 1.131(1),(2) both $\mathrm{dom}(\mathfrak{s})$ and $N_\mathfrak{s}$ are linear spaces. Using again Definition 1.131(1),(2) and polarization we will easily find a sesquilinear form $\langle \cdot, \cdot \rangle_\mathfrak{s}$ such that $\mathfrak{s}(\xi) = \langle \xi, \xi \rangle_\mathfrak{s}$. Let $q_\mathfrak{s} : \mathrm{dom}(\mathfrak{s}) \to \mathrm{dom}(\mathfrak{s})/N_\mathfrak{s}$ be the quotient map, and let L be the completion of $\mathrm{dom}(\mathfrak{s})/N_\mathfrak{s}$ with respect to the inner product induced from $\langle \cdot, \cdot \rangle_\mathfrak{s}$, that is $\langle q_\mathfrak{s}\xi, q_\mathfrak{s}\eta \rangle_L = \langle \xi, \eta \rangle_\mathfrak{s}$ for any $\xi, \eta \in \mathrm{dom}(\mathfrak{s})$. If we denote by $\| \cdot \|_L$ the corresponding Hilbert space norm on L, then $\mathfrak{s}(\xi) = \|q_\mathfrak{s}\xi\|_L^2$ for $\xi \in \mathrm{dom}(q_\mathfrak{s})$ and $\mathfrak{s}(\xi) = \infty$ for $\xi \notin \mathrm{dom}(q_\mathfrak{s})$. The operator $q_\mathfrak{s}$, as an operator from K to L, is closed. To see this, let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence from $\mathrm{dom}(\mathfrak{s})$ converging to some $\xi \in K$ and such that the sequence $(q_\mathfrak{s}\xi_n)_{n \in \mathbb{N}}$ from L converges to some $\eta \in L$. Then, using Definition 1.131(3), we get $\mathfrak{s}(\xi) \le \sup_{n \in \mathbb{N}} \mathfrak{s}(\xi_n) = \sup_{n \in \mathbb{N}} \|q_\mathfrak{s}\xi_n\|_L^2 < \infty$, so that $\xi \in \mathrm{dom}(q_\mathfrak{s})$. For a fixed $\varepsilon > 0$, there is some n_0 such that, for $n, m \ge n_0$, $\|q_\mathfrak{s}\xi_n - q_\mathfrak{s}\xi_m\|_L \le \varepsilon$, that is $\mathfrak{s}(\xi_n - \xi_m) \le \varepsilon^2$. Hence, again by lower semicontinuity of $\mathfrak{s}$, we get $\mathfrak{s}(\xi_n - \xi) \le \sup_m \mathfrak{s}(\xi_n - \xi_m) \le \varepsilon^2$, so that $\|q_\mathfrak{s}\xi_n - q_\mathfrak{s}\xi\|_L \le \varepsilon^2$, that is $q_\mathfrak{s}\xi = \eta$. By a result fully analogous to that of Proposition 1.112, but for operators between different Hilbert spaces (see [**KR83**, Theorem 2.7.8(v)]), we get that $q_\mathfrak{s}^* q_\mathfrak{s}$ is an (obviously positive) self-adjoint operator on K. Put $x_\mathfrak{s} = q_\mathfrak{s}^* q_\mathfrak{s}$. Then $x_\mathfrak{s}^{1/2} = |q_\mathfrak{s}|$ and

$$\|x_\mathfrak{s}^{1/2}\xi\| = \|q_\mathfrak{s}\xi\| \quad \text{for all } \xi \in \mathrm{dom}(x_\mathfrak{s}^{1/2}) = \mathrm{dom}(q_\mathfrak{s}).$$

Since $u'\mathrm{dom}(\mathfrak{s}) \subseteq \mathrm{dom}(\mathfrak{s})$ by Definition 1.131(4), we have $u'K \subseteq K$ for any unitary $u' \in \mathcal{M}'$. Similarly, $x_\mathfrak{s} \in {}^\eta\mathcal{M}$.

(c) $\Rightarrow$ (a). Let K and x be as in (c). Denote by p the projection from H on $H \ominus K$ and let $x = \int_0^\infty \lambda de_\lambda$ with $e_\lambda \nearrow \mathbb{1} - p$ for $\lambda \to \infty$. By Proposition 1.61(7) and Definition 1.124, we have $e_\lambda \in \mathcal{M}$ for all $\lambda \in [0, \infty)$ and $p \in \mathcal{M}$. Moreover, $x^{1/2} = \int_0^\infty \lambda^{1/2} de_\lambda$ and $\mathrm{dom}(x^{1/2}) =$

$\{\xi \in K : \int_0^\infty \lambda d(e_\lambda \xi, \xi) < \infty\}$. Put $m_{(K,x)}(\omega) = \int_0^\infty \lambda d(\omega(e_\lambda)) + \infty \cdot \omega(p)$. For $\omega = \omega_\xi$ we have $m_{(K,x)}(\omega_\xi) = \|x^{1/2}\xi\|^2$ for $\xi \in \mathrm{dom}(x^{1/2})$, $m(\omega_\xi) = \infty$ otherwise. The proof of lower semicontinuity is the same as in Remark 1.127. The uniqueness on ω_ξ is clear, by Theorem 1.55(6), other ω's are sums of ω_ξ. □

THEOREM 1.133. *Each $m \in \widehat{\mathcal{M}}_+$ has a unique spectral decomposition of the form*

$$m(\omega) = \int_0^\infty \lambda d\omega(e_\lambda) + \infty \cdot \omega(p), \quad \omega \in \mathcal{M}_*^+, \qquad (1.4)$$

which we write simply as

$$m = \int_0^\infty \lambda de_\lambda + \infty \cdot p,$$

where (e_λ) is an increasing, right-continuous family of projections from $\mathcal{M}$ and $p = \mathbb{1} - e_\infty$ with $e_\infty = \lim_{\lambda \to \infty} e_\lambda$. Moreover

(1) *$e_0 = 0$ if and only if $m(\omega) > 0$ for all $\omega \in \mathcal{M}_*^+ \setminus \{0\}$;*
(2) *$p = 0$ if and only if $\{\omega \in \mathcal{M}_*^+ : m(\omega) < \infty\}$ is dense in $\mathcal{M}_*^+$.*

PROOF. The representation (1.4) is a consequence of Theorem 1.132 and its proof. For the proof of (1) and (2) assume that $\mathcal{M}$ acts in such a Hilbert space H that any state ω is a vector state. By Theorem 1.132, $e_0 = 0$ if and only if x is injective if and only if $m(\omega_\xi) > 0$ for all $\xi \in H \setminus \{0\}$, which proves (1). If $p = 0$, then $H = K$, so that $\mathrm{dom}(x^{1/2}) = \{\xi : m(\omega_\xi) < \infty\}$ is dense in H. Let $\omega \in \mathcal{M}_*^+$, $\omega = \omega_\xi$. Choose $\xi_n \in \mathrm{dom}(x^{1/2})$ so that $\xi_n \to \xi$. Then $\omega_n = \omega_{\xi_n} \to \omega$ and $m(\omega_n) < \infty$. This shows "⇒" in (2). If $p \neq 0$, the set $\{\omega \in \mathcal{M}_*^+ : \omega(p) = 0\}$, which contains the set $\{\omega \in \mathcal{M}_*^+ : m(\omega) < \infty\}$, is not dense in $\mathcal{M}_*^+$ (as it is closed). □

REMARK 1.134. Any normal $*$-isomorphism $\mathscr{I}$ from $\mathcal{M}$ onto $\mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras, in a very natural way admits a map from $\widehat{\mathcal{M}}_+$ onto $\widehat{\mathcal{N}}_+$. Given $m \in \widehat{\mathcal{M}}_+$ with

$$m(\omega) = \int_0^\infty \lambda d\omega(e_\lambda) + \infty \cdot \omega(p), \quad \omega \in \mathcal{M}_*^+,$$

one simply defines $\mathscr{I}(m)$ by means of the prescription

$$\mathscr{I}(m)(\omega) = \int_0^\infty \lambda d\omega(\mathscr{I}(e_\lambda)) + \infty \cdot \omega(\mathscr{I}(p)), \quad \omega \in \mathcal{N}_*^+.$$

REMARK 1.135. Many useful aspects of the Borel functional calculus are also applicable to the generalized positive operators. If for example we are given an increasing function $f : [0, \infty] \to [0, \infty]$ which satisfies $f(0) = 0$ and $\lim_{s \nearrow \infty} f(s) = \infty$, and which is continuous on $[0, b_f]$ where $b_f = \sup\{s \in [0, \infty) : f(s) < \infty\}$, then given $m \in \widehat{\mathcal{M}}_+$ with

$$m(\omega) = \int_0^\infty \lambda d\omega(e_\lambda) + \infty \cdot \omega(p), \quad \omega \in \mathcal{M}_*^+,$$

we may define $f(m)$ by means of the prescription

$$f(m)(\omega) = \int_0^{b_f} f(\lambda) d\omega(e_\lambda) + \infty \cdot \omega(p + (e_\infty - e_{b_f})), \quad \omega \in \mathcal{M}_*^+.$$

REMARK 1.136. If, for some $m \in \widehat{\mathcal{M}}_+$, $m(\omega) < \infty$ for all $\omega \in \mathcal{M}_*^+$, then there is an $a \in \mathcal{M}_+$ such that $m(\omega) = \omega(a)$ for all $\omega \in \mathcal{M}_*^+$. This is an easy consequence of Theorem 1.132.

PROPOSITION 1.137. *Any $m \in \widehat{\mathcal{M}}_+$ is a pointwise limit of an increasing sequence of bounded operators from $\mathcal{M}_+$.*

PROOF. If $m = \int_0^\infty \lambda de_\lambda + p \cdot \infty$, then $a_n := \int_0^n \lambda de_\lambda + np \nearrow m$. □

Since we can treat $\overline{\mathcal{M}}_+$ as a subset of $\widehat{\mathcal{M}}_+$, it is important to see whether the inclusion $x \mapsto m_x$ respects the order introduced in Definition 1.125.

PROPOSITION 1.138. *For $x, y \in \overline{\mathcal{M}}_+$, $x \leq y$ if and only if $m_x \leq m_y$. In other words, $\leq$ is the order obtained by restricting the usual order of generalized positive operators.*

PROOF. By definition, $x \leq y$ implies $m_x(\omega_\xi) \leq m_y(\omega_\xi)$ for each $\xi \in \mathrm{dom}(y^{1/2})$. Since an arbitrary $\omega \in \mathcal{M}_*$ can be represented as a sum $\omega = \sum_{n \in \mathbb{N}} \omega_{\xi_n}$, and $m_y(\omega_\xi) = \infty$ whenever $\xi \notin \mathrm{dom}(y^{1/2})$, we have $m_x \leq m_y$. Now, if $m_x \leq m_y$ and $\xi \in \mathrm{dom}(y^{1/2})$, then $m_y(\omega_\xi) < \infty$, hence $m_x(\omega_\xi) < \infty$ and $\xi \in \mathrm{dom}(x^{1/2})$. Thus $x \leq y$.   □

Hence, the order on $\overline{\mathcal{M}}_+$ inherited from $\widehat{\mathcal{M}}_+$ coincides with the order in $\overline{\mathcal{M}}_+$. Moreover, $\overline{\mathcal{M}}_+$ is hereditary with respect to the order of generalized positive operators:

PROPOSITION 1.139. *Let $m, n \in \widehat{\mathcal{M}}_+$ with $m \leq n$. If $n = m_y$ with $y \in \overline{\mathcal{M}}_+$, then $m = m_x$ for some $x \in \overline{\mathcal{M}}_+$.*

PROOF. Let $m := \int_0^\infty \lambda de_\lambda + p \cdot \infty$ and $n := \int_0^\infty \lambda df_\lambda + q \cdot \infty$. It is clear that $m \leq n$ implies $p \leq q$. Since $n = m_y$, we have $q = 0$, hence $p = 0$, which yields the result. $\qquad\square$

We also have:

LEMMA 1.140. *If $m_i \nearrow m$ in $\widehat{\mathcal{M}}_+$ and $a \in \mathcal{M}$, then $a^* m_i a \nearrow a^* m a$.*

PROOF. Obvious from definitions. $\qquad\square$

EXAMPLE 1.141. Let $\mathcal{M}$ be commutative. Then, by Theorem 1.100, it admits an additive and positively homogenous functional $\tau \colon \mathcal{M}_+ \to [0, \infty]$ such that $(\mathbb{P}(M), \mu)$, with $\mu = \tau \upharpoonright \mathbb{P}(M)$, is a localizable measure algebra. Let $(\mathcal{X}, \mathfrak{S}, \nu)$ be the corresponding measure space, and $L^\infty(\mathcal{X}, \mathfrak{S}, \nu)$ the W^*-algebra isomorphic to the von Neumann algebra $\mathcal{M}$. We can identify $\mathcal{M}_*^+$ with $L_+^1(\mathcal{X}, \mathfrak{S}, \nu)$. Let $f \colon \mathcal{X} \to [0, \infty]$ be a $\mathfrak{S}$-measurable function. Put

$$m_f(\omega) = \int_\Omega f \cdot \omega \, d\mu \text{ for } \omega \in \mathcal{M}_*^+.$$

Then $m_f \in \widehat{\mathcal{M}}_+$ by Fatou lemma. Moreover, $m_f = m_g$ implies $f = g$ μ-a.e. On the other hand, if $m \in \widehat{\mathcal{M}}_+$, we can find a sequence $f_k \in L^\infty(\mathcal{X}, \mathfrak{S}, \nu)$ s.t. $\int f_k \omega \, d\mu \nearrow m(\omega)$. Put $f = \sup f_k$. Then $m = m_f$. Hence, $\widehat{\mathcal{M}}_+$ can be identified with the set of equivalence classes of measurable functions $\mathcal{X} \to [0, \infty]$.

CHAPTER 2

Noncommutative measure theory
— semifinite case

2.1 Traces

The existence of traces on semifinite algebra is a non-trivial matter. The key is the following lemma:

LEMMA 2.1. [**SZ79**, 7.10] *Let $\mathcal{M}$ be a finite von Neumann algebra. Any functional $\mu \in \mathcal{Z}(\mathcal{M})_*$ has a unique extension to a tracial form $\tau_\mu \in \mathcal{M}_*$. If $\mu \in \mathcal{Z}(\mathcal{M})_*^+$, then $\tau_\mu \in \mathcal{M}_*^+$.*

The above lemma leads to an easy construction of a trace-like map that characterizes finite von Neumann algebras.

DEFINITION 2.2. [**SZ79**, 7.11, 7.12] A *(canonical) centre-valued* (or *central*) *trace* on $\mathcal{M}$ is a linear bounded map $\mathcal{T}$ from $\mathcal{M}$ onto $\mathcal{Z}(\mathcal{M})$ such that

(1) $\mathcal{T}(ab) = \mathcal{T}(ba)$ for all $a, b \in \mathcal{M}$;
(2) $\mathcal{T}(c) = c$ for $c \in \mathcal{Z}(\mathcal{M})$.

THEOREM 2.3. [**SZ79**, 7.11] *A von Neumann algebra $\mathcal{M}$ is finite if and only if there exists a centre-valued trace $\mathcal{T}$ on $\mathcal{M}$. The centre-valued trace $\mathcal{T}$ is unique and has the following additional properties:*

(1) $\|\mathcal{T}\| = 1$;
(2) $\mathcal{T}(ca) = c\mathcal{T}(a)$ *for* $c \in \mathcal{Z}(\mathcal{M}), a \in \mathcal{M}$ *(i.e. $\mathcal{T}$ is a $\mathcal{Z}(\mathcal{M})$-module map);*
(3) $\mathcal{T}(\mathcal{M}_+) \subseteq \mathcal{Z}(\mathcal{M})_+$ *(i.e. $\mathcal{T}$ is positive);*
(4) $\mathcal{T}$ *is σ-weakly continuous (in particular, normal);*
(5) *if $a \in \mathcal{M}_+$ and $\mathcal{T}(a) = 0$, then $a = 0$ (i.e. $\mathcal{T}$ is faithful);*
(6) *For each $a \in \mathcal{M}$, $\mathcal{T}(a)$ is in the norm-closed convex hull of the set $\{uau^* \colon u \in \mathbb{U}(\mathcal{M})\}$.*

The next corollary states that the centre-valued trace restricted to projections constitute a (centre-valued) dimension function.

COROLLARY 2.4. *Let $\mathcal{M}$ be a finite von Neumann algebra with the centre-valued trace $\mathscr{T}$. Then, for $e, f \in \mathbb{P}(\mathcal{M})$, we have $e \sim f$ (resp. $e \precsim f$) if and only if $\mathscr{T}(e) = \mathscr{T}(f)$ (resp. $\mathscr{T}(e) \leq \mathscr{T}(f)$)).*

PROOF. It follows from condition (1), (2) of Definition 2.2 and (3) of Theorem 2.3 that $e \sim f$ (resp. $e \precsim f$) implies $\mathscr{T}(e) = \mathscr{T}(f)$ (resp. $\mathscr{T}(e) \leq \mathscr{T}(f)$)). Let now $e, f \in \mathbb{P}(\mathcal{M})$ and assume that $\mathscr{T}(e) \leq \mathscr{T}(f)$. By Comparability Theorem 1.76 there is a central projection $z \in \mathcal{M}$ such that $ez \precsim fz$ and $fz^{\perp} \precsim ez^{\perp}$. From the assumption, $\mathscr{T}(ez^{\perp}) = \mathscr{T}(e)z^{\perp} \leq \mathscr{T}(f)z^{\perp} = \mathscr{T}(fz^{\perp})$. On the other hand, from $fz^{\perp} \precsim ez^{\perp}$ it follows that $\mathscr{T}(ez^{\perp}) \geq \mathscr{T}(fz^{\perp})$. Thus $\mathscr{T}(ez^{\perp}) = \mathscr{T}(fz^{\perp})$. If $ez^{\perp} \not\sim fz^{\perp}$, then $ez^{\perp} \sim g \leq fz^{\perp}$ and $g \neq fz^{\perp}$. This implies $\mathscr{T}(ez^{\perp}) = \mathscr{T}(g) \leq \mathscr{T}(fz^{\perp})$ and $\mathscr{T}(g) \neq \mathscr{T}(fz^{\perp})$, by the faithfulness of $\mathscr{T}$ (see Theorem 2.3(5)), a contradiction. Hence $ez^{\perp} \sim fz^{\perp}$ and $e \precsim f$. If $\mathscr{T}(e) = \mathscr{T}(f)$, then $e \precsim f$ and $f \precsim e$, and by Proposition 1.74, $e \sim f$. $\square$

LEMMA 2.5. *Let $\mathcal{M}$ be a finite von Neumann algebra, and let $\mathscr{T}$ be the centre-valued trace on $\mathcal{M}$. Then $\mathscr{T}(p \vee q) \leq \mathscr{T}(p) + \mathscr{T}(q)$ for all $p, q \in \mathbb{P}(\mathcal{M})$.*

PROOF. By Kaplansky's parallelogram law (Proposition 1.77) $\mathscr{T}(p \vee q - p) = \mathscr{T}(q - p \wedge q)$, so that $\mathscr{T}(p \vee q) - \mathscr{T}(p) = \mathscr{T}(q) - \mathscr{T}(p \wedge q)$. Hence $\mathscr{T}(p \vee q) = \mathscr{T}(p) + \mathscr{T}(q) - \mathscr{T}(p \wedge q) \leq \mathscr{T}(p) + \mathscr{T}(q)$. $\square$

DEFINITION 2.6. A functional $\tau \colon \mathcal{M}_{+} \to [0, \infty]$ is called a *trace* on $\mathcal{M}$ if it satisfies the following conditions:

    (1) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in \mathcal{M}_{+}$ (additivity)
    (2) $\tau(\lambda a) = \lambda \tau(a)$ for all $a \in \mathcal{M}_{+}$ and $\lambda \geq 0$, with the provision that $0 \cdot \infty = 0$ (homogeneity);
    (3) $\tau(a^*a) = \tau(aa^*)$ for all $a \in \mathcal{M}$.

DEFINITION 2.7. We say that a trace τ is:

    (1) *normal* if for any $a \in \mathcal{M}_{+}$ and any net (a_i) with $a_i \in \mathcal{M}_{+}$ such that $a_i \nearrow a$, we have $\tau(a_i) \nearrow \tau(a)$.
    (2) *faithful* if $\tau(a) = 0$ for some $a \in \mathcal{M}_{+}$ implies $a = 0$.
    (3) *semifinite* if the linear span of the set $\{a \in \mathcal{M}_{+} : \tau(a) < \infty\}$ is σ-weakly dense in $\mathcal{M}$.

DEFINITION 2.8. For a trace τ, we introduce the following standard notation:

$$N_\tau = \{a \in \mathcal{M} : \tau(a^*a) = 0\}$$
$$\mathfrak{p}_\tau = \{a \in \mathcal{M}_+ : \tau(a) < \infty\}$$
$$\mathfrak{n}_\tau = \{a \in \mathcal{M} : a^*a \in \mathfrak{p}_\tau\}$$
$$\mathfrak{m}_\tau = \text{linear span of } \mathfrak{p}_\tau$$

PROPOSITION 2.9. *Let τ by a trace on $\mathcal{M}$. Then:*

(1) *$\mathfrak{p}_\tau$ is a hereditary subcone of $\mathcal{M}_+$;*
(2) *N_τ and $\mathfrak{n}_\tau$ are two-sided ideals in $\mathcal{M}$;*
(3) *$\mathfrak{m}_\tau$ is the linear span of $\mathfrak{n}_\tau^*\mathfrak{n}_\tau$ and $\mathfrak{m}_\tau \subseteq \mathfrak{n}_\tau \cap \mathfrak{n}_\tau^*$;*
(4) *$\mathfrak{m}_\tau$ is a *-subalgebra of $\mathcal{M}$;*
(5) *$\tau \upharpoonright \mathfrak{p}_\tau$ extends to a positive linear form on $\mathfrak{m}_\tau$ (denoted also by τ) and $\mathfrak{m}_\tau \cap \mathcal{M}_+ = \mathfrak{p}_\tau$.*

PROOF. (1) is obvious.

(2) Let $a, b \in \mathcal{M}$. Then

$$(a + b)^*(a + b) = 2(a^*a + b^*b) - (a - b)^*(a - b) \le 2(a^*a + b^*b)$$

which shows that both N_τ and $\mathfrak{n}_\tau$ are linear spaces. Moreover,

$$\tau(((ba)^*(ba)) = \tau(a^*b^*ba) \le \|b\|\tau(a^*a),$$

so that both N_τ and $\mathfrak{n}_\tau$ are left ideals. But condition (3) in Definition 2.6 ensures that N_τ and $\mathfrak{n}_\tau$ are in fact two-sided ideals.

(3) If $a \in \mathfrak{p}_\tau$, the $a^{1/2} \in \mathfrak{n}_\tau$, so that $a = (a^*)^{1/2}a^{1/2} \in \mathfrak{n}_\tau^*\mathfrak{n}_\tau$ and $\mathfrak{p}_\tau \subseteq \mathfrak{n}_\tau^*\mathfrak{n}_\tau$. On the other hand, if $a, b \in \mathfrak{n}_\tau$, then the polarization identity

$$b^*a = \frac{1}{4}\sum_{n=0}^{3} i^n(a + i^nb)^*(a + i^nb)$$

shows that $b^*a \in \mathfrak{m}_\tau$ and consequently $\mathfrak{n}_\tau^*\mathfrak{n}_\tau \subseteq \mathfrak{m}_\tau$. Finally, since $\mathfrak{n}_\tau$ is a two-sided ideal, $\mathfrak{p}_\tau \subseteq \mathfrak{n}_\tau^*\mathfrak{n}_\tau \subseteq \mathfrak{n}_\tau \cap \mathfrak{n}_\tau^*$, which ends the proof of (3).

(4) It is obvious that $\mathfrak{m}_\tau$ is a linear space. That $ab \in \mathfrak{m}_\tau$ whenever $a, b \in \mathfrak{m}_\tau$ follows easily from (3) and the fact that $\mathfrak{n}_\tau$ is a left ideal. Finally, $a^* \in \mathfrak{m}_\tau$ for $a \in \mathfrak{m}_\tau$ follows immediately from (3).

(5) If $a = a_1 - a_2 + ia_3 - ia_4 = b_1 - b_2 + ib_3 - ib_4 \in \mathfrak{m}_\tau$ with $a_i, b_i \in \mathfrak{p}_\tau$ for $i = 1, \ldots, 4$, we have $a_1 - a_2 = b_1 - b_2$ and $a_3 - a_4 = b_3 - b_4$, which yields $\tau(a_1) - \tau(a_2) = \tau(b_1) - \tau(b_2)$ and $\tau(a_3) - \tau(a_4) = \tau(b_3) - \tau(b_4)$. Hence the number $\tau(a_1) - \tau(a_2) + i\tau(a_3) - i\tau(a_4)$ does not depend on representation

of a as a linear combination of four elements of $\mathfrak{p}_\tau$ and we define $\tau(a)$ as that number. τ extended in this manner is clearly linear and hermitian. If $a \in \mathfrak{m}_\tau \cap \mathcal{M}_+$, then $0 \le a = a_1 - a_2 \le a_1 \in \mathfrak{p}_\tau$, so that $a \in \mathfrak{p}_\tau$ and τ is positive on $\mathfrak{m}_\tau$.

$\square$

Here is a list of easy properties of the trace:

PROPOSITION 2.10. *Let τ be a trace on $\mathcal{M}$. Then:*

(1) *$\tau(u^*au) = \tau(a)$ for any $a \in \mathcal{M}_+$ and $u \in \mathbb{U}(\mathcal{M})$;*
(2) *if $e, f \in \mathbb{P}(\mathcal{M})$ and $e \sim f$ (resp. $e \precsim f$), then $\tau(e) = \tau(f)$ (resp. $\tau(e) \le \tau(f)$);*
(3) *if τ is faithful, $\mathcal{M}$ is a factor and $\tau(e) = \tau(f)$ (resp. $\tau(e) \le \tau(f)$, $\tau(e) < \tau(f)$) for some $e, f \in \mathbb{P}(\mathcal{M})$, then $e \sim f$ (resp. $e \precsim f$, $e \prec f$).*
(4) *if τ is faithful and $\tau(e) < \infty$ for some $e \in \mathbb{P}(\mathcal{M})$, then e is finite;*

PROOF. (1) We have $\tau(u^*au) = \tau((a^{1/2}u)(a^{1/2}u)^*) = \tau(a)$.

(2) $e \sim f$ means $e = u^*u$, $f = uu^*$ for some $u \in \mathcal{M}$, hence $\tau(e) = \tau(f)$. The other part follows by Definition 1.73 and the monotonicity of the trace.

(3) Since $\mathcal{M}$ is a factor, $e \nsim f$, implies $e \prec f$ or $f \prec e$. But then $\tau(e) < \tau(f)$ or $\tau(f) < \tau(e)$, contradiction. The other parts follow as in (2).

(4) Let $f \in \mathbb{P}(\mathcal{M})$ be such that $f \le e$ and $f \sim e$. Then $\tau(e - f) = 0$, hence $e = f$. That implies finiteness of e. $\square$

PROPOSITION 2.11. *If τ is a normal trace on $\mathcal{M}$, then there is a unique central projection $z \in \mathcal{M}$ such that τ is faithful on $(\mathcal{M}z)_+$ and (the extension of) τ is zero on $\mathcal{M}z^\perp$.*

PROOF. By Kaplansky's law 1.77 and Proposition 2.10(2), we have $\tau(p \vee q) + \tau(p \wedge q) = \tau(p) + \tau(q)$ for any $p, q \in \mathbb{P}(\mathcal{M})$. Hence $p, q \in N_\tau$ implies $p \vee q \in N_\tau$. Hence the family of projections in N_τ is upward directed. Since τ is normal, $\tau(\sup \mathbb{P}(N_\tau)) = 0$. Put $z := \sup \mathbb{P}(N_\tau)^\perp$. Then $z^\perp$ is the largest projection which is annihilated by τ, from which it is clear that τ is faithful on $(z\mathcal{M}z)_+$ and (the extension of) τ is zero on $z^\perp \mathcal{M}z^\perp$. It can now easily be checked that $\tau(uz^\perp u^*) = \tau(z^\perp) = 0$, so that $uz^\perp u^* \le z^\perp$, for all $u \in \mathbb{U}(\mathcal{M})$. But this means that $uz^\perp u^* = z^\perp$, for all $u \in \mathbb{U}(\mathcal{M})$, and hence that $z \in \mathcal{Z}(\mathcal{M})$. $\square$

DEFINITION 2.12. The projection z from the proposition above is called the *support (projection)* of τ and is denoted by $\operatorname{supp}\tau$. The orthogonal complement of $\operatorname{supp}\tau$ is called the *null projection* of τ and is denoted by $e_0(\tau)$.

PROPOSITION 2.13. *Let τ be a faithful normal trace on a von Neumann algebra $\mathcal{M}$. The following conditions are equivalent:*

(1) $\mathfrak{p}_\tau$ *generates $\mathcal{M}$ as a von Neumann algebra (equivalently, the *-algebra $\mathfrak{m}_\tau$ is dense in $\mathcal{M}$ in any of the following topologies: weak, σ-weak, strong, σ-strong, strong*, σ-strong*);*

(2) *there exists an orthogonal family $\{e_i\}_{i\in I}$ such that $e_i \in \mathbb{P}(\mathcal{M})\cap\mathfrak{p}_\tau$ for all $i \in I$, and $\sum_{i\in I} e_i = \mathbb{1}$;*

(3) *for each non-zero $e \in \mathbb{P}(\mathcal{M})$ there exists $f \in \mathbb{P}(\mathcal{M})$ such that $0 \neq f \le e$ and $\tau(f) < \infty$;*

(4) *there exists a family $\{\omega_i\}_{i\in I}$ with $\omega_i \in \mathcal{M}_*^+$ for all $i \in I$, with pairwise orthogonal supports, such that $\sum_{i\in I} \operatorname{supp}\omega_i = \mathbb{1}$ and $\tau = \sum_{i\in I} \omega_i$ (pointwise).*

PROOF. $(3) \Rightarrow (2)$ Choose $\{e_i\}$ to be a maximal family of mutually orthogonal non-zero projections of finite trace (use Zorn's lemma and condition (3) to show its existence).

$(2) \Rightarrow (1)$ Let $\{e_i\}_{i\in I}$ be the family from condition (2). For finite $J \subseteq I$ put $f_J = \sum_{i\in J} e_i$. We will show that for $a \in \mathcal{M}$, $f_J a f_J \in \mathfrak{m}_\tau$ and $f_J a f_J \to a$ strongly, which shows (1) by Theorem 1.59. In fact, for any $i,j \in I$ we have $e_i, e_j \in \mathfrak{n}_\tau \cap \mathfrak{n}_\tau^*$ and $\mathfrak{n}_\tau$ is a left ideal in $\mathcal{M}$, so that $e_i a e_j \in \mathfrak{n}_\tau \mathfrak{n}_\tau^* \subseteq \mathfrak{m}_\tau$. Now, $f_J \to 1$ strongly and, for any $\xi \in H$,

$$\|(f_J a f_J - a)\xi\| \le \|(f_J a(1 - f_J))\xi\| + \|(1 - f_J)a\xi\|$$
$$\le \|a\|\|(1 - f_J)\xi\| + \|(1 - f_J)a\xi\| \to 0,$$

which ends the proof.

$(1) \Rightarrow (3)$. Let $e \in \mathcal{M}$. By the Kaplansky density theorem 1.60, there is a net $(a_i)_{i\in I}$, with $a_i \in \mathcal{M}_+, \|a_i\| \le 1$ such that $a_i \to e$ strongly and $\tau(a_i) < \infty$ for all i. Then $\tau(e a_i e) = \tau(a_i^{1/2} e a_i^{1/2}) \le \tau(a_i) < \infty$ for all i. Since $e a_i e \to e$ strongly, there must exist an i_0 such that $e a_{i_0} e \neq 0$. Hence, for some $\epsilon > 0$, $0 \neq \epsilon \chi_{[\epsilon,\infty)}(e a_{i_0} e) \le e a_{i_0} e \le e$. Put $f = \chi_{[\epsilon,\infty)}(e a_{i_0} e)$. Then $0 \neq f \le (1/\epsilon)e$, so that $f \le e$ and $\tau(f) < \infty$.

$(2) \Rightarrow (4)$. Let $\{e_i\}$ be an orthogonal family of projections from $\mathfrak{p}_\tau$ with $\sum e_i = \mathbb{1}$. Then, as in $(2) \Rightarrow (1)$, for all i,j and $a \in \mathcal{M}_+$, $e_j a e_i \in \mathfrak{m}_\tau$

and $\tau(a) = \sum_{i,j} \tau(e_j a e_i) = \sum_i \tau(e_i a e_i)$. Put $\omega_i = e_i \tau e_i$. Then $\operatorname{supp} \omega_i = e_i$ and $\tau = \sum \omega_i$.

$(4) \Rightarrow (2)$. Choose $e_i = \operatorname{supp} \omega_i$. Then $e_i \in \mathbb{P}(\mathcal{M}) \cap \mathfrak{p}_\tau$ for all $i \in I$, and $\sum_{i \in I} e_i = \mathbb{1}$, whence (2). $\qquad\square$

DEFINITION 2.14. We say that a faithful normal trace τ is *semifinite* if it satisfies one of the equivalent conditions from Proposition 2.13. We write *f.n.s.* instead of *faithful normal semifinite*.

COROLLARY 2.15. *A semifinite normal trace τ is σ-weakly lower semicontinuous.*

PROOF. Proposition 2.13(4) shows that τ is a sum of σ-weakly continuous functionals from $\mathcal{M}_*^+$, which implies that it is σ-weakly lower semicontinuous. $\qquad\square$

PROPOSITION 2.16. *If τ is a normal trace on $\mathcal{M}$, then there exists a unique central projection $z \in \mathcal{M}$ such that τ is semifinite on $\mathcal{M}z$ and $\tau(a) = \infty$ for every non-zero $a \in \mathcal{M}_+ z^\perp$.*

PROOF. Let $\bar{\mathfrak{n}}$ be the σ-weak closure of $\mathfrak{n}_\tau$. By Proposition 2.9(2), $\bar{\mathfrak{n}}$ is then also a two-sided ideal in $\mathcal{M}$, which by Proposition 1.61(5) ensures that there is a central projection $z \in \mathcal{M}$ such that $\bar{\mathfrak{n}} = \mathcal{M}z$. Notice that then

$$\operatorname{span}(\mathfrak{p}_\tau) = \mathfrak{m}_\tau = \operatorname{span}(\mathfrak{n}_\tau^* \mathfrak{n}_\tau) \subset \operatorname{span}(\bar{\mathfrak{n}}^* \bar{\mathfrak{n}}) = \mathcal{M}z.$$

So $\overline{\operatorname{span}(\mathfrak{p}_\tau)} \subset \mathcal{M}z$. But we conversely also have that $\mathcal{M}z = \operatorname{span}(\bar{\mathfrak{n}}^* \bar{\mathfrak{n}}) \subset \overline{\operatorname{span}(\mathfrak{p}_\tau)}$. To see this let $a, b \in \bar{\mathfrak{n}}$ be given and select nets (a_γ), (b_ρ) in $\mathfrak{n}_\tau$ converging to a and b respectively. For each fixed ρ, the net $(b_\rho^* a_\gamma)_\gamma \subset \operatorname{span}(\mathfrak{n}_\tau^* \mathfrak{n}_\tau) = \operatorname{span}(\mathfrak{p}_\tau)$ converges to $b_\rho^* a$. Hence the net $(b_\rho^* a)$ belongs to $\overline{\operatorname{span}(\mathfrak{p}_\tau)}$. But then the limit $b^* a$ also belongs to $\overline{\operatorname{span}(\mathfrak{p}_\tau)}$. This then ensures that $\mathcal{M}z = \operatorname{span}(\bar{\mathfrak{n}}^* \bar{\mathfrak{n}}) \subset \overline{\operatorname{span}(\mathfrak{p}_\tau)}$, and hence that $\overline{\operatorname{span}(\mathfrak{p}_\tau)} = \mathcal{M}z$, thereby proving the first claim.

If, on the other hand, there existed a non-zero $a \in \mathcal{M}_+ z^\perp$, then also $a^{1/2} \in \mathcal{M}z^\perp$, so that $a^{1/2} \notin \mathfrak{n}_\tau$ and $\tau(a) = \infty$. $\qquad\square$

DEFINITION 2.17. The projection z from the previous proposition is called the *semifinite projection* of τ and is denoted by $e_\infty(\tau)$.

Obviously, $e_0(\tau) \le e_\infty(\tau)$ (cf. Definition 2.12).

PROPOSITION 2.18. *Let τ be a trace on factor von Neumann algebra $\mathcal{M}$. If τ is f.n.s. and $e \in \mathbb{P}(\mathcal{M})$ is finite, then $\tau(e) < \infty$.*

PROOF. Let $\tau(e) = \infty$. By Proposition 2.13 there exists a projection f with $f \leq e$ and $\tau(f) < \tau(e)$. Let $\{e_i\}_{i \in I}$ be a maximal orthogonal family of non-zero equivalent projections in $\mathcal{M}$ with $\tau(e_i) < \infty$ and $e' := \sum_{i \in I} e_i \leq e$. This family must be infinite since if it was finite, then given i_0 we would have that $\tau(e_{i_0}) < \infty = \tau(e - e')$, whence $e_{i_0} \prec e - e'$ by Proposition 2.13. Thus there would then exist some projection e_0 equivalent to all the other e_i's with $e_0 \leq e - \sum_i e_i \leq e$, clearly contradicting the maximality. By maximality, $e - e' \prec e_i$ (for all i). Choose a specific $i_0 \in I$. Then $e - e' \prec e_{i_0}$ and, by Proposition 1.80, $\sum_{i \in I, i \neq i_0} e_i \sim e'$. Hence

$$e = (e - e') + e' \prec e_{i_0} + \sum_{i \in I, i \neq i_0} e_i = e' \leq e,$$

which means that $\sum_{i \in I, i \neq i_0} e_i \sim e' \sim e$. This is impossible if e is finite, hence the result. $\qquad\square$

DEFINITION 2.19. A family $\{\tau_i\}_{i \in I}$ of non-zero traces on $\mathcal{M}$ is called *sufficient* if for any non-zero $a \in \mathcal{M}_+$ there is $i_0 \in I$ such that $\tau_{i_0}(a) \neq 0$.

THEOREM 2.20. *An algebra $\mathcal{M}$ is finite if and only if it possesses a sufficient family of finite normal traces. An algebra $\mathcal{M}$ is finite and σ-finite if and only if it admits a faithful normal tracial state.*

PROOF. "$\Rightarrow$" Assume $\mathcal{M}$ is finite. Let $\{\mu_i\}_{i \in I}$ be a maximal family of non-zero elements of $\mathcal{Z}(\mathcal{M})_*^+$ with mutually orthogonal supports. It is clear that $\sum_{i \in I} \operatorname{supp} \mu_i = \mathbb{1}$. Let $\tau_i := \mu_i \circ \mathscr{T}$, where $\mathscr{T}$ is the centre-valued trace on $\mathcal{M}$. Then $\{\tau_i\}_{i \in I}$ is a sufficient family of finite normal traces on $\mathcal{M}$. If $\mathcal{M}$ is σ-finite, then $\mathcal{Z}(\mathcal{M})$ is σ-finite, so by Corollary 1.101 there is a state μ on it, so that $\tau := \mu \circ \mathscr{T}$ is a tracial state on $\mathcal{M}$.

"$\Leftarrow$" Let $0 \neq p \in \mathbb{P}(\mathcal{M})$ be such that $p \sim \mathbb{1}$. Let $\{\tau_i\}_{i \in I}$ be a sufficient family of finite normal traces on $\mathcal{M}$. Put $z_i := \operatorname{supp} \tau_i$. Then $\tau_i(pz_i) = \tau_i(z_i)$ for each $i \in I$, hence $pz_i = z_i$ for each $i \in I$. Thus $p = \mathbb{1}$, which implies that $\mathbb{1}$ is finite. Hence $\mathcal{M}$ is finite. If $\mathcal{M}$ admits a faithful normal tracial state, then it must obviously be σ-finite. $\qquad\square$

THEOREM 2.21. *A von Neumann algebra admits a faithful normal semifinite trace if and only if it is semifinite.*

PROOF. "$\Rightarrow$" Assume $\mathcal{M}$ admits a faithful normal semifinite trace τ. We need to show that there are no purely infinite projections in the centre of $\mathcal{M}$ (see Definitions 1.78(6) and 1.84(7)). Let $0 \neq z \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$. By the definition of semifiniteness (see Definition 2.14) and Proposition 2.13(3),

there is a non-zero projection $q \in \mathcal{M}$ such that $q \leq z$ and $\tau(q) < \infty$. By Proposition 2.10(4), q is finite. Hence $\mathcal{M}$ is semifinite.

"$\Leftarrow$" Assume $\mathcal{M}$ is semifinite. Let $\{\tau_i\}_{i \in I}$ be a maximal family of normal semifinite traces on $\mathcal{M}$ with mutually orthogonal supports. Put $\tau := \sum_{i \in I} \tau_i$. We claim that τ is a normal semifinite trace on $\mathcal{M}$. In fact, the trace property $\tau(a^*a) = \tau(aa^*)$ is evident. Normality of τ follows from normality of τ's and the fact that we can exchange sup and $\sum$ for nonnegative terms. Let now $p \in \mathbb{P}(\mathcal{M})$ be such that $\tau(p) = \infty$. Then for some i_0 we have $pz_{i_0} \neq 0$. By Proposition 2.13, there is a non-zero projection $q \in \mathcal{M}z_{i_0}$ such that $\tau_{i_0}(q) < \infty$. But $q \leq p$ and $\tau(q) = \tau_{i_0}(q)$, so that again by Proposition 2.13, τ is semifinite.

We will show that $\mathrm{supp}\,\tau = \mathbb{1}$. Suppose $0 \neq z := \mathbb{1} - \mathrm{supp}\,\tau$. Since $\mathcal{M}_z$ is semifinite, by Structure Theorem for von Neumann algebras there is a non-zero central projection $z_0 \leq z$ in $\mathcal{M}$ such that $\mathcal{M}_{z_0} = \mathcal{N} \overline{\otimes} B(H_0)$ with a finite von Neumann algebra $\mathcal{N}$ and a Hilbert space H_0. By Theorem 2.20, there is a non-zero finite normal trace $\tau_{\mathcal{N}}$ on $\mathcal{N}$. By reducing the projection z_0 even further, to the support of $\tau_{\mathcal{N}}$, we can assume that $\tau_{\mathcal{N}}$ is faithful. For $a \in \mathcal{M}_{z_0}$, write $a = (a_{i,j})_{i,j \in I}$ using Notation 1.89. Define τ_0 on $\mathcal{M}_{z_0}$ by $\tau_0(a) = \sum_{i \in I} \tau(a_{i,i})$. It is clear that τ_0 is additive and positively homogeneous. By Proposition 1.90, we have: $(a^*a)_{i,j} = \sum_{j \in I} a_{j,i}^* a_{j,i}$ and $(aa^*)_{i,j} = \sum_{j \in I} a_{i,j} a_{i,j}^*$ with the sums σ-strong*-convergent. Hence, using normality of $\tau_{\mathcal{N}}$ and positivity of the terms, we get

$$
\begin{aligned}
\tau_0(a^*a) &= \sum_{i \in I} \tau_{\mathcal{N}} \left(\sum_{j \in I} a_{j,i}^* a_{j,i} \right) \\
&= \sum_{i \in I} \sum_{j \in I} \tau_{\mathcal{N}}(a_{j,i}^* a_{j,i}) = \sum_{j \in I} \sum_{i \in I} \tau_{\mathcal{N}}(a_{j,i}^* a_{j,i}) \\
&= \sum_{j \in I} \tau_{\mathcal{N}} \left(\sum_{i \in I} a_{j,i}^* a_{j,i} \right) = \sum_{j \in I} \tau_{\mathcal{N}}((aa^*)_{j,j}) = \tau_0(aa^*).
\end{aligned}
$$

Thus τ_0 is a trace on $\mathcal{M}_{z_0}$. Normality of τ_0 follows from normality of $\tau_{\mathcal{N}}$ together with the possibility of exchanging sup and $\sum$ for positive terms. Note that $\tau_0(a^*a) = 0$ implies $\tau_{\mathcal{N}}(a_{j,i}^* a_{j,i}) = 0$ for all $i, j \in I$. Hence by faithfulness of $\tau_{\mathcal{N}}$ one has $a_{j,i} = 0$ for all $i, j \in I$, so that $a = 0$. Thus τ_0 is faithful. If τ_0 is not semifinite, than for some projection $w \in \mathcal{Z}(\mathcal{M}_{z_0})$ we have $\tau_0(a) = \infty$ for any non-zero positive $a \in \mathcal{M}$, $a \leq w$. But $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{N}) \overline{\otimes} \mathbb{C}\mathbb{1}_{H_0}$, so that $w = w_0 \otimes \mathbb{1}$ with $w_0 \in \mathcal{Z}(\mathcal{N})$. Fix

$i_0, j_0 \in I$ and let $a = (a_{i,j})$ with $a_{i,j} = w_0$ for $i = i_0$ and $j = j_0$, and $a_{i,j} = 0$ otherwise. Then $\tau_0(a) = \tau_{\mathcal{N}}(w_0) < \infty$, a contradiction. Hence τ_0 is a f.n.s. trace on $\mathcal{M}z_0$, which contradicts the maximality of the family $\{\tau_i\}$. So finally, $\operatorname{supp} \tau = \mathbb{1}_{\mathcal{M}}$, so that τ is a f.n.s. trace on $\mathcal{M}$. $\qquad\square$

PROPOSITION 2.22. *If $\mathcal{M}$ is finite and τ is a normal semifinite trace on $\mathcal{M}$, then the restriction of τ to the centre of $\mathcal{M}$ is semifinite.*

PROOF. Let τ be a normal semifinite trace on $\mathcal{M}$ and let $a \in \mathcal{M}_+$. By Proposition 2.3(6),

$$\mathscr{T}(a) = \operatorname{norm-lim} \sum_{k=1}^{k_n} \lambda_k^{(n)} u_k^{(n)*} a u_k^{(n)},$$

for some unitaries $u_k^{(n)}$ and non-negative numbers $\lambda_k^{(n)}$ with $\sum_{k=1}^{k_n} \lambda_k^{(n)} = 1$. Since τ is σ-weakly (so also norm) lower semicontinuous (see Corollary 2.15), we have

$$(\tau \circ \mathscr{T})(a) \leq \liminf_n \sum_{k=1}^{k_n} \lambda_k^{(n)} \tau(u_k^{(n)*} a u_k^{(n)}) = \tau(a).$$

We are ready to prove that τ is semifinite on the centre of $\mathcal{M}$. Let $0 \neq p \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$. By semifiniteness of τ on $\mathcal{M}$, there is a non-zero $q \in \mathbb{P}(\mathcal{M})$ such that $q \leq p$. Now $\mathscr{T}(q) \leq \mathscr{T}(p) = p$, and for some $\epsilon > 0$ the spectral projection $r := \chi_{(\epsilon,\infty)}(\mathscr{T}(q)) \neq 0$. We have $\epsilon r \leq r\mathscr{T}(q) \leq \mathscr{T}(q)$, so that $\tau(r) \leq (1/\epsilon)\tau(\mathscr{T}(q)) \leq (1/\epsilon)\tau(q) < \infty$. This means that τ is semifinite. $\qquad\square$

COROLLARY 2.23. *Any normal semifinite trace τ on a finite von Neumann algebra $\mathcal{M}$ is of the form $\mu \circ \mathscr{T}$, where μ is a normal semifinite trace on $\mathcal{Z}(\mathcal{M})$.*

PROOF. It is clear that if μ is a normal semifinite trace on $\mathcal{Z}(\mathcal{M})$, then $\mu \circ \mathscr{T}$ is a normal semifinite trace on $\mathcal{M}$ (just note that $\mathscr{T}$ acts like identity on $\mathcal{Z}(\mathcal{M})$). If τ is a normal semifinite trace on $\mathcal{M}$, then by the proposition its restriction μ to $\mathcal{Z}(\mathcal{M})_+$ is semifinite. Hence it is a sum of a family $\{\mu_i\}$ with $\mu_i \in \mathcal{Z}(\mathcal{M})_*^+$ with pairwise orthogonal supports. By Lemma 2.1, $\mu_i \circ \mathscr{T} = \mu_i$ on $\mathcal{Z}(\mathcal{M})$ for all i, and by uniqueness in the same lemma, $\tau \upharpoonright (\operatorname{supp}(\mu_i)\mathcal{M}) = \mu_i \circ \mathscr{T}$. Hence $\tau = \mu \circ \mathscr{T}$. $\qquad\square$

DEFINITION 2.24. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. We say that τ is *bounded away from* 0 if

$$\inf\{\tau(p) \colon p \in \mathbb{P}(\mathcal{M}), p \neq 0\} > 0. \tag{2.1}$$

LEMMA 2.25. *A f.n.s. trace τ on $\mathcal{M}$ can be bounded away from 0 only if $\mathcal{M}$ is of type I and the centre of $\mathcal{M}$ is purely atomic. If τ is not bounded away from zero, then we can find an orthogonal sequence (p_n) of non-zero projections from $\mathcal{M}$ such that $\tau(p_n) \to 0$.*

PROOF. By Theorem 2.21, $\mathcal{M}$ must be semifinite. If $\mathcal{M}$ contains a continuous summand, we may assume that $\mathcal{M}$ actually is continuous. Let $p \in \mathbb{P}(\mathcal{M})$ in $\mathcal{M}$ be such that $0 < \tau(p) < \infty$. Using the Halving Lemma 1.83(2), we can construct an orthogonal sequence (p_n) of projections from $\mathcal{M}$ starting with p and such that $\tau(p_n) \to 0$; in particular, τ is not bounded away from 0.

Assume now that $\mathcal{M}$ is discrete and contains a direct summand of type I_α of the form $\mathcal{K}\overline{\otimes}B(H)$ with non-atomic K and α-dimensional H. Let r be any minimal projection in the $B(H)$. For a projection p in $\mathcal{K}$, any subprojection of $p \otimes r$ must be of the form $q \otimes r$ with $q \leq p$ being a projection in $\mathcal{K}$. Hence the formula $\mu(a) := \tau(a \otimes r)$ defines a semifinite trace on $\mathcal{K}$. We can now start with any non-zero projection p in $\mathcal{K}$ of finite trace, and divide it into two non-zero projections, of which at least one will have trace $\leq \tau(p)/2$. It is clear that we can continue the process, producing an orthogonal sequence (p_n) of projections from $\mathcal{K}$ such that $\tau(p_n \otimes r) = \mu(p_n) \to 0$. We have produced orthogonal sequence $(p_n \otimes r)$ in $\mathcal{M}$ with $\tau(p_n \otimes r) \to 0$; in particular, τ is not bounded away from 0.

Finally, let $\mathcal{M}$ be an infinite direct sum $\sum_{i \in I}^{\oplus} \mathcal{F}_i$ of type I_α factors. Let r_i be a minimal projection in $\mathcal{F}_i$. If τ is not bounded away from 0, there is a sequence (p_n) of (different) projections from the family $\{r_i\}$ such that $\tau(p_n) \to 0$. The sequence is clearly orthogonal. $\qquad\square$

LEMMA 2.26. *If the algebra $\mathcal{M}$ is infinite-dimensional, then there exists an infinite orthogonal family (p_n) of non-zero projections from $\mathcal{M}$.*

PROOF. This follows from Lemma 2.25 if τ is not bounded away from zero. If τ is bounded away from zero, than $\mathcal{M}$ is an infinite direct sum of type I_α factors, and the units of the factors form the desired family. $\qquad\square$

2.2 **Measurability**

In this section we deal with an important concept of measurable operators. The class of measurable operators will have a structure of a topological $*$-algebra and will contain all the noncommutative function spaces we are going to consider.

The number of different operator classes we introduce here may seem overwhelming. Nevertheless, some of them are given here only for the sake of completeness. The differences between various classes of measurable operators disappear if we deal with finite trace or if we restrict our attention to factors.

DEFINITION 2.27. Let $\mathcal{M}$ be a semifinite von Neumann algebra endowed with a faithful normal semifinite trace τ. Then:

(1) $F(\mathcal{M}, \tau)$ denotes the set $\{a \in \mathcal{M} \colon \tau(|a|) < \infty\}$; its elements are called τ-*finite*.

(2) $K(\mathcal{M}, \tau)$ denotes the set

$$\{a \in \overline{\mathcal{M}} \colon \tau(\chi_{(\epsilon, \infty)}(|a|)) < \infty \text{ for each } \epsilon > 0\};$$

its elements are called τ-*compact*.

(3) $S(\mathcal{M}, \tau)$ denotes the set

$$\{a \in \overline{\mathcal{M}} \colon \tau(\chi_{(\epsilon, \infty)}(|a|)) < \infty \text{ for some } \epsilon > 0\};$$

its elements are called τ-*measurable*.

(4) $S(\mathcal{M})$ denotes the set

$$\{a \in \overline{\mathcal{M}} \colon \chi_{(\epsilon, \infty)}(|a|) \text{ is finite for some } \epsilon > 0\};$$

its elements are called *(Segal) measurable*.

(5) $LS(\mathcal{M}, \tau)$ denotes the set $\{a \in \overline{\mathcal{M}} \colon$ there exists a sequence $z_n \in \mathcal{Z}(\mathcal{M})$ such that $z_n \nearrow \mathbb{1}$ and $az_n \in S(\mathcal{M}, \tau)$ for each $n\}$; its elements are called *locally τ-measurable*.

(6) $LS(\mathcal{M})$ denotes the set $\{a \in \overline{\mathcal{M}} \colon$ there exists a sequence $z_n \in \mathcal{Z}(\mathcal{M})$ such that $z_n \nearrow \mathbb{1}$ and $az_n \in S(\mathcal{M})$ for each $n\}$; its elements are called *locally (Segal) measurable*.

Of the sets defined above, the most important for our purposes is that of τ-measurable operators. Whenever it does not lead to confusion, we will denote it by $\widetilde{\mathcal{M}}$. The notion was introduced by Nelson [**Nel74**]. It turned out to be most useful, in particular, the class is large enough for the theory of non-commutative L^p- and Orlicz spaces, both in the semifinite

and the general case. The class of measurable operators was introduced much earlier by Segal [**Seg53**]. Although not used much in the sequel, its obvious advantage is its lack of dependence on the trace. Many results are proved both for τ measurable and measurable operators, since the proofs are essentially the same, *mutatis mutandis*. The definition of locally measurable operators was given by Sankaran [**San59**], and the "lacking" class of τ-locally measurable operators appeared in a paper by Cecchini [**Cec78**]. The importance of the class of locally measurable operators will be explained in Section 2.3.

τ-compact operators were introduced by Fack and Kosaki [**FK86**], and investigated thoroughly by Stroh and West [**SW93**]. An inquisitive reader will certainly notice the lack of operators compact with respect to a von Neumann algebra. This class was introduced and explored by Kaftal [**Kaf77, Kaf78**] and further investigated in type III factors by Halpern and Kaftal [**HK86, HK87**]. It is not described here as it does not fit the scheme we are using.

We know that the sum of two unbounded densely defined operators can happen to have domain consisting only of 0. Thus, to have a non-trivial algebraic structure for our operators we have to assume that their domains are "large enough", so that intersections of the domains are again dense in H. Below you will find the proper definitions:

DEFINITION 2.28. Let τ be a faithful normal semifinite trace on $\mathcal{M}$. A subspace D of H is called τ-*dense* (resp. *strongly dense*) in H if there is a sequence (p_n) of projections from $\mathcal{M}$ such that $p_n H \subseteq D, p_n \nearrow \mathbb{1}$ and, for each $n, \tau(p_n^\perp)$ is finite (resp. $p_n^\perp$ is finite). A sequence (p_n) from the definition is called a *determining* sequence for D.

REMARK 2.29. It is clear that if (p_n) is a determining sequence for a τ-dense (resp. strongly dense) subspace of H, then $\tau(p_n^\perp) \to 0$ (resp. $p_n \to 0$ strongly) as $n \to \infty$. Moreover, a τ-dense subspace is strongly dense, and a strongly dense subspace is dense in H.

PROPOSITION 2.30.(1) *The intersection of a countable number of τ-dense subspaces of H is τ-dense in H.*

   (2) *The intersection of a finite number of strongly dense subspaces of H is strongly dense in H.*

PROOF. (1) Let $\{D_k\}$ be a countable family of τ-dense subspaces of H. Let $D := \bigcap D_k$ and let $(p_n^{(k)})$ be a determining sequence for D_k, for

each k. By going to a subsequence, we can assume that for each n and k, $\tau(p_n^{(k)\perp}) < 1/2^{n+k}$. Put $q_n = \bigwedge_{k=1}^{\infty} p_n^{(k)}$. Then

$$\tau(q_n^{\perp}) = \tau(\bigvee_{k=1}^{\infty} p_n^{(k)\perp}) \leq \sum_{k=1}^{\infty} \tau(p_n^{(k)\perp}) \leq \sum_{k=1}^{\infty} 1/2^{k+n} = 1/2^n \to 0$$

as $n \to \infty$. Obviously, $q_n \nearrow$, and the faithfulness of τ ensures that $q_n \nearrow 1$. It is also clear that, for each n, $q_n H \subseteq D$. Thus (q_n) is a determining sequence for D, and D is τ-dense.

(2) Assume now that D_1 and D_2 are strongly dense subspaces of H, with determining sequences (p_n) and (q_n). We will show that (r_n), where $r_n := p_n \wedge q_n$, is a determining sequence for $D := D_1 \cap D_2$. Note first that $r_1^{\perp} = p_1^{\perp} \vee q_1^{\perp}$ is a finite projection in $\mathcal{M}$, and that for each n, $r_n^{\perp} \leq r_1^{\perp}$, so that $p_n, q_n, r_n^{\perp} \in \mathcal{M}_{r_1^{\perp}}$. Let $\mathscr{T}$ be the centre-valued trace on $\mathcal{M}_{r_1^{\perp}}$. Then $\mathscr{T}(p_n^{\perp}), \mathscr{T}(q_n^{\perp}) \searrow 0$ and, by Lemma 2.5,

$$\mathscr{T}(r_n^{\perp}) = \mathscr{T}(p_n^{\perp} \vee q_n^{\perp}) \leq \mathscr{T}(p_n^{\perp}) + \mathscr{T}(q_n^{\perp}) \searrow 0.$$

Together with the obvious fact that $r_n^{\perp} \searrow$, the faithfulness of $\mathscr{T}$ yields $r_n^{\perp} \searrow 0$. It is clear that $r_n H \subseteq D$, which means that r_n is a determining sequence for D. $\qquad \square$

DEFINITION 2.31. A preclosed (closable) operator $a \in {}^{\eta}\mathcal{M}$ is called τ-*premeasurable* (resp. *premeasurable*) if it has a τ-dense (resp. strongly dense) domain.

We need the following simple lemma.

LEMMA 2.32.(1) *Let* $p, q \in \mathbb{P}(\mathcal{M})$. *If* $p \wedge q = 0$, *then* $p \preceq q^{\perp}$.
   (2) *Let* D *be strongly dense and let* (p_n) *be a determining sequence for* D. *If* $q \in \mathbb{P}(\mathcal{M})$ *is such that* $q \wedge p_n = 0$ *for each* n, *then* $q = 0$.
   (3) *Let* D *be strongly dense and let* (p_n) *be a determining sequence for* D. *If* $q, r \in \mathbb{P}(\mathcal{M})$ *is such that* $q \wedge p_n = r \wedge p_n$ *for each* n, *then* $q = r$.

PROOF. (1) By Kaplansky's parallelogram law (see Proposition 1.77),

$$p = p - p \wedge q \sim p \vee q - q \leq q^{\perp}.$$

(2) By (1), $q \preceq p_n^{\perp} \searrow 0$. Hence $\mathscr{T}(q) \leq \mathscr{T}(p_n) \to 0$ yields $\mathscr{T}(q) = 0$ and, by faithfulness of $\mathscr{T}$, $q = 0$.

(3) Put $p = q - q \wedge r$. By assumption, for each n we have $q \wedge p_n = (q \wedge r) \wedge p_n$, which implies $p \wedge p_n = 0$. By (2), $p = 0$, so that $q = q \wedge r$. Similarly, $r = q \wedge r$, which yields $q = r$. $\qquad \square$

LEMMA 2.33. *Let* $a \in \overline{\mathcal{M}}$. *If* $\xi \in \chi_{[0,\epsilon]}(|a|)H$, *then* $\|a\xi\| \leq \epsilon\|\xi\|$. *If, on the other hand,* $0 \neq \xi \in \chi_{(\epsilon,\infty)}(|a|)H$, *then* $\|a\xi\| > \epsilon\|\xi\|$.

PROOF. In the first case, $\|a\xi\| = \||a|\xi\| = \||a|\chi_{[0,\epsilon]}(|a|)\xi\| \leq \epsilon\|\xi\|$. For the second part, by right continuity of the spectral decomposition there is an $\epsilon' > \epsilon$ such that $\xi \in \chi_{[\epsilon',\infty)}(|a|)$. Let $\int_{-\infty}^{\infty} \lambda de_\lambda$ be the spectral decomposition of $|a|$. Then

$$\|a\xi\|^2 = \||a|\xi\|^2 = \int_{-\infty}^{\infty} \lambda^2 d(\langle e_\lambda \xi, \xi\rangle)$$

$$\geq \int_{[\epsilon',\infty)} \lambda^2 d(\langle e_\lambda \xi, \xi\rangle) \geq \int_{[\epsilon',\infty)} \epsilon'^2 d(\langle e_\lambda \xi, \xi\rangle)$$

$$= \epsilon'^2\|\xi\|^2 > \epsilon^2\|\xi\|^2.$$

$\square$

LEMMA 2.34.(1) *Let* $a \in {}^\eta\mathcal{M}$ *be preclosed and let* $p \in \mathbb{P}(\mathcal{M})$ *be such that* $pH \subseteq \mathrm{dom}(a)$. *Then* $ap \in \mathcal{M}$. *In particular, if* (p_n) *is a determining sequence for the domain* $\mathrm{dom}(a)$ *of a* τ-*premeasurable (resp. premeasurable) operator* a, *then* ap_n *is bounded for each* n.

    (2) *Let* $a \in \overline{\mathcal{M}}$. *Assume* $p \in \mathbb{P}(\mathcal{M})$ *is such that* $pH \subseteq \mathrm{dom}(a)$. *Then* $\chi_{(\|ap\|,\infty)}(|a|) \preceq p^\perp$.

PROOF. (1) If a is preclosed and $pH \subseteq \mathrm{dom}(a)$, then ap is closed and everywhere defined, hence bounded by the closed graph theorem.

(2) The operator ap is closed and everywhere defined, hence bounded. We have $ap \in {}^\eta\mathcal{M}$, so that $ap \in \mathcal{M}$. Put $\epsilon := \|ap\|$. By Lemma 2.33, $p \wedge \chi_{(\epsilon,\infty)}(|a|) = 0$. By Lemma 2.32(1), $\chi_{(\epsilon,\infty)}(|a|) \preceq p^\perp$

$\square$

PROPOSITION 2.35. *Let* $a \in \overline{\mathcal{M}}$. *The following conditions are equivalent:*

    (1) a *is* τ-*measurable (resp. a is measurable);*
    (2) $|a|$ *is* τ-*measurable (resp. $|a|$ is measurable);*
    (3) *the domain of* a *is* τ-*dense (resp. strongly dense) in* H;
    (4) *there is a projection* $p \in \mathcal{M}$ *such that* $pH \subseteq \mathrm{dom}(a)$ *and* $\tau(p^\perp)$ *(resp. $p^\perp$) is finite.*

PROOF. (1)$\Rightarrow$(2) is true by definition.

(2)$\Rightarrow$(3) Let $a \in \widetilde{\mathcal{M}} = S(\mathcal{M},\tau)$ (resp. $a \in S(\mathcal{M})$), and let $\epsilon > 0$ be such that $\tau(\chi_{(\epsilon,\infty)}(|a|))$ (resp. $\chi_{(\epsilon,\infty)}(|a|)$) is finite. Choose an increasing

sequence $\epsilon_n \to \infty$ with $\epsilon_1 := \epsilon$. Then the sequence $p_n := \chi_{(\epsilon_n,\infty)}(|a|)$ satisfies (3).

(3)$\Rightarrow$(4) is obvious.

(4)$\Rightarrow$(1) This follows directly by Lemma 2.34(2). $\qquad\square$

COROLLARY 2.36. *The closure $[a]$ of a τ-premeasurable (resp. premeasurable) operator a is τ-measurable (resp. measurable).*

LEMMA 2.37. *Let $a, b \in {}^\eta\mathcal{M}$, and let $p, q \in \mathbb{P}(\mathcal{M})$ be such that $pH \subseteq$ $\mathrm{dom}(a)$, $qH \subseteq \mathrm{dom}(b)$ and $ap, bq \in \mathcal{M}$. Put $e := \mathfrak{n}(p^\perp bq)$. Then $e^\perp \precsim$ $p^\perp$, $p^\perp bqe = 0$ and $e \wedge q \subseteq \mathrm{dom}(ab)$.*

PROOF. We have $e^\perp = \mathfrak{s}_r(p^\perp bq) \sim \mathfrak{s}_l(p^\perp bq) \le p^\perp$. The equality $p^\perp bqe = 0$ is evident from the definition of e. Thus $\xi \in eH$ implies $p^\perp bq\xi = 0$, so that $bq\xi = pbq\xi \subseteq pH \subseteq \mathrm{dom}(a)$. Hence $eH \subseteq \mathrm{dom}(abq)$ and $(e \wedge q)H \subseteq \mathrm{dom}(ab)$. $\qquad\square$

LEMMA 2.38. *If $a, b \in {}^\eta\mathcal{M}$ are τ-premeasurable (resp. premeasurable), then $\mathrm{dom}(ab)$ is τ-dense (resp. strongly dense).*

PROOF. Let (p_n) be a determining sequence for $\mathrm{dom}(a)$ and let (q_n) be a determining sequence for $\mathrm{dom}(b)$. Put $e_n := \mathfrak{n}(p_n^\perp bq_n)$. By Lemma 2.37, for each n, $\tau(e_n^\perp)$ (resp. $e_n^\perp$) is finite. Put $f_n := e_n \wedge q_n$. Again by Lemma 2.37, $f_n \subseteq \mathrm{dom}(ab)$ for each n. Clearly $\tau(f_n^\perp)$ (resp. $f_n^\perp$) are finite. We shall show that $f_n \nearrow 1$. First, by Lemma 2.37,

$$p_{n+1}^\perp bq_{n+1}f_n = p_{n+1}^\perp bq_{n+1}q_ne_nf_n = p_{n+1}^\perp p_n bq_n f_n = 0.$$

Hence $f_n \le \mathfrak{n}(p_{n+1}^\perp bq_{n+1}) = e_{n+1}$. We also have $f_n \le q_n \le q_{n+1}$, hence $f_n \le f_{n+1}$. Suppose $f_n \nearrow f$. Using the normality of the centre-valued trace $\mathscr{T}$ on the finite algebra $f_1^\perp \mathcal{M} f_1^\perp$, we get $\mathscr{T}(e_n^\perp) \le \mathscr{T}(p_n^\perp) \searrow 0$ and $\mathscr{T}(q_n^\perp) \searrow 0$, so that by Lemma 2.5 $\mathscr{T}(f_n^\perp) \searrow \mathscr{T}(f^\perp) = 0$ and $f = \mathbb{1}$. Hence, ab has a τ-dense (resp. strongly dense) domain. $\qquad\square$

COROLLARY 2.39. *If $a \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a \in S(\mathcal{M})$) and $b \in \mathcal{M}$, then $a + b, ab \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a + b, ab \in S(\mathcal{M})$).*

PROOF. Since a is closed, both $a + b$ and ab are closed. Note that $\mathrm{dom}(a + b) = \mathrm{dom}(a)$, hence measurability of $a + b$ follows from 2.35, (1)$\Leftrightarrow$(3). By Lemma 2.38, ab has a strongly dense domain, which implies its measurability (see 2.35, (1)$\Leftrightarrow$(3)). $\qquad\square$

An important property of τ-measurable (resp. measurable) operators stated in the next proposition is their rigidity: if they agree on a τ-dense

(resp. strongly dense) domains, they are equal. This property will be further generalized in Proposition 2.51.

PROPOSITION 2.40.(1) *If $a, b \in {}^{\eta}\mathcal{M}$ are premeasurable, D is a strongly dense (in particular, a τ-dense) subspace of $\mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a \restriction D = b \restriction D$, then $[a] = [b]$.*

(2) *If $a \in {}^{\eta}\mathcal{M}$ is premeasurable and D is a strongly dense (in particular, a τ-dense) subspace of $\mathrm{dom}(a)$, then D is a core for $[a]$.*

(3) *If $a \in {}^{\eta}\mathcal{M}$ is premeasurable and (p_n) is a determining sequence for $\mathrm{dom}(a)$, then $D_0 := \bigcup_n p_n H$ is a core for $[a]$.*

PROOF. (1) We can assume that $a, b \in S(\mathcal{M})$. Let $\mathcal{N}$ be the von Neumann algebra $\mathcal{M} \overline{\otimes} B(\mathbb{C}^2)$ acting in $H \oplus H$. Denote by p_a, p_b the projections onto the graphs $\mathcal{G}(a), \mathcal{G}(b) \subseteq H \oplus H$ of a and b, respectively. Then $\mathcal{G}(a)$ and $\mathcal{G}(b)$ are invariant under all elements of $\mathcal{N}' = \mathcal{M}' \otimes \mathbb{1}_{\mathbb{C}^2}$, hence $p_a, p_b \in \mathcal{N}$. Let (p_n) be a determining sequence for D and let $q_n := p_n \otimes \mathbb{1}_{\mathbb{C}^2}$ for each n. One easily checks that (q_n) is a determining sequence for a strongly dense subspace of $H \oplus H$. Moreover,

$$\begin{aligned}
\mathcal{G}(a) \cap (q_n H \oplus q_n H) &= \{(\xi, a\xi) : \xi \in q_n H, a\xi \in q_n H\} \\
&= \{(\xi, b\xi) : \xi \in q_n H, b\xi \in q_n H\} \\
&= \mathcal{G}(b) \cap (q_n H \oplus q_n H).
\end{aligned}$$

Hence $p_a \wedge q_n = p_b \wedge q_n$ for each n, and by Lemma 2.32(3), $p_a = p_b$, so that $[a] = [b]$.

(2) It is clear that $a \restriction D$ is premeasurable and that $[a \restriction D] \restriction D = [a] \restriction D$. By (1), $[a \restriction D] = [a]$, hence also $[[a] \restriction D] = [a]$, so that D is a core for $[a]$.

(3) follows immediately from (2). $\qquad\square$

COROLLARY 2.41. *$a \in LS(\mathcal{M})$ (resp. $a \in LS(\mathcal{M}, \tau)$) if and only if $|a| \in LS(\mathcal{M})$ (resp. $|a| \in LS(\mathcal{M}, \tau)$)*

PROOF. Obvious by definition. $\qquad\square$

COROLLARY 2.42. *If $a \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a \in S(\mathcal{M})$, $a \in LS(\mathcal{M}, \tau)$, $a \in LS(\mathcal{M})$), then $a^* \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a^* \in S(\mathcal{M})$, $a^* \in LS(\mathcal{M}, \tau)$, $a^* \in LS(\mathcal{M})$)*

PROOF. Let $a \in \overline{\mathcal{M}}$ have polar decomposition $a = u|a|$. Then $a^* = |a|u^*$, and the statements follow from Proposition 2.35, $(1) \Rightarrow (2)$ and Corollaries 2.41, 2.39. $\qquad\square$

The relation between various classes of measurable operators is elucidated in Propositions 2.43, 2.44, 2.45 and Theorem 2.46. Most of the implications have appeared in the literature in one form or another, although a few seem new. The diagram 2.2 should facilitate the reader's orientation. Those of you with deeper interest in measurability in von Neumann algebras will find the survey of Muratov and Chilin [**MC16**] invaluable, where they report on their own work on the subject and cite many other sources. In particular, they research the dependence of the class $\widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ on the trace τ. It is worth noting that the class $LS(\mathcal{M}, \tau)$ of τ-locally measurable operators does not depend on the choice of τ (see [**Cec78**]).

PROPOSITION 2.43. *Let τ be a faithful normal semifinite trace on $\mathcal{M}$. We have*

$$
\begin{array}{ccccccc}
& \mathcal{M} & & & S(\mathcal{M}) & & \\
& & & & & & \\
F(\mathcal{M},\tau) & & S(\mathcal{M},\tau) & & & LS(\mathcal{M}) \subseteq \overline{\mathcal{M}} & \quad (2.2) \\
& & & & & & \\
& K(\mathcal{M},\tau) & & LS(\mathcal{M},\tau) & & &
\end{array}
$$

PROOF. The inclusion $F(\mathcal{M}, \tau) \subseteq K(\mathcal{M}, \tau)$ follows from Chebyshev's inequality:

$$\epsilon \chi_{(\epsilon,\infty)}(|a|) \leq |a| \chi_{(\epsilon,\infty)}(|a|) \leq |a|.$$

Once you observe that projections from $\mathcal{M}$ having finite trace are necessarily finite, all the other inclusions become completely obvious. □

We are going to describe the dependence of the classes on dimensionality, factoriality and finiteness of $\mathcal{M}$, and on properties of τ, such as being finite or bounded away from zero.

PROPOSITION 2.44. *Let τ be a faithful normal semifinite trace on $\mathcal{M}$.*

(1) *If $\mathcal{M}$ is a factor, then*

$$S(\mathcal{M}, \tau) = S(\mathcal{M}) = LS(\mathcal{M}, \tau) = LS(\mathcal{M}).$$

(2) *If $\mathcal{M}$ is finite, then*

$$S(\mathcal{M}) = LS(\mathcal{M}) = \overline{\mathcal{M}}.$$

(3) *The trace τ is bounded away from zero if and only if $K(\mathcal{M}, \tau) \subseteq \mathcal{M}$, and if and only if $\mathcal{M} = S(\mathcal{M}, \tau)$.*

Proof. (1) The centre of a factor is trivial, and hence there is no difference between local and non-local measurability. Moreover, in a factor the trace of a projection is finite if and only if the projection is finite, hence there is no difference between versions with and without τ.

(2) If $\mathcal{M}$ is finite, then every densely defined operator affiliated with $\mathcal{M}$ is measurable, i.e. $S(\mathcal{M}) = \overline{\mathcal{M}}$.

(3) If τ is bounded away from zero, then no unbounded operator $a \in \overline{\mathcal{M}}$ is τ-measurable. In fact, assume that $a \in \overline{\mathcal{M}}$ is not bounded and let $e_n = \chi_{[n,n+1)}(|a|)$. Then infinitely many e_n's are non-zero, which shows that a is not τ-measurable. Hence $\mathcal{M} = S(\mathcal{M}, \tau)$, which implies $K(\mathcal{M}, \tau) \subseteq \mathcal{M}$. If τ is not bounded away from zero, then the algebra $\mathcal{M}$ is infinite-dimensional, and there is, by Lemma 2.26, an orthogonal sequence of non-zero projections (p_n) such that $\tau(p_n) \to 0$. We can assume that $\sum_{n=1}^{\infty} \tau(p_n) < \infty$. Then the operator $\sum_{n=1}^{\infty} n p_n$ belongs to $K(\mathcal{M}, \tau)$ and is unbounded. Hence $K(\mathcal{M}, \tau) \not\subseteq \mathcal{M}$, and consequently $\mathcal{M} \neq S(\mathcal{M}, \tau)$. $\qquad\square$

Proposition 2.45.(1) *If $\mathcal{M}$ is finite-dimensional (so that the trace τ is necessarily finite), then*

$$F(\mathcal{M}, \tau) = \mathcal{M} = K(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)$$
$$= S(\mathcal{M}) = LS(\mathcal{M}, \tau) = LS(\mathcal{M}) = \overline{\mathcal{M}}.$$

(2) *If $\mathcal{M}$ is infinite-dimensional, but the trace τ is finite, then*

$$F(\mathcal{M}, \tau) = \mathcal{M} \subsetneqq K(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)$$
$$= S(\mathcal{M}) = LS(\mathcal{M}, \tau) = LS(\mathcal{M}) = \overline{\mathcal{M}}.$$

(3) *If the trace τ is infinite, and $\mathcal{M}$ is of type I with finite-dimensional centre, then*

$$F(\mathcal{M}, \tau) \subsetneqq K(\mathcal{M}, \tau) \subsetneqq \mathcal{M} = S(\mathcal{M}, \tau)$$
$$= S(\mathcal{M}) = LS(\mathcal{M}, \tau) = LS(\mathcal{M}) \subsetneqq \overline{\mathcal{M}}.$$

Proof. (1) is obvious, since in a finite-dimensional algebra $F(\mathcal{M}, \tau) = \mathcal{M} = \overline{\mathcal{M}}$.

(2) If the trace τ is finite, then $F(\mathcal{M}, \tau) = \mathcal{M}$ and $K(\mathcal{M}, \tau) = \overline{\mathcal{M}}$. For τ to be bounded away from zero, $\mathcal{M}$ would have to be a direct sum of type I factors (see Lemma 2.25). But τ is finite, so none of these factors can be infinite-dimensional. Hence $\mathcal{M}$ is an infinite direct sum of type I_n factors with $n < \infty$. Since τ is finite, it cannot be bounded away from zero. By Proposition 2.44(3), $\mathcal{M} \neq S(\mathcal{M}, \tau)$.

(3) If $\mathcal{M}$ is of type I with finite-dimensional centre, then $\mathcal{M}$ is a finite direct sum of type I factors, with at least one of them infinite. Thus τ is bounded away from zero, and by Proposition 2.44, (1) and (3), $\mathcal{M} = S(\mathcal{M}, \tau) = S(\mathcal{M}) = LS(\mathcal{M}, \tau) = LS(\mathcal{M})$. Hence $K(\mathcal{M}, \tau) \subseteq \mathcal{M}$, but $\tau(\mathbb{1}) = \infty$, so that $\mathbb{1} \notin K(\mathcal{M}, \tau)$ and $K(\mathcal{M}, \tau) \neq \mathcal{M}$. Let now (p_n) be an infinite orthogonal sequence of projections from $\mathcal{M}$. Then the operator $\sum_{n=1}^{\infty} (1/n) p_n$ belongs to $K(\mathcal{M}, \tau)$, but not to $F(\mathcal{M}, \tau)$. Obviously, $\mathcal{M} \neq \overline{\mathcal{M}}$. $\qquad\square$

The next theorem clarifies which inclusions in diagram 2.2 of Proposition 2.43 are, in fact, equalities:

THEOREM 2.46.(1) $F(\mathcal{M}, \tau) = \mathcal{M}$ iff τ is finite.
  (2) $\mathcal{M} = S(\mathcal{M}, \tau)$ iff τ is bounded away from 0.
  (3) $F(\mathcal{M}, \tau) = K(\mathcal{M}, \tau)$ iff $\mathcal{M}$ is finite-dimensional.
  (4) $K(\mathcal{M}, \tau) = S(\mathcal{M}, \tau)$ iff τ is finite.
  (5) $S(\mathcal{M}, \tau) = S(\mathcal{M})$ iff finite projections from $\mathcal{M}$ have finite trace.
  (6) $S(\mathcal{M}) = LS(\mathcal{M})$ iff the centre of the properly infinite part of $\mathcal{M}$ is finite-dimensional.
  (7) $S(\mathcal{M}, \tau) = LS(\mathcal{M}, \tau)$ iff the restriction of τ to the finite part of $\mathcal{M}$ is finite, and the centre of the properly infinite part of $\mathcal{M}$ is finite-dimensional.
  (8) $LS(\mathcal{M}, \tau) = LS(\mathcal{M})$ iff the centre of the type II part of $\mathcal{M}$ is σ-finite.
  (9) $LS(\mathcal{M}) = \overline{\mathcal{M}}$ iff $\mathcal{M}$ is finite.

PROOF. (1) is clear from the definition.
(2) This is part of Proposition 2.44(3).
(3) "$\Leftarrow$" is obvious, since in finite-dimensional algebras $F(\mathcal{M}, \tau) = \mathcal{M} = \overline{\mathcal{M}}$.
"$\Rightarrow$" If $\mathcal{M}$ is infinite-dimensional, then we can construct the operator a such that $a \in K(\mathcal{M}, \tau)$ and $a \notin F(\mathcal{M}, \tau)$ as in the proof of Proposition 2.45(3).
(4) "$\Leftarrow$" If τ is finite, then clearly $K(\mathcal{M}, \tau) = \overline{\mathcal{M}}$, hence the result.
"$\Rightarrow$" If τ is infinite, then $\mathbb{1} \in S(\mathcal{M}, \tau)$ and $\mathbb{1} \notin K(\mathcal{M}, \tau)$.
(5) "$\Leftarrow$" is clear from the definitions.
"$\Rightarrow$" Suppose $p \in \mathbb{P}(\mathcal{M})$ is such that p is finite and $\tau(p) = \infty$. Take a maximal orthogonal family of non-zero subprojections of p of finite trace (it exists and sums up to p, by Proposition 2.13). It is clear that we can choose from it a countable subfamily $\{p_k\}$ such that $\sum_{k=1}^{\infty} \tau(p_k) = \infty$. Let

$a := \sum_{k=1}^{\infty} kp_k$. Then $a \in S(\mathcal{M})$, since $\chi_{(0,\infty)}(a) = \sum_{k=1}^{\infty} p_k \leq p$, which is finite, but for each natural n, $\chi_{(n,\infty)}(a) = \sum_{k=n+1}^{\infty} p_k$ with $\tau(\sum_{k=n+1}^{\infty} p_k) = \infty$. Hence $a \in S(\mathcal{M})$, but $a \notin S(\mathcal{M}, \tau)$.

(6) "$\Leftarrow$" If the algebra $\mathcal{M}$ is a direct sum of a finite von Neumann algebra and a finite number of infinite factors, then $S(\mathcal{M}) = LS(\mathcal{M})$ follows from (1) and (2) of Proposition 2.44.

"$\Rightarrow$" We assume that $S(\mathcal{M}) = LS(\mathcal{M})$ and that the centre of the properly infinite part of $\mathcal{M}$ is infinite dimensional. We are going to construct an operator a such that $a \in LS(\mathcal{M})$ and $a \notin S(\mathcal{M})$. There is no loss of generality in assuming additionally that $\mathcal{M}$ is properly infinite. Let (z_k) be an orthogonal sequence of non-zero central projections from $\mathcal{M}$ such that $\sum_{k=1}^{\infty} z_k = \mathbb{1}$. Define $a := \sum_{k=1}^{\infty} kz_k$. If $w_n = \sum_{k=1}^{n} z_k$, then $w_n \nearrow \mathbb{1}$ and $w_n^\perp$ is infinite (as a central projection in a properly infinite algebra). Note that $aw_n \in \mathcal{M} \subseteq S(\mathcal{M})$, so that $a \in LS(\mathcal{M})$. On the other hand, for any $\epsilon > 0$, the spectral projection $\chi_{(\epsilon,\infty)}$ contains some $w_n^\perp$, so it must be infinite. Hence, $a \notin S(\mathcal{M})$, which ends the proof.

(7) "$\Leftarrow$" Let $z \in \mathcal{Z}(\mathcal{M})$ be such that $z\mathcal{M}$ is finite and $(\mathbb{1} - z)\mathcal{M}$ is properly infinite. It is clear that $S(z\mathcal{M}, \tau) = zS(\mathcal{M}, \tau)$ and $S((\mathbb{1} - z)\mathcal{M}, \tau) = (\mathbb{1} - z)S(\mathcal{M}, \tau)$. Take $a \in LS(\mathcal{M}, \tau)$. Then $a \in LS(\mathcal{M})$, so that by (6) $a \in S(\mathcal{M})$. Note that $za \in S(z\mathcal{M}, \tau)$ by (5) applied to $z\mathcal{M}$, and $(\mathbb{1}-z)a \in S((\mathbb{1}-z)\mathcal{M}, \tau)$ by Proposition 2.44(1). Hence $a \in S(\mathcal{M}, \tau)$.

"$\Rightarrow$" Assume $S(\mathcal{M}, \tau) = LS(\mathcal{M}, \tau)$. It is enough to show that if either the restriction of τ to the finite part of $\mathcal{M}$ is infinite or the centre of the properly infinite part of $\mathcal{M}$ is infinite-dimensional, we get a contradiction. In both cases, there is an orthogonal sequence of non-zero projections $z_k \in \mathcal{Z}(\mathcal{M})$ such that for each $n \in \mathbb{N}$, $\sum_{k=n}^{\infty} \tau(z_k) = \infty$. Indeed, if the restriction of τ to the finite part of $\mathcal{M}$ is infinite, we can choose z_k so that $\tau(z_k) < \infty$ for each k, but $\sum_{k=1}^{\infty} \tau(z_k) = \infty$, as in (5)"$\Rightarrow$". If, on the other hand, there is an orthogonal sequence of non-zero projections in the centre of the properly infinite part of $\mathcal{M}$, then all the projections z_k are infinite, so that $\tau(z_k) = \infty$ for each k. Put $w_n = \sum_{k=1}^{n} z_k$ and $w = \sup_n w_n$. Then $v_n := w_n + (\mathbb{1} - w) \nearrow \mathbb{1}$. Let $a \in {}^\eta \mathcal{M}$ be the operator $\sum_{k=1}^{\infty} kz_k$. For each n we have then $v_n a = \sum_{k=1}^{n} kz_k \in \mathcal{M} \subseteq S(\mathcal{M}, \tau)$, so that $a \in LS(\mathcal{M}, \tau)$, but $a \notin S(\mathcal{M}, \tau)$ (cf. the proof of (5)).

(8) "$\Leftarrow$" Suppose first that $\mathcal{M}$ is of type I_n with $n < \infty$. Take $a \in LS(\mathcal{M})$. By (7), $a \in S(\mathcal{M})$. Let $\{f_k\}$ be an orthogonal family of abelian projections from $\mathcal{M}$ such that $\sum_{k=1}^{n} f_k = \mathbb{1}$ (which implies $\mathbb{z}(f_k) = \mathbb{1}$ for each k), and denote by $v_{i,k}$ partial isometries from $\mathcal{M}$ such that $v_{i,k}^* v_{i,k} =$

f_k and $v_{i,k}v_{i,k}^* = f_i$. By Lemma 2.39, the operators $af_k \in S(\mathcal{M})$, and by Proposition 2.35, also $|af_k|$, are measurable. Let $e_{k,m} := \chi_{(m,\infty)}(|af_k|)$ and $z_{k,m} := \mathbb{z}(e_{k,m})$. Note that $\mathbb{n}(|af_k|) = \mathbb{n}(af_k) \geq f_k^\perp$, so that $e_{k,m} \leq \mathbb{s}_l(|af_k|) \leq f_k$. Since $f_k\mathcal{M}f_k = \mathcal{Z}(\mathcal{M})f_k$ (see Proposition 1.68), $e_{k,m} = z_{k,m}f_k$. We check that for a fixed k, $z_{k,m} \searrow 0$ as $m \to \infty$. In fact,

$$z_{k,m} = \sum_{j=1}^n f_j z_{k,m} = \sum_{j=1}^n v_{j,k}f_k v_{j,k}^* z_{k,m} = \sum_{j=1}^n v_{j,k}e_{k,m}v_{j,k}^*,$$

and so $e_{k,m} \searrow 0$ as $m \to \infty$ implies $z_{k,m} \searrow 0$ as $m \to \infty$.

Put $z_m := z_{1,m}^\perp z_{2,m}^\perp \dots z_{n,m}^\perp$. Clearly, $z_m \nearrow \mathbb{1}$ as $m \to \infty$. Since $z_{k,m}^\perp f_k = e_{k,m}^\perp f_k$, we have that

$$f_k z_m = f_k e_{k,m}^\perp w_k,$$

where $w_k := z_{1,m}^\perp \dots z_{k-1,m}^\perp z_{k+1,m}^\perp \dots z_{n,m}^\perp$. Hence the closed operators $af_k z_m = af_k e_{k,m}^\perp w_k$ are bounded with $\|af_k z_m\| \leq m$. Therefore the closed operators $az_m = a(\sum_{k=1}^n f_k z_m) \supseteq \sum_{k=1}^n af_k z_m$ (see 1.104) are also bounded, and consequently $a \in LS(\mathcal{M},\tau)$.

Assume now that $\mathcal{M}$ is a σ-finite algebra of type II_1, and let $a \in LS(\mathcal{M})$. Again, by (7), $a \in S(M)$. If τ is finite, there is nothing to prove (see Proposition 2.44), so assume τ is infinite. By Lemma 2.22, the restriction of τ to $\mathcal{M}$ is semifinite. By Proposition 2.13(2), there is an orthogonal family of non-zero projections $\{w_i\}_{i \in I}$ in the centre $\mathcal{Z}(\mathcal{M})$ such that $\tau(w_i) < \infty$ and $\sum_{i \in I} w_i = \mathbb{1}$. Since the algebra is σ-finite, I is countable. Put $z_n := \sum_{k=1}^n w_k$. Then $z_n \nearrow \mathbb{1}$ as $n \to \infty$, and $az_n \in S(\mathcal{M}z_n, \tau)$, by (5). Obviously, az_n is τ-measurable in $\mathcal{M}$ as well, hence $a \in LS(\mathcal{M},\tau)$.

Let now $a \in S(\mathcal{M})$ and $\mathcal{M}$ be properly infinite. Let $p := \chi_{[0,\lambda]}(|a|)$ be such that $ap \in \mathcal{M}$ and $p^\perp$ is finite. Using Lemma 2.39 and the inclusion $a = a(p + p^\perp) \supseteq ap + ap^\perp$, we easily get $a = ap + ap^\perp$, with $ap^\perp \in S(M)$. If the centre of type II part of $\mathcal{M}$ is σ-finite, the same is true of $\mathcal{M}_{p^\perp}$. If this is the case, $ap^\perp \in LS(\mathcal{M},\tau)$ by the previous part of the proof. Again by Lemma 2.39, $a \in LS(\mathcal{M},\tau)$.

It is clear from the above proof that whenever $a \in S(\mathcal{M})$ and $\mathcal{M}$ is either of type I_n (with an arbitrary n) or of type II with σ-finite centre, one can find a sequence of central projections z_n such that az_n is bounded and $z_n \nearrow \mathbb{1}$. If now $a \in LS(\mathcal{M})$, then there is a sequence (w_k) of central projections such that $aw_k \in S(\mathcal{M})$ and $w_k \nearrow \mathbb{1}$. Let $v_n^{(k)} \in \mathcal{Z}(\mathcal{M})$ be such that $a(w_k - w_{k-1})v_n^{(k)} \in \mathcal{M}$ and $v_n^{(k)} \nearrow w_k - w_{k-1}$ (with $w_0 = 0$) as

$n \to \infty$. Put $z_n^{(k)} = \sum_{j=1}^{k} v_n^{(k)}$. It is routine to check that $z_k^k \nearrow \mathbb{1}$, with the boundedness of az_k^k implying that $a \in LS(\mathcal{M}, \tau)$.

"$\Rightarrow$" Assume now $\mathcal{M}$ is a non-σ-finite von Neumann algebra of type II. Let $p \in \mathbb{P}(\mathcal{M})$ be finite with central support $\mathbb{1}$; this can easily be done using an exhaustion argument, remembering that the sum of centrally orthogonal finite projections is again finite. Let now (p_k) be an orthogonal sequence of non-zero subprojections of p with central support $\mathbb{1}$ such that $\sum_{k=1}^{\infty} p_k = p$; this could be done by consecutive halving of projections (see Lemma 1.83)(2), starting with p. Note that $p_k z \neq 0$ for all non-zero $z \in \mathcal{Z}(\mathcal{M})$. Let $a := \sum_{k=1}^{\infty} kp_k$. Then $a \in S(\mathcal{M})$, since $\chi_{(n,\infty)}(|a|) = \sum_{k=n+1}^{\infty} p_k \leq p$ is finite. Let (z_n) be a sequence of central projections such that $z_n \nearrow \mathbb{1}$. For sufficiently large n the projection z_n is a sum of uncountable number of orthogonal central projections, say z_α (otherwise, $(z_n - z_{n-1})$ with $z_0 = 0$ would be a countable orthogonal sequence of σ-finite central projections with sum $\mathbb{1}$). For such n, az_n cannot be τ-measurable, since $\tau(\chi_{(n,\infty)}(|az_n|)) = \tau(\sum_{k=n+1}^{\infty} p_k)$ and $\tau(p_{n+1}) = \sum_\alpha \tau(p_{n+1}z) = \infty$, as an uncountable sum of positive numbers. Hence, $a \notin LS(\mathcal{M}, \tau)$, which ends the proof.

(9) "$\Leftarrow$" This is part of Proposition 2.44.

"$\Rightarrow$" Suppose $\mathcal{M}$ is not finite. We are going to build a closed and densely defined operator a which is not locally measurable. Let $\mathcal{M}z$, for a non-zero central projection z, be the properly infinite part of $\mathcal{M}$. If $a \notin LS(\mathcal{M}z, \tau)$, then $a \notin LS(\mathcal{M}, \tau)$, so we can assume that $z = \mathbb{1}$. Choose an orthogonal sequence of properly infinite projections (p_k) from $\mathcal{M}$ with central support $\mathbb{1}$ (for example, by consecutive halving, starting with $\mathbb{1}$), and let $a := \sum_{k=1}^{\infty} kp_k$. Then $a \in \overline{\mathcal{M}}$, but $a \notin LS(M)$, since $\mathbb{z}(a) = \mathbb{1}$ and for any central projection z the projection $\chi_{(n,\infty)}(|az|) = \sum_{k=n+1}^{\infty} p_k z)$ is infinite. □

2.3   Algebraic properties of measurable operators

In this section we show (in Theorem 2.50) that the space $LS(\mathcal{M})$ of locally measurable operators forms, with strong sum and strong product (see Theorem 2.48), a *-algebra, and that $LS(M, \tau), S(\mathcal{M})$ and $\widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ are its *-subalgebras. The importance of locally measurable operators stems from the fact that they form, in a sense, the largest class that is worth considering for measurability. If we want to include an operator $x \in \overline{\mathcal{M}}$ in such a class, we should clearly require that ax is closable for

any $a \in \mathcal{M}$, so that the strong product of a and x can be defined. As shown by Yeadon in a paper [**Yea75**] generalizing a former result of Dixon [**Dix71**], this implies that x is, in fact, locally measurable. Another important application of the notion will be found in Section 3.3.

LEMMA 2.47. *If $a, b \in {}^{\eta}\mathcal{M}$ are τ-premeasurable (resp. premeasurable), then $a + b$ and ab are τ-premeasurable (resp. premeasurable).*

PROOF. By Proposition 2.30 and Lemma 2.38, both $a + b$ and ab have τ-dense (resp. strongly dense) domains, so it is enough to show that they are closable. By Corollary 2.42, both $[a]^*$ and $[b]^*$ are measurable, hence $\mathrm{dom}([a]^* + [b]^*)$ and $\mathrm{dom}([b]^*[a]^*)$ are dense and the operators $([a]^* + [b]^*)^*$ and $([b]^*[a]^*)^*$ exist and are closed. Consequently, both

$$a + b \subseteq [a] + [b] \subseteq ([a]^* + [b]^*)^*$$

and

$$ab \subseteq [a][b] \subseteq ([b]^*[a]^*)^*$$

are closable. $\qquad\square$

THEOREM 2.48. *The space $\widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $S(\mathcal{M})$) with the operations $(a, b) \mapsto [a + b]$ of strong sum, $(a, b) \mapsto [ab]$ of strong product, together with the operation of multiplication by (complex) scalar $(\lambda, a) \mapsto \lambda a$ and the adjoint operation $*$ forms a $*$-algebra.*

PROOF. If $a, b, c \in {}^{\eta}\mathcal{M}$ are τ-premeasurable (resp. premeasurable), then by Lemma 2.47 $(a + b) + c$, $a + (b + c)$, $(ab)c$, $a(bc)$, $(a + b)c$, $ac + bc$, $c(a + b)$, $ca + cb$ are all τ-premeasurable (resp. premeasurable), and by Proposition 2.40 one has

$$[[a + b] + c] = [a + [b + c]], \qquad\qquad [[ab]c] = [a[bc]],$$
$$[[a + b]c] = [[ac] + [bc]], \qquad\qquad [c[a + b]] = [[ca] + [cb]],$$
$$[a + b]^* = [a^* + b^*], \qquad\qquad [ab]^* = [b^*a^*].$$

It is obvious that λa is τ-premeasurable (resp. premeasurable) if a is τ-premeasurable (resp. premeasurable), for any $\lambda \in \mathbb{C}$. Similarly, $\lambda a \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $\lambda a \in S(\mathcal{M})$) for $a \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a \in S(\mathcal{M})$), for any $\lambda \in \mathbb{C}$. The result follows. $\qquad\square$

NOTATION 2.49. In the sequel, we will denote the operations of the strong sum and strong product by $\bar{+}$ and $\bar{\cdot}$, respectively. That is, $a\bar{+}b :=$ $[a + b]$ and $a\bar{\cdot}b := [ab]$. Similarly, we write $a\bar{-}b$ for $[a - b]$. We shall write

simply $a + b$ for the strong sum and ab for the strong product whenever the meaning is obvious from the context and does not lead to confusion.

THEOREM 2.50. *If $a, b \in LS(\mathcal{M}, \tau)$ (resp. $LS(\mathcal{M})$), then $a + b, ab$ are both closable, $[a + b], [ab] \in LS(\mathcal{M}, \tau)$ (resp. $LS(\mathcal{M})$), and the space $LS(\mathcal{M}, \tau)$ (resp. $LS(\mathcal{M})$) with the operations of strong sum, strong product, multiplication by (complex) scalar $(\lambda, a) \mapsto \lambda a$ and the adjoint operation $*$ forms a $*$-algebra.*

PROOF. If $a, b \in LS(\mathcal{M}, \tau)$ (resp. $LS(\mathcal{M})$), then we can find one sequence (z_n) of central projections with $z_n \nearrow \mathbb{1}$ such that $az_n, bz_n \in S(\mathcal{M}, \tau)$ (resp. $S(\mathcal{M})$). Since $(a+b)z_n = az_n + bz_n$, $a \bar{+} b \in LS(\mathcal{M}, \tau)$ (resp. $LS(\mathcal{M})$). For the product, note that $az_n bz_n \in S(\mathcal{M}, \tau)$ (resp. $S(\mathcal{M})$), and that $z_n b \subseteq bz_n$ by Lemma 1.121, which implies $az_n bz_n \subseteq abz_n$. By Proposition 2.40, $az_n bz_n = abz_n$. This means that $abz_n \in S(\mathcal{M}, \tau)$ (resp. $abz_n \in S(\mathcal{M})$) and $ab \in LS(\mathcal{M}, \tau)$ (resp. $ab \in LS(\mathcal{M})$). Finally, $a \in \overline{\mathcal{M}}$ implies that $z_n a$ is densely defined and $z_n a \subseteq az_n$. Hence $(az_n)^*$ exists and $(az_n)^* \subseteq (z_n a)^* = a^* z_n$. By Proposition 2.40, $a^* z_n = (az_n)^* \in S(\mathcal{M}, \tau)$ (resp. $a^* z_n \in S(\mathcal{M})$) and $a \in LS(\mathcal{M}, \tau)$ (resp. $a \in LS(\mathcal{M})$). $\qquad\square$

The following proposition strengthens the conclusions of Proposition 2.40:

PROPOSITION 2.51. *Let $a, b \in {}^\eta\mathcal{M}$ be τ-premeasurable (resp. premeasurable), and let $x \in {}^\eta\mathcal{M}$. Then*

(1) *If D is a dense subspace of $\mathrm{dom}(a) \cap \mathrm{dom}(b)$ and $a \upharpoonright D = b \upharpoonright D$, then $[a] = [b]$; in particular, if $a, b \in \widetilde{\mathcal{M}} = S(\mathcal{M}, \tau)$ (resp. $a, b \in S(\mathcal{M})$) agree on a dense subspace, then $a = b$.*
(2) *If x is closable and $a \subseteq x$, then $[x] = [a]$.*
(3) *If x is densely defined and $x \subseteq a$, then $[x] = [a]$.*

PROOF. (1) $[a] \bar{-} [b]$ is τ-measurable (resp. measurable) and $[a] \bar{-} [b] \supseteq 0 \upharpoonright D$, so $([a] \bar{-} [b])^* \subseteq 0$. On the other hand, $([a] \bar{-} [b])^*$ is τ-measurable (resp. measurable), so it must be 0, which implies $[a] = [b]$ by Theorem 2.50.

(2) The assumptions ensure that x is τ-premeasurable (resp. premeasurable), so we get the result from (1).

(3) follows from (2) by taking adjoints. $\qquad\square$

2.4 **Topological properties of measurable operators**

In this section, we will introduce the so-called *measure* topology (respectively the topology of *convergence in measure*) in the *-algebra $\widetilde{\mathcal{M}}$ and show that it turns $\widetilde{\mathcal{M}}$ into a complete topological *-algebra.

One can obtain similar results for other classes of measurability, but that would require introduction of another topology, that of local convergence in measure (see [**Yea73**]), which will not be used in the sequel.

NOTATION 2.52. We denote by $\mathcal{N}(\epsilon, \delta)$ the set $\{a \in \widetilde{\mathcal{M}}:$ there exists a projection $p = p_{\epsilon,\delta} \in \mathcal{M}$ such that $pH \subseteq \mathrm{dom}(a)$, $\|ap\| \leq \epsilon$ and $\tau(p^{\perp}) \leq \delta\}$.

LEMMA 2.53. *Let $a \in \widetilde{\mathcal{M}}$. Then:*

(1) $a \in \mathcal{N}(\epsilon, \delta)$ *iff* $\tau(\chi_{(\epsilon,\infty)}(|a|)) \leq \delta$. *In particular,* $a \in \mathcal{N}(\epsilon, \delta)$ *iff* $|a| \in \mathcal{N}(\epsilon, \delta)$.
(2) *for all $a \in \widetilde{\mathcal{M}}$ and $\delta > 0$ there is an $\epsilon > 0$ such that $a \in \mathcal{N}(\epsilon, \delta)$.*

PROOF. (1)"$\Leftarrow$" Put $p := \chi_{(\epsilon,\infty)}(|a|)$ in the definition of $\mathcal{N}(\epsilon, \delta)$. (1)"$\Rightarrow$" This follows directly from Lemma 2.34(2).

(2) follows immediately from (1), the definition of τ-measurability and the normality of the trace. $\square$

LEMMA 2.54. *For all $\epsilon, \epsilon_1, \epsilon_2, \delta, \delta_1, \delta_2 > 0$ and $\lambda \in \mathbb{C}$ we have:*

(1) $\mathcal{N}(\epsilon, \delta)^* = \mathcal{N}(\epsilon, \delta)$,
(2) $\mathcal{N}(|\lambda|\epsilon, \delta) = \lambda\mathcal{N}(\epsilon, \delta)$,
(3) *if $\epsilon_1 \leq \epsilon_2$ and $\delta_1 \leq \delta_2$, then $\mathcal{N}(\epsilon_1, \delta_1) \subseteq \mathcal{N}(\epsilon_2, \delta_2)$,*
(4) $\mathcal{N}(\epsilon_1, \delta_1) \cap \mathcal{N}(\epsilon_2, \delta_2) \supseteq \mathcal{N}(\min(\epsilon_1, \epsilon_2), \min(\delta_1, \delta_2))$,
(5) $\mathcal{N}(\epsilon_1, \delta_1) \dot{+} \mathcal{N}(\epsilon_2, \delta_2) \subseteq \mathcal{N}(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$,
(6) $\mathcal{N}(\epsilon_1, \delta_1) \dot{\cdot} \mathcal{N}(\epsilon_2, \delta_2) \subseteq \mathcal{N}(\epsilon_1\epsilon_2, \delta_1 + \delta_2)$.

PROOF. (1) Assume $a \in \widetilde{\mathcal{M}}$, and let $a = u|a|$ be its polar decomposition. Then u maps $\mathrm{s}(|a|)$ isometrically onto $\mathrm{s}(|a^*|)$ and $|a^*| = u|a|u^*$. For any $\xi \in H$, we have

$$\langle|a^*|\xi, \xi\rangle = \int_{(0,\infty)} \lambda d\langle e_{\lambda}(|a^*|)\xi, \xi\rangle,$$

and

$$\langle |a^*|\xi, \xi \rangle = \langle |a|u^*\xi, u^*\xi \rangle = \int_{(0,\infty)} \lambda d\langle e_\lambda(|a|)u^*\xi, u^*\xi \rangle$$
$$= \int_{(0,\infty)} \lambda d\langle (ue_\lambda(|a|)u^*)\xi, \xi \rangle.$$

By the uniqueness of the spectral decomposition, $e_\lambda(|a^*|) = ue_\lambda(|a|)u^*$ for each $\lambda \geq 0$. Hence $e_{(\epsilon,\infty)}(|a^*|) = ue_{(\epsilon,\infty)}(|a|)u^*$ and

$$\tau(\chi_{(\epsilon,\infty)}(|a^*|)) = \tau(u\chi_{(\epsilon,\infty)}(|a|)u^*) = \tau(u^*u\chi_{(\epsilon,\infty)}(|a|))$$
$$= \tau(\mathbb{s}(|a|)\chi_{(\epsilon,\infty)}(|a|)) = \tau(\chi_{(\epsilon,\infty)}(|a|))$$

for each $\epsilon > 0$, which implies (1).

(2) and (3) are obvious, and (4) follows immediately from (3).

(5) If $a \in \mathcal{N}(\epsilon_1, \delta_1)$ and $b \in \mathcal{N}(\epsilon_2, \delta_2)$, then there exist projections $p, q \in \mathcal{M}$ such that $pH \subseteq \text{dom}(a)$, $qH \subseteq \text{dom}(b)$, $\|ap\| \leq \epsilon_1$, $\|bq\| \leq \epsilon_2$, and $\tau(p^\perp) \leq \delta_1$, $\tau(q^\perp) \leq \delta_2$. Put $r := p \wedge q$. Then $rH \subseteq \text{dom}(a\bar{+}b)$, $\|(a\bar{+}b)r\| = \|ar + br\| \leq \epsilon_1 + \epsilon_2$ and $\tau(r^\perp) \leq \delta_1 + \delta_2$, which implies (5).

(6) Assume $a \in \mathcal{N}(\epsilon_1, \delta_1)$ and $b \in \mathcal{N}(\epsilon_2, \delta_2)$. Choose projections $p, q \in \mathcal{M}$ such that $pH \subseteq \text{dom}(a)$, $qH \subseteq \text{dom}(b)$, $\|ap\| \leq \epsilon_1$, $\|bq\| \leq \epsilon_2$, and $\tau(p^\perp) \leq \delta_1$, $\tau(q^\perp) \leq \delta_2$. Let $e := \mathbb{m}(p^\perp bq)$ and $f := e \wedge q$. Then, by Lemma 2.37, $fH \subseteq \text{dom}(ab)$ and $pbqe = bqe$, so that $abf = abqef = apbqef = apbqf$. Thus $\|(a\bar{\cdot}b)f\| = \|abf\| \leq \epsilon_1\epsilon_2$. Similarly, again by Lemma 2.37, $\tau(f^\perp) \leq \tau(e^\perp) + \tau(q^\perp) \leq \tau(p^\perp) + \tau(q^\perp) \leq \delta_1 + \delta_2$, which ends the proof. $\quad\square$

PROPOSITION 2.55. *The sets $\mathcal{N}(\epsilon, \delta)$ form a basis of neighbourhoods of 0 for a vector space topology on $\widetilde{\mathcal{M}}$, called the* measure *topology.*

PROOF. By definition, a set $A \subseteq \widetilde{\mathcal{M}}$ is open in the measure topology if, for each $a \in A$, there are $\epsilon, \delta > 0$ such that $a\bar{+}\mathcal{N}(\epsilon, \delta) \subseteq A$. It follows from (4) of Lemma 2.54 that the sets $\mathcal{N}(\epsilon, \delta)$ form a basis of neighbourhoods of 0 for a translation-invariant topology on $\widetilde{\mathcal{M}}$. The neighbourhoods $\mathcal{N}(\epsilon, \delta)$ are balanced (or circled) by (2) and (3) of the lemma, and absorbing by using additionally Lemma 2.53(2). Together with Lemma 2.54, this yields the result (see, for example, [**SW99**, 1.2]). $\quad\square$

THEOREM 2.56. *The algebra $\widetilde{\mathcal{M}}$ of τ-measurable operators endowed with the measure topology, is a complete metrizable topological *-algebra in which $\mathcal{M}$ is dense.*

PROOF. The joint continuity of the multiplication follows easily from Lemma 2.54(6) and Lemma 2.53(2). The continuity of the * operation follows from Lemma 2.54(1). Note that $\{\mathcal{N}(1/n, 1/n) \colon n \in \mathbb{N}\}$ forms a countable base of neighbourhoods at 0 for the measure topology. The topology is Hausdorff, since $\bigcap_{\epsilon, \delta > 0} \mathcal{N}(\epsilon, \delta) = \{0\}$. In fact, if, for a fixed ϵ, $a \in \mathcal{N}(\epsilon, \delta)$ for each $\delta > 0$, then by Lemma 2.53(1), $\tau(\chi_{(\epsilon, \infty)}(|a|)) = 0$. Since this is true for all $\epsilon > 0$, we get $a = 0$. The countability of the base at 0 of a Hausdorff vector topology implies its metrizability, see [**SW99**, 6.1].

Let us show that $\mathcal{M}$ is dense in $\widetilde{\mathcal{M}}$ in the measure topology. Let $a \in \widetilde{\mathcal{M}}$ and let (p_n) be a determining sequence for $\mathrm{dom}(a)$. The normality of τ implies $\tau(p_n^\perp) \to 0$. Take any $\epsilon, \delta > 0$, and choose n_0 so that $\tau(p_n^\perp) \leq \delta$ for $n \geq n_0$. Then $a - ap_n \in \mathcal{N}(\epsilon, \delta)$ for $n \geq n_0$. In fact, it is enough to take $p_{\epsilon, \delta} := p_n$ in the definition of $\mathcal{N}(\epsilon, \delta)$ (see 2.52).

To show that $\widetilde{\mathcal{M}}$ is complete in the measure topology, it is enough to show that every sequence in $\widetilde{\mathcal{M}}$ Cauchy in measure converges in measure to an element of $\widetilde{\mathcal{M}}$. This follows directly from the metrizability of $\widetilde{\mathcal{M}}$ (or, even simpler, the existence of a countable base of neighbourhoods of 0 in $\widetilde{\mathcal{M}}$). We can assume that the Cauchy sequence is taken from $\mathcal{M}$. In fact, for a sequence (a_n) from $\widetilde{\mathcal{M}}$ we can pick a sequence (a_n') from $\mathcal{M}$ in such a way that $a_n - a_n' \in \mathcal{N}(1/n, 1/n)$ for all $n \in \mathbb{N}$. Then (a_n') is Cauchy, and its limit (if any) is also the limit of (a_n). By going to a subsequence, we can also assume that $a_n - a_{n+1} \in \mathcal{N}(1/2^{n+1}, 1/2^{n+1})$. So let (a_n) be a Cauchy sequence from $\mathcal{M}$ satisfying the above condition. Then, by Lemma 2.54, also $a_n^* - a_{n+1}^* \in \mathcal{N}(1/2^{n+1}, 1/2^{n+1})$. Choose projections p_n' and p_n'' from $\mathcal{M}$ so that $\|(a_{n+1} - a_n)p_n'\| \leq 1/2^{n+1}$, $\|(a_{n+1}^* - a_n^*)p_n''\| \leq 1/2^{n+1}$, $\tau(p_n'^\perp) \leq 1/2^{n+1}$ and $\tau(p_n''^\perp) \leq 1/2^{n+1}$. Put $p_n := p_n' \wedge p_n''$. Then $\|(a_{n+1} - a_n)p_n\| \leq 1/2^{n+1}$ and $\tau(p_n^\perp) \leq 1/2^n$. For each $n \in \mathbb{N}$, put $q_n = \bigwedge_{k=n+1}^\infty p_k$, so that $\tau(q_n^\perp) \leq \sum_{k=n+1}^\infty 1/2^k = 1/2^n$.

We calculate, for $m \geq n + 1$ and $\ell \in \mathbb{N}$,

$$\|(a_{m+\ell} - a_m)q_n\| \leq \sum_{k=m}^{m+\ell-1} \|(a_{k+1} - a_k)q_n\| \leq \sum_{k=m}^{m+\ell-1} \|(a_{k+1} - a_k)p_k\|$$

$$\leq \sum_{k=m}^{m+\ell-1} 1/2^{k+1} \leq 1/2^m,$$

and similarly
$$\|(a_{m+\ell}^* - a_m^*)q_n\| \leq 1/2^m.$$
Hence, if $\xi \in q_n H$, then both $(a_n \xi)$ and $(a_n^* \xi)$ are Cauchy sequences in H. Thus we can define operators a_0 and b_0 with domain $D = \bigcup_{n \in \mathbb{N}} q_n H$ by $a_0 \xi := \lim_{m \to \infty} a_m \xi$ and $b_0 \xi := \lim_{m \to \infty} a_m^* \xi$. Clearly $a_0, b_0 \in {}^\eta \mathcal{M}$ and their domain D is τ-dense in H. Moreover, for all $\xi, \eta \in D$ we have $\langle a_0 \xi, \eta \rangle = \lim_{m \to \infty} \langle a_m \xi, \eta \rangle = \lim_{m \to \infty} \langle \xi, a_m^* \eta \rangle = \langle \xi, b_0 \eta \rangle$, so that $a_0 \subseteq b_0^*$ and a_0 is premeasurable. Put $a = [a_0]$, so that $a \in \widetilde{\mathcal{M}}$.

We will show now that (a_n) actually converges in measure to a. Take any $\epsilon, \delta > 0$, and let n_0 be such that $\epsilon \geq 1/2^{n_0+1}$ and $\delta \geq 1/2^{n_0}$ We claim that $a - a_m \in \mathcal{N}(\epsilon, \delta)$ for $m \geq n_0 + 1$. In fact, it is enough to take $p := q_{n_0}$ in the definition of $\mathcal{N}(\epsilon, \delta)$. We have $\tau(q_{n_0}^\perp) \leq 1/2^{n_0} \leq \delta$. Moreover,

$$
\begin{aligned}
\|(a - a_m)q_{n_0}\| &= \sup_{\xi \in H, \|\xi\| \leq 1} \|(a - a_m)q_{n_0}\xi\| \\
&\leq \sup_{\xi \in H, \|\xi\| \leq 1} \limsup_{\ell \to \infty} \|(a_{m+\ell} - a_m)q_{n_0}\xi\| \\
&\leq \sup_{\xi \in H, \|\xi\| \leq 1} \limsup_{\ell \to \infty} \|(a_{m+\ell} - a_m)q_{n_0}\|\|\xi\| \\
&\leq \sup_{\xi \in H, \|\xi\| \leq 1} \limsup_{\ell \to \infty} (1/2^m)\|\xi\| \leq \epsilon.
\end{aligned}
$$

This ends the proof of the theorem. $\qquad\square$

2.5 Order properties of measurable operators

Up to now, we have defined the order $\leq$ in a C^*-algebra, in the set of bounded and unbounded operators acting on a Hilbert space H, and another order $\hat{\leq}$ in the set of generalized positive operators $\widehat{\mathcal{M}}_+$, associated with a von Neumann algebra $\mathcal{M}$. For semifinite von Neumann algebras and measurable operators associated with them, the natural order is given by the following:

NOTATION 2.57. For $a, b \in S(\mathcal{M})_h$, we write $a \bar{\leq} b$ if $b \bar{-} a \in S(\mathcal{M})_+$.

LEMMA 2.58. *For $a, b \in S(\mathcal{M}, \tau)_+$ (or $a, b \in S(\mathcal{M})_+$) we have $a \leq b$ iff $a \bar{\leq} b$.*

PROOF. We have $\mathrm{dom}(x) \subseteq \mathrm{dom}(x^{1/2})$ for any $x \in \overline{\mathcal{M}}_+$. If $a \leq b$, then $\langle a\xi, \xi \rangle \leq \langle b\xi, \xi \rangle$ for all $\xi \in \mathrm{dom}(b^{1/2}) \supseteq \mathrm{dom}(a) \cap \mathrm{dom}(b)$, so that $a \bar{\leq} b$. To prove the reverse implication, note that by Propositions 2.30 and 2.40(2), $\mathrm{dom}(a) \cap \mathrm{dom}(b)$ is a core for $b^{1/2}$. Choose $\xi \in \mathrm{dom}(b^{1/2})$.

Let $\xi_n \in \operatorname{dom}(a) \cap \operatorname{dom}(b)$ be such that $\xi_n \to \xi$ and $b^{1/2}\xi_n \to b^{1/2}\xi$. Then $\langle a(\xi_n - \xi_m), \xi_n - \xi_m \rangle \le \langle b(\xi_n - \xi_m), \xi_n - \xi_m \rangle$ for all n, m, so that $\|a^{1/2}\xi_n - a^{1/2}\xi_m\| \le \|b^{1/2}\xi_n - b^{1/2}\xi_m\|$ and $(a^{1/2}\xi_n)$ is a Cauchy sequence, and since $a^{1/2}$ is closed, $\xi \in \operatorname{dom}(a^{1/2})$. The inequality $\|a^{1/2}\xi\| \le \|b^{1/2}\xi\|$ holds for all $\xi \in \operatorname{dom}(a) \cap \operatorname{dom}(b)$, and $\operatorname{dom}(a) \cap \operatorname{dom}(b)$ is a core for $b^{1/2}$, so the inequality holds for $\xi \in \operatorname{dom}(b^{1/2})$ as well. $\qquad\square$

COROLLARY 2.59. *The inclusions* $\widetilde{\mathcal{M}}_+ = S(\mathcal{M}, \tau)_+ \hookrightarrow S(\mathcal{M})_+ \hookrightarrow \overline{\mathcal{M}}_+ \hookrightarrow \widehat{\mathcal{M}}_+$ *are order-preserving.*

It should be noted that one of the consequences of Lemma 2.58 is that to prove $a \le b$ for (τ-)measurable a, b it is enough to show that $\langle a\xi, \xi \rangle \le \langle b\xi, \xi \rangle$ for $\xi \in \operatorname{dom}(a) \cap \operatorname{dom}(b)$.

We will now show that all the classes of measurable operators that we have considered are hereditary with respect to the order.

PROPOSITION 2.60. *If* $b \in \widetilde{\mathcal{M}}_+ = S(\mathcal{M}, \tau)_+$ *(resp.* $b \in S(\mathcal{M})_+$, $b \in LS(\mathcal{M}, \tau)_+$, $b \in LS(\mathcal{M})_+$) *and* $a \in \overline{\mathcal{M}}_+$ *is such that* $a \le b$, *then* $a \in \widetilde{\mathcal{M}}_+ = S(\mathcal{M}, \tau)_+$ *(resp.* $a \in S(\mathcal{M})_+$, $a \in LS(\mathcal{M}, \tau)_+$, $a \in LS(\mathcal{M})_+$).

PROOF. Since $b \in \widetilde{\mathcal{M}}_+$ (resp. $b \in S(\mathcal{M})_+$), the domain of $b^{1/2}$ is τ-dense (resp. strongly dense), so that the domain of $a^{1/2}$ is τ-dense (resp. strongly dense). Consequently, $a = (a^{1/2})^2$ is τ-measurable (resp. measurable). If $b \in LS(\mathcal{M}, \tau)_+$ (resp. $b \in LS(\mathcal{M})_+$), then for some sequence (z_n) of central projections from $\mathcal{M}$ increasing to $\mathbb{1}$, $bz_n \in S(\mathcal{M}, \tau)_+$ (resp. $bz_n \in S(\mathcal{M})_+$). The first part of the proof shows that $az_n \in S(\mathcal{M}, \tau)_+$ (resp. $az_n \in S(\mathcal{M})_+$), so that $a \in LS(\mathcal{M}, \tau)_+$ (resp. $a \in LS(\mathcal{M})_+$). $\qquad\square$

PROPOSITION 2.61. *Let* $b \in \widetilde{\mathcal{M}}_+$ *(resp.* $b \in S(\mathcal{M})_+$) *and let* (a_i) *be an increasing net in* $\widetilde{\mathcal{M}}_+$ *(resp.* $S(\mathcal{M})_+$) *with* $a_i \le b$ *for all* i. *Then* $a := \sup_i a_i$ *exists in* $\widetilde{\mathcal{M}}$ *(resp.* $S(\mathcal{M})$). *Moreover, we have* $\operatorname{dom}(a^{1/2}) = \{\xi \colon \sup_i \|a_i^{1/2}\xi\| < \infty\}$ *and* $\|a^{1/2}\xi\| = \sup_i \|a_i^{1/2}\xi\|$ *for* $\xi \in \operatorname{dom}(a^{1/2})$.

PROOF. Let $m \in \widehat{\mathcal{M}}_+$ be the supremum of m_{a_i}. Then $m \le m_b$, so by Proposition 1.139, there is an $a \in \widetilde{\mathcal{M}}_+$ (resp. $a \in S(\mathcal{M})_+$) such that $m = m_a$. Evidently, $a = \sup_i a_i$. We have $\operatorname{dom}(a) = \{\xi \in H \colon m_a(\omega_\xi) < \infty\} = \{\xi \in H \colon \sup_i m_{a_i}(\omega_\xi) < \infty\} = \{\xi \in H \colon \sup_i \|a_i^{1/2}\xi\| < \infty\}$. The equality $\|a^{1/2}\xi\| = \sup_i \|a_i^{1/2}\xi\|$ follows from $\|a^{1/2}\xi\| = m_a(\omega_\xi) = \sup_i m_{a_i}(\omega_\xi) = \sup_i \|a_i^{1/2}\xi\|$. $\qquad\square$

LEMMA 2.62. *If $a, b \in LS(\mathcal{M})_+$ with $a \leq b$ and $d \in LS(\mathcal{M})$, then $d^{*\cdot}a^{\cdot}d \leq d^{*\cdot}b^{\cdot}d$.*

PROOF. We need to check that

$$\operatorname{dom}(|b^{1/2\cdot}d|) \subseteq \operatorname{dom}(|a^{1/2\cdot}d|) \text{ and } \||b^{1/2\cdot}d|\xi\| \leq \||b^{1/2\cdot}d|\xi\|$$

for all $\xi \in \operatorname{dom}(|b^{1/2\cdot}d|)$. Since $\operatorname{dom}(b^{1/2}) \subseteq \operatorname{dom}(a^{1/2})$, we have that

$$\operatorname{dom}(b^{1/2}d) \subseteq \operatorname{dom}(a^{1/2}d) \subseteq \operatorname{dom}(|a^{1/2\cdot}d|)$$

and

$$\|b^{1/2}d\xi\| \leq \|a^{1/2}d\xi\| \leq \||b^{1/2\cdot}d|\xi\|$$

for $\xi \in \operatorname{dom}(b^{1/2}d)$. If (ξ_n) is a sequence from $\operatorname{dom}(b^{1/2}d)$ convergent to $\xi \in \operatorname{dom}(b^{1/2\cdot}d)$ and $b^{1/2}d\xi_n \to (b^{1/2\cdot}d)\xi$, then $(a^{1/2}d\xi_n)$ is Cauchy in H and since $(a^{1/2\cdot}d)$ is closed, we have $\xi \in \operatorname{dom}(a^{1/2\cdot}d)$. The required norm inequality now easily follows from the convergence of $(a^{1/2}d\xi_n)$ to $(a^{1/2\cdot}d)\xi$. $\qquad\square$

PROPOSITION 2.63. *If we have $a_i, a \in \widetilde{\mathcal{M}}_+$ (resp. $a_i, a \in S(\mathcal{M})_+$) with $a_i \nearrow a$, then for all $b \in \widetilde{\mathcal{M}}$ (resp. $b \in S(\mathcal{M})$), $b^{*\cdot}a_i^{\cdot}b \nearrow b^{*\cdot}a^{\cdot}b$.*

PROOF. It follows from Lemma 2.62 that $b^{*\cdot}a_i^{\cdot}b \nearrow d$ for some $d \in \widetilde{\mathcal{M}}_+$ (resp. $d \in S(\mathcal{M})_+$), and by Proposition 2.61 we have $d \leq b^{*\cdot}a^{\cdot}b$,

$$\|(a_i^{1/2\cdot}b)\xi\| = \|(b^{*\cdot}a_i^{\cdot}b)^{1/2}\xi\| \nearrow \|d^{1/2}\xi\| \text{ for all } \xi \in D(d^{1/2}),$$

and

$$\|a_i^{1/2}b\xi\| \nearrow \|a^{1/2}b\xi\| \text{ for all } \xi \in \operatorname{dom}(a^{1/2}b) \subseteq \operatorname{dom}((b^{*\cdot}a^{\cdot}b)^{1/2}) \subseteq D(d^{1/2}).$$

Note that for all i and all $\xi \in \operatorname{dom}(a^{1/2}b)$, $\operatorname{dom}(a^{1/2}b) \subseteq \operatorname{dom}(a_i^{1/2}b)$ and

$$\|(b^{*\cdot}a^{\cdot}b)^{1/2}\xi\| = \||a_i^{1/2\cdot}b|\xi\| = \|a_i^{1/2}b\xi\|.$$

Hence $\|d^{1/2}\xi\| = \|a^{1/2\cdot}b\xi\|$ for all $\xi \in \operatorname{dom}(a^{1/2}b)$, which implies $\langle d\xi, \xi \rangle = \langle (b^{*\cdot}a^{\cdot}b)\xi, \xi \rangle$ for all $\xi \in \operatorname{dom}(b^*ab)$. This means that the τ-measurable (resp. measurable) operators $b^{*\cdot}a^{\cdot}b$ and d are equal on a τ-dense (resp. strongly dense) domain. By Proposition 2.51(1), $b^{*\cdot}a^{\cdot}b = d$. $\qquad\square$

As an easy corollary, we get the consistency of notation when an operator is treated both as a measurable one and as a generalized one.

COROLLARY 2.64. *If $d \in \mathcal{M}$ and $a \in \widetilde{\mathcal{M}}_+$ (resp. $a \in S(\mathcal{M})_+$), then*

$$d^* m_a d = m_{d^{*\cdot}a^{\cdot}d}.$$

Proof. Let (a_n) be a sequence from $\mathcal{M}$ such that $a_n \nearrow a$. Then $d^* m_{a_n} d \nearrow d^* m_a d$ and $d^* a_n d \nearrow d^{*-} \bar{a} \cdot d$. We have $d^* m_{a_n} d = m_{d^* a_n d} \nearrow m_{d^{*-} \bar{a} \cdot d}$, so the result follows from Lemma 1.140. $\qquad \square$

PROPOSITION 2.65. [**Sch86**, 2.2D] *If* $a, b \in \widetilde{\mathcal{M}}_+$ *(resp.* $a, b \in S(\mathcal{M})_+$*) with* $a \leq b$*, then there exists an operator* $d \in \mathcal{M}_+$ *with* $d \leq \mathfrak{s}(b)$ *such that* $a = b^{1/2} d b^{1/2}$.

PROOF. Assume first that $\mathfrak{s}(b) = \mathbb{1}$. By assumption, $\mathrm{dom}(b^{1/2}) \subseteq \mathrm{dom}(a^{1/2})$. By definition (see 1.103(7)), we have the equality $\mathrm{dom}(b^{1/2}) = b^{-1/2} \mathrm{dom}(b^{-1/2})$. For $\xi \in \mathrm{dom}(b^{1/2})$ put $x(b^{1/2}\xi) := a^{1/2}\xi$. Then $x = a^{1/2} b^{-1/2}$ satisfies $\|x\xi\| \leq 1$ for all ξ in a dense subspace $\mathrm{dom}(x) = \mathrm{dom}(b^{-1/2})$. Thus $[x]$ is bounded. Since $x \in {}^\eta \mathcal{M}$ (see Lemma 1.123), we have $[x] \in \mathcal{M}$. Thus $[x]b^{1/2} = a^{1/2}$ on a τ-dense (resp. strongly dense) domain $\mathrm{dom}(b^{1/2})$, hence they are equal (see Proposition 2.40). By Proposition 1.111(7), $([x]b^{1/2})^* = b^{1/2}[x]^*$, so that, with $d := [x]^*[x] \in \mathcal{M}_+$, we get $d \leq \mathbb{1}$ and $a = b^{1/2} d b^{1/2}$.

If $\mathfrak{s}(b)$ is not equal to $\mathbb{1}$, we can repeat the proof on the Hilbert subspace $\mathfrak{s}(b)H$, and note that a must vanish on the orthogonal complement of the subspace. $\qquad \square$

LEMMA 2.66. *Let* $a \in \widetilde{\mathcal{M}}_+$ *be given with* $a \in \mathcal{N}(\epsilon, \delta)$*. Then* $a^{1/2} \in \mathcal{N}(\sqrt{\epsilon}, \delta)$

PROOF. Note that if $a \in \mathcal{N}(\epsilon, \delta)$, then of course $\tau(\chi_{(\epsilon,\infty)}(a)) \leq \delta$. Since by the Borel functional calculus $\chi_{(\epsilon,\infty)}(a) = \chi_{(\sqrt{\epsilon},\infty)}(a^{1/2})$, we in fact have that $a^{1/2} \in \mathcal{N}(\sqrt{\epsilon}, \delta)$ $\qquad \square$

LEMMA 2.67. *Let* $a, b \in \widetilde{\mathcal{M}}_+$ *with* $a \leq b$*. If* $b \in \mathcal{N}(\epsilon, \delta)$*, then also* $a \in \mathcal{N}(\epsilon, \delta)$.

PROOF. Assume $b \in \mathcal{N}(\epsilon, \delta)$. If $0 \neq \xi \in \chi_{(\epsilon,\infty)}(a)H$, then by Lemmas 2.33 and 2.66, $\|b^{1/2}\xi\| \geq \|a^{1/2}\xi\| > \epsilon^{1/2}\|\xi\|$. If, on the other hand, $\xi \in \chi_{[0,\epsilon]}(b) = \chi_{[0,\epsilon^{1/2}]}(b^{1/2})$, then $\|b^{1/2}\xi\| \leq \epsilon^{1/2}\|\xi\|$. Hence, $\chi_{(\epsilon,\infty)}(a)) \wedge \chi_{[0,\epsilon]}(b)) = 0$, which by Lemma 2.32(1) yields $\tau(\chi_{(\epsilon,\infty)}(b)) \leq \tau(\chi_{(\epsilon,\infty)}(a)) \leq \delta$, so that $b \in \mathcal{N}(\epsilon, \delta)$. $\qquad \square$

PROPOSITION 2.68. *The positive cone* $\widetilde{\mathcal{M}}_+$ *is closed in measure in* $\widetilde{\mathcal{M}}$.

PROOF. Since $\widetilde{\mathcal{M}}$ is metrizable in the measure topology, it is enough to show that if a sequence (a_n) from $\widetilde{\mathcal{M}}_+$ converges to a, then $a \in \widetilde{\mathcal{M}}_+$.

Since the adjoint operation is continuous, we must have $a \in \widetilde{\mathcal{M}}_h$. Assume that $a \notin \widetilde{\mathcal{M}}_+$. Then $a = \int_{\mathbb{R}} \lambda de_\lambda$ with $\int_{(-\infty, -\epsilon]} \lambda de_\lambda \neq 0$ for some $\epsilon > 0$. Put $p := \chi_{(-\infty, -\epsilon]}(a)$. Since $pa_np \to pap$, we can take pap in place of a and assume that $a \leq -\epsilon\mathbb{1}$. Put now $\delta := \min(1, \tau(1))$ If $b \in \widetilde{\mathcal{M}}_+$ belongs to $a + N(\epsilon/2, \delta/2)$, then by Lemma 2.53(1)

$$\delta/2 \geq \tau(\chi_{(\epsilon/2,\infty)}(|b-a|)) = \tau(\chi_{(\epsilon/2,\infty)}(b-a)) \geq \tau(\chi_{(\epsilon/2,\infty)}(b+\epsilon\mathbb{1})) = \tau(1),$$

which is impossible. Hence we cannot have $a_n \to a$, and consequently the assumption that a is not positive leads to a contradiction. $\qquad\square$

LEMMA 2.69. *Let (a_n) be a sequence in $\widetilde{\mathcal{M}}_+$ converging to 0 in measure. Then there exists a subsequence (a_{n_k}) of (a_n) and some $b \in \widetilde{\mathcal{M}}_+$ such that $2^k a_{n_k} \leq b$ for all $k \in \mathbb{N}$.*

PROOF. Suppose that (a_n) is a sequence in $\widetilde{\mathcal{M}}_+$ converging to 0 in the topology of convergence in measure. Observe that the collection of sets $U_n = \mathcal{N}(2^{-n}, 2^{-n})$ ($n \in \mathbb{N}$) constitutes a countable neighbourhood base at 0 for the topology of convergence in measure on $\widetilde{\mathcal{M}}$. The convergence of (a_n) to 0 in measure now ensures that we may select natural numbers n_k such that $a_n \in 2^{-k}U_k$ for each $n \geq n_k$. Since for each k we clearly have that $2^{-(k+1)}U_{k+1} \subseteq 2^{-k}U_k$, we may select the n_k's to be increasing. The subsequence we seek is then (a_{n_k}). We proceed to show that this sequence satisfies the hypothesis of the lemma. Firstly note that by construction $2^k x_{n_k} \in U_k$ for each $k \in \mathbb{N}$. Now consider the sequence $w_m = \sum_{k=1}^m 2^k x_{n_k}$ ($m \in \mathbb{N}$). Let $\mathcal{N}$ be an arbitrary neighbourhood of 0 in the measure topology. For N large enough, we will then have that $U_m \subseteq \mathcal{N}$ for each $m \geq N$. Observe that by Lemma 2.54(5), we have

$$
\begin{aligned}
w_{m+n} - w_m &= \sum_{k=m+1}^{m+n} 2^k x_{n_k} \\
&\in \sum_{k=m+1}^{m+n} \mathcal{N}(2^{-k}, 2^{-k}) \\
&\subseteq \mathcal{N}\left(\sum_{k=m+1}^{m+n} 2^{-k}, \sum_{k=m+1}^{m+n} 2^{-k}\right) \\
&\subseteq \mathcal{N}(2^{-m}, 2^{-m}) = U_m \\
&\subseteq \mathcal{N}
\end{aligned}
$$

whenever $m \geq N$. This clearly shows that (w_n) is a Cauchy sequence in $\widetilde{\mathcal{M}}$. By the completeness of $\widetilde{\mathcal{M}}$, (w_n) must converge to some $b \in \widetilde{\mathcal{M}}$. But since (w_m) is by construction an increasing sequence in $\widetilde{\mathcal{M}}_+$, we not only have that $b \geq 0$ (by the closedness of $\widetilde{\mathcal{M}}_+$ in the topology of convergence in measure), but also that $w_m \leq b$ for all m. This in particular also ensures that $2^k x_{n_k} \leq b$ for all $k \in \mathbb{N}$, as required. $\qquad\square$

COROLLARY 2.70. *Let $\mathcal{M}, \mathcal{N}$ be semifinite von Neumann algebras endowed with f.n.s. traces $\tau_{\mathcal{M}}, \tau_{\mathcal{N}}$. Assume $T : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ is a positive map, and let $(a_n) \subseteq \widetilde{\mathcal{M}}_+$ be a sequence converging to 0 in $\tau_{\mathcal{M}}$-measure. Then there exists a subsequence (a_{n_k}) of (a_n) such that $T(a_{n_k}) \to 0$ in $\tau_{\mathcal{N}}$-measure.*

PROOF. By Lemma 2.69 there exists a subsequence (a_{n_k}) of (a_n) and some $b \in \widetilde{\mathcal{M}}_+$ such that $2^k a_{n_k} \leq b$ for all $k \in \mathbb{N}$. The positivity of T then ensures that $0 \leq T(a_{n_k}) \leq 2^{-k} T(b)$ for all $k \in \mathbb{N}$. The sequence $(2^{-k} T(b))$ trivially converges to 0 in $\tau_{\mathcal{N}}$-measure. By Lemma 2.67, the same must then be true of $(T(a_{n_k}))$. $\qquad\square$

2.6   Jordan morphisms on $\widetilde{\mathcal{M}}$

PROPOSITION 2.71. *Let $\mathcal{M}_1, \mathcal{M}_2$ be semifinite von Neumann algebras with f.n.s. traces τ_1 and τ_2 respectively, and let $\mathscr{J} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ be a Jordan *-morphism (see Remark 1.12). Then $\mathscr{J}$ is positivity preserving. Moreover $\mathscr{J}$ maps $\mathcal{M}_1$ into $\mathcal{M}_2$ in a uniformly continuous manner.*

PROOF. Since $\mathscr{J}$ preserves squares of self-adjoint elements of $\widetilde{\mathcal{M}}_1$, it easily follows from Lemma 2.66 that it is positivity preserving with $\mathscr{J}(\mathbb{1})$ a projection commuting with all elements of $\mathscr{J}(\widetilde{\mathcal{M}}_1)$ since

$$\mathscr{J}(a) = \mathscr{J}(\mathbb{1})\,\mathscr{J}(a)\,\mathscr{J}(\mathbb{1}) \text{ for all } a \in \widetilde{\mathcal{M}}_1.$$

For any $a \in \mathcal{M}_1^+$ we of course have that $0 \leq a \leq \|a\|\mathbb{1}$. It therefore follows from what we have just noted that then $0 \leq \mathscr{J}(a) \leq \|a\|\mathscr{J}(\mathbb{1})$. Therefore $\mathscr{J}(a) \in \mathcal{M}_2^+$ with $\|\mathscr{J}(a)\| \leq \|a\| < \infty$ whenever $a \in \mathcal{M}_1^+$. We then clearly have that $\mathscr{J}(\mathcal{M}_1) \subseteq \mathcal{M}_2$ and also that the stated continuity claim holds. $\qquad\square$

COROLLARY 2.72. *Let $\mathcal{M}_1, \mathcal{M}_2$ be semifinite von Neumann algebras with f.n.s. traces τ_1 and τ_2 respectively, and let $\mathscr{J} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ be a Jordan *-morphism. Then $\mathscr{J}$ maps projections in $\mathcal{M}_1$ onto projections in $\mathcal{M}_2$.*

PROOF. Easy consequence of the above proposition. $\square$

PROPOSITION 2.73. *Let $\mathcal{M}_1, \mathcal{M}_2$ be semifinite von Neumann algebras with f.n.s. traces τ_1 and τ_2 respectively, and let $\mathscr{J} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ be a Jordan *-morphism. Then $\mathscr{J}$ is continuous in measure if and only if $\mathscr{J}$ is the continuous (in measure) extension of a Jordan *-morphism $\mathscr{J}_0 : \mathcal{M}_1 \to \mathcal{M}_2$ for which $\tau_2 \circ \mathscr{J}_0$ is $\epsilon - \delta$ absolutely continuous with respect to τ_1, in the sense that for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\tau_2(\mathscr{J}_0(e)) \leq \epsilon$ whenever $e \in \mathcal{M}_1$ is a projection satisfying $\tau_1(e) \leq \delta$.*

PROOF. Suppose that $\mathscr{J}$ is continuous with respect to the topology of convergence in measure. We know from Proposition 2.71 that $\mathscr{J}$ maps $\mathcal{M}_1$ into $\mathcal{M}_2$ in a uniformly continuous manner with norm $\|\mathscr{J}\|$. So we only need to verify the claim about absolute continuity. Given a basic neighbourhood of 0 of the form

$$\mathscr{N}_2(\epsilon_2, \epsilon) = \{a \in \widetilde{\mathcal{M}}_2 \colon \tau_2(\chi_{(\epsilon_2, \infty)}(|a|)) \leq \epsilon\},$$

there exists $\epsilon_1, \delta > 0$ so that $\mathscr{J}$ maps $\mathscr{N}_1(\epsilon_1, \delta) \cap \mathcal{M}_1$ where

$$\mathscr{N}_1(\epsilon_1, \delta) = \{a \in \widetilde{\mathcal{M}}_1 \colon \tau_1(\chi_{(\epsilon_1, \infty)}(|a|)) \leq \delta\}$$

into $\mathscr{N}_2(\epsilon_2, \epsilon)$. So given a non-zero projection $e \in \mathcal{M}_1$ with $\tau_1(e) \leq \delta$ it is clear that $(1 + \epsilon_2)e \in \mathscr{N}_1(\epsilon_1, \delta)$ and hence that $(1 + \epsilon_2)\mathscr{J}(e) \in \mathscr{N}_2(\epsilon_2, \epsilon)$. Since $\mathscr{J}(e)$ is again a projection, this is sufficient to imply $\tau_2(\mathscr{J}(e)) \leq \epsilon$.

Conversely suppose that we are given a Jordan morphism $\mathscr{J}_0 : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\tau_2 \circ \mathscr{J}_0$ is $\epsilon - \delta$ absolutely continuous with respect to τ_1. As a positive map, $\mathscr{J}_0$ is of course uniformly continuous with norm say $\|\mathscr{J}_0\|$. Let $0 < \tilde{\epsilon}, \tilde{\delta}$ be given and select $\delta > 0$ so that $\tau_2(\mathscr{J}_0(e)) < \tilde{\delta}$ whenever $e \in \mathcal{M}_1$ is a projection with $\tau_1(e) < 2\delta$. We show that then

$$\mathscr{J}_0(\mathscr{N}_1(\epsilon, \delta) \cap \mathcal{M}_1) \subseteq \mathscr{N}_2(\tilde{\epsilon}, \tilde{\delta})$$

whenever $\sqrt{2\|\mathscr{J}_0\|}\epsilon \leq \tilde{\epsilon}$. This will be sufficient to establish the continuity of $\mathscr{J}_0$ (at 0). Thus let $a \in \mathscr{N}_1(\epsilon, \delta) \cap \mathcal{M}_1$ be given. Then of course $a^* \in \mathscr{N}_1(\epsilon, \delta)$. Consequently we may find projections $e, f \in \mathcal{M}_1$ with

$$\tau_1(e), \tau_1(f) < \delta \quad \text{and} \quad \|a(1 - e)\|, \|a^*(1 - f)\| \leq \epsilon.$$

Now let $g = e \vee f$. Then since τ_1 is a trace, $\tau_1(g) \leq \tau_1(e) + \tau_1(f) < 2\delta$, and hence $\tau_2(\mathscr{J}_0(g)) < \tilde{\delta}$ by the τ_1-absolute continuity of $\tau_2 \circ \mathscr{J}_0$. It remains to show that $\|\mathscr{J}_0(a)(1 - \mathscr{J}_0(g))\| \leq \sqrt{2\|\mathscr{J}_0\|}\epsilon$. Now since $1 - g = 1 - (e \vee f) = (1 - e) \wedge (1 - f)$ is majorised by both $1 - e$ and

$\mathbb{1} - f$, it follows that $\|a(\mathbb{1} - g)\| = \|a(\mathbb{1} - e)(\mathbb{1} - g)\| \leq \|a(\mathbb{1} - e)\| \leq \epsilon$ and similarly that $\|a^*(\mathbb{1} - g)\| \leq \epsilon$. The identities on p212 of [**BR87a**] can easily be verified in the present context, and hence on using positivity where necessary it follows from these identities that

$$
\begin{aligned}
|\mathscr{J}_0(a)(\mathbb{1} - \mathscr{J}_0(g))|^2 &= (\mathbb{1} - \mathscr{J}_0(g))|\mathscr{J}_0(a)|^2(\mathbb{1} - \mathscr{J}_0(g)) \\
&\leq (\mathbb{1} - \mathscr{J}_0(g))(|\mathscr{J}_0(a)|^2 + |\mathscr{J}_0(a^*)|^2)(\mathbb{1} - \mathscr{J}_0(g)) \\
&= (\mathbb{1} - \mathscr{J}_0(g))(\mathscr{J}_0(|a|^2 + |a^*|^2))(\mathbb{1} - \mathscr{J}_0(g)) \\
&= \mathscr{J}_0((\mathbb{1} - g)(|a|^2 + |a^*|^2)(\mathbb{1} - g)) \\
&= \mathscr{J}_0(|a(\mathbb{1} - g)|^2 + |a^*(\mathbb{1} - g)|^2).
\end{aligned}
$$

The fact that both $a(\mathbb{1}-g)$ and $a^*(\mathbb{1}-g)$ are bounded, will when combined with the above inequality, ensure that $|\mathscr{J}_0(a)(\mathbb{1} - \mathscr{J}_0(g))|^2$, and hence $\mathscr{J}_0(a)(\mathbb{1} - \mathscr{J}_0(g))$, is bounded. In fact it follows from the above that

$$
\begin{aligned}
\|\mathscr{J}_0(a)(\mathbb{1} - \mathscr{J}_0(g))\|^2 &= \||\mathscr{J}_0(a)(\mathbb{1} - \mathscr{J}_0(g))|^2\| \\
&\leq \|\mathscr{J}_0(|a(\mathbb{1} - g)|^2 + |a^*(\mathbb{1} - g)|^2)\| \\
&\leq \|\mathscr{J}_0\|(\||a(\mathbb{1} - g)|^2\| + \||a^*(\mathbb{1} - g)|^2\|) \\
&\leq 2\|\mathscr{J}_0\|\epsilon.
\end{aligned}
$$

Since both $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are complete linear metric spaces with $\mathcal{M}_1$ dense in $\widetilde{\mathcal{M}}_1$, $\mathscr{J}_0$ then allows for a continuous extension of $\mathscr{J}_0$ to all of $\widetilde{\mathcal{M}}_1$. It is an exercise to see that the extension is still a Jordan *-morphism $\qquad\square$

In closing we present an automatic continuity result of Weigt [**Wei09**], effectively showing that all Jordan morphisms $\mathscr{J} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ are continuous extensions of Jordan morphisms $\mathscr{J}_0 : \mathcal{M}_1 \to \mathcal{M}_2$.

DEFINITION 2.74. For a linear map $T : \mathcal{A} \to \mathcal{B}$ between linear metric spaces $\mathcal{A}$ and $\mathcal{B}$, we define the *separating space* $\mathcal{S}(T, \mathcal{B})$ to be

$$\mathcal{S}(T, \mathcal{B}) = \{b \in \mathcal{B} \colon x_n \to 0 \text{ and } T(x_n) \to b \text{ for some sequence } (x_n) \subseteq \mathcal{A}\}.$$

It can easily be verified that $\mathcal{S}(T, \mathcal{B})$ is a vector subspace of $\mathcal{B}$. The following version of the Closed Graph Theorem is valid in this context:

THEOREM 2.75 ([**KN63**, p.101]). *Let $T : \mathcal{A} \to \mathcal{B}$ be a linear map between complete linear metric spaces $\mathcal{A}$ and $\mathcal{B}$. Then T is continuous whenever $\mathcal{S}(T, \mathcal{B}) = \{0\}$.*

We finally come to the promised automatic continuity result. Alongside Proposition 2.73, this then provides a complete depiction of the nature of Jordan *-morphisms from $\widetilde{\mathcal{M}}_1$ to $\widetilde{\mathcal{M}}_2$.

THEOREM 2.76. *Every Jordan *-morphism $\widetilde{\mathscr{J}} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ is automatically continuous with respect to the topologies of convergence in measure, and is therefore the continuous extension of a Jordan $*$-morphism $\mathscr{J}$ from $\mathcal{M}_1$ to $\mathcal{M}_2$ for which $\tau_2 \circ \mathscr{J}$ is $\epsilon - \delta$ absolutely continuous with respect to τ_1.*

PROOF. Once the first claim has been established, the second will trivially follow from Proposition 2.73 and Proposition 2.71. We therefore only need to prove the first claim. The Closed Graph Theorem is known to hold for complete linear metric spaces [**KN63**, p 101]. Having established earlier that $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ belong to this category of spaces, the proof of automatic continuity consists of nothing more than showing that $\widetilde{\mathscr{J}}$ fulfills the prerequisites of this particular flavour of the Closed Graph Theorem. This amounts to showing that $\mathcal{S}(\widetilde{\mathscr{J}}, \widetilde{\mathcal{M}}_2) = \{0\}$ where $\mathcal{S}(\widetilde{\mathscr{J}}, \widetilde{\mathcal{M}}_2)$ is the set consisting of all elements $b \in \widetilde{\mathcal{M}}_2$ for which we can find a sequence $(a_n) \subseteq \widetilde{\mathcal{M}}_1$ such that $a_n \to 0$ and $\widetilde{\mathscr{J}}(a_n) \to b$ in measure. So let $b \in \mathcal{S}(\widetilde{\mathscr{J}}, \widetilde{\mathcal{M}}_2)$ be given and let $(a_n) \subseteq \widetilde{\mathcal{M}}_1$ be a sequence such that $a_n \to 0$ and $\widetilde{\mathscr{J}}(a_n) \to b$ in measure. We will show that $b = 0$. Notice that we now have that $\mathrm{Re}(a_n) = \frac{1}{2}(a_n + a_n^*) \to 0$ and $\widetilde{\mathscr{J}}(\mathrm{Re}(a_n)) = \frac{1}{2}(\widetilde{\mathscr{J}}(a_n) + \widetilde{\mathscr{J}}(a_n)^*) \to \mathrm{Re}(b)$, and similarly $\mathrm{Im}(a_n) \to 0$ and $\widetilde{\mathscr{J}}(\mathrm{Im}(a_n)) \to \mathrm{Im}(b)$. Thus for the task of verifying that $b = 0$, we may without loss of generality assume that $b = b^*$, and $a_n = a_n^*$ for all $n \in \mathbb{N}$. By continuity of multiplication and the fact that $\widetilde{\mathscr{J}}$ preserves squares of self-adjoint elements, we have that $|a_n|^2 = a_n^2 \to 0$ and $\widetilde{\mathscr{J}}(a_n^2) = \widetilde{\mathscr{J}}(a_n)^2 \to |b|^2 = b^2$ in measure. But by Corollary 2.70, there exists a subsequence (a_{n_k}) of (a_n) such that $\widetilde{\mathscr{J}}(a_{n_k}^2) \to 0$ in measure. For this subsequence we also surely still have that $\widetilde{\mathscr{J}}(a_{n_k}^2) \to |b|^2$. Thus the uniqueness of the limit ensures that $|b|^2 = 0$, or equivalently that $b = 0$, as required. $\qquad\square$

CHAPTER 3

Weights and densities

3.1 Weights

Let $\mathcal{M}$ be a von Neumann algebra acting in the Hilbert space H.

DEFINITION 3.1. A map $\varphi : \mathcal{M}_+ \to [0, \infty]$ is a *weight* on $\mathcal{M}$ if

(1) $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in \mathcal{M}_+$;
(2) $\varphi(\lambda a) = \lambda \varphi(a)$ for all $a \in \mathcal{M}_+$, $\lambda \geq 0$.

A weight φ is *finite* if $\varphi(\mathbb{1}) < \infty$.

NOTATION 3.2. For a weight φ on $\mathcal{M}$

$$N_\varphi := \{a \in \mathcal{M} : \varphi(a^*a) = 0\}$$
$$\mathfrak{p}_\varphi := \{a \in \mathcal{M}_+ : \varphi(a) < \infty\}$$
$$\mathfrak{n}_\varphi := \{a \in \mathcal{M} : \varphi(a^*a) < \infty\}$$
$$\mathfrak{m}_\varphi := \text{linear span of } \mathfrak{p}_\varphi.$$

PROPOSITION 3.3.(1) $\mathfrak{p}_\varphi$ *is a hereditary subcone of* $\mathcal{M}_+$;
(2) N_φ *and* $\mathfrak{n}_\varphi$ *are left ideals in* $\mathcal{M}$;
(3) $\mathfrak{m}_\varphi$ *is the linear span of* $\mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$ *and* $\mathfrak{m}_\varphi \subseteq \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$;
(4) $\mathfrak{m}_\varphi$ *is a* *-subalgebra of* $\mathcal{M}$;
(5) $\varphi \upharpoonright \mathfrak{p}_\varphi$ *extends to a positive linear form on* $\mathfrak{m}_\varphi$ *(denoted also by* φ*) and* $\mathfrak{m}_\varphi \cap \mathcal{M}_+ = \mathfrak{p}_\varphi$.

PROOF. The proof is almost identical to the proof of Proposition 2.13, the only difference is that both N_φ and $\mathfrak{n}_\varphi$ are only left ideals. This is in fact the only place where the tracial property (see Definition 2.6(3)) is used in the proof of Proposition 2.13.

$\square$

THEOREM 3.4 ([**Haa75a**]). *For any weight* φ *on* $\mathcal{M}$ *the following conditions are equivalent:*

(1) *If $(a_i)_{i \in I}$ is a bounded from above, increasing net from $\mathcal{M}_+$, then*
    $\varphi(\sup_{i \in I} a_i) = \sup_{i \in I} \varphi(a_i)$.

(2) *φ is completely additive; that is, whenever $\{a_i\}_{i \in I}$ is a family
    from $\mathcal{M}_+$ such that $\sum_{i \in I} a_i$ exists in the strong topology of $\mathcal{M}$,
    we have $\varphi(\sum_{i \in I} a_i) = \sum_{i \in I} \varphi(a_i)$.*

(3) *φ is σ-weakly lower semicontinuous (i.e. for all $\lambda \geq 0$ the set
    $\{a \in \mathcal{M}_+ : \varphi(a) \leq \lambda\}$ is σ-weakly closed);*

(4) *$\varphi(a) = \sup_{\omega \in F} \omega(a)$ for some $F \subseteq \mathcal{M}_*^+$, for all $a \in \mathcal{M}_+$;*

(5) *$\varphi(a) = \sup\{\omega(a) : \omega \in \mathcal{M}_*^+, \omega \leq \varphi\}$ for all $a \in \mathcal{M}_+$;*

(6) *$\varphi(a) = \sum_{i \in I} \varphi_i(a)$ for all $a \in \mathcal{M}_+$, for a family $\{\varphi_i\}_{i \in I}$ with
    $\varphi_i \in \mathcal{M}_*^+$ for all $i \in I$.*

DEFINITION 3.5. A weight φ is *normal* if it satisfies any of (1)–(6). φ
is *faithful* if $\varphi(a) = 0$ implies $a = 0$ for $a \in \mathcal{M}_+$.

EXAMPLE 3.6 ([**Haa75a**]). Complete additivity on projections does
not imply normality. Indeed, let $\mathcal{M} = l^\infty$ with $\varphi((\alpha_n)_{n \in \mathbb{N}}) = \sum_{n \in \mathbb{N}} \alpha_n$
if $\#\{n : \alpha_n \neq 0\} < \infty$ and $\varphi((\alpha_n)_{n \in \mathbb{N}}) = \infty$ otherwise. Then φ is not
normal: for $a = (\alpha_n) := (2^{-n})$ and $a^{(n)} = (\alpha_1, \ldots \alpha_n, 0, \ldots)$, we have
$\varphi(a^{(n)}) = 1 - 2^{-n}$ and $\varphi(a) = \infty$, so that $a^{(n)} \nearrow a$ but $\varphi(a^{(n)})$ does not
converge to $\varphi(a)$. On the other hand, condition $\varphi(\sum_{i \in I} p_i) = \sum_{i \in I} \varphi(p_i)$
is satisfied for any orthogonal family of projections from $\mathcal{M}$.

PROPOSITION 3.7. *If φ is a normal weight on $\mathcal{M}$, then there exists a
unique projection $p \in \mathcal{M}$ such that φ is faithful on $\mathcal{M}_{p^\perp}$ and (the extension
of) φ equals zero on $\mathcal{M}_p$.*

PROOF. Note that, according to Theorem 3.4(5), we have

$$N_\varphi = \bigcap_{\omega \in \mathcal{M}_*, \omega \leq \varphi} N_\omega.$$

Since all N_ω's are σ-weakly closed (see Lemma 1.62), N_φ is also σ-weakly
closed. By Proposition 3.3(2) and Proposition 1.61(5), there is a projection
$p \in \mathcal{M}$ such that $N_\varphi = \mathcal{M}p$. If $\varphi(a) = 0$ for some $0 \neq a \in p^\perp \mathcal{M}_+ p^\perp$, then
$a^{1/2} \in N_\varphi$, so that $a \in p\mathcal{M}p$ — a contradiction. We leave the proof of
uniqueness as an exercise. $\qquad\square$

DEFINITION 3.8. For a normal weight φ, the projection p from the
proposition above is called the *null projection* of φ and is denoted by $e_0(\varphi)$.
The orthogonal complement of $e_0(\varphi)$ is called the *support projection* of φ
and is denoted by $\operatorname{supp} \varphi$.

It is clear that φ is faithful if and only if $e_0(\varphi) = 0$.

The definition of semifiniteness of a weight might not seem the most natural at first. Below we introduce versions of the notion that correspond to various conditions in Proposition 2.13.

DEFINITION 3.9. A normal weight φ on $\mathcal{M}$ is called:

(1) *semifinite* if $\mathfrak{p}_\varphi$ generates $\mathcal{M}$ as a von Neumann algebra.

(2) *orthogonally semifinite* if there is an orthogonal family $\{e_i\}_{i \in I}$ of projections from $\mathcal{M}$ such that $\sum_{i \in I} e_i = \mathbb{1}$ and for all i, $\varphi(e_i) < \infty$.

(3) *strictly semifinite* if $\varphi = \sum_{i \in I} \varphi_i$ for a family of functionals from $\mathcal{M}_*^+$ with orthogonal supports.

(4) *strongly semifinite* if for every non-zero projection e from $\mathcal{M}$ there is a non-zero subprojection f of e such that $\varphi(f) < \infty$; in other words, if the set of projections of finite weight is order-dense in the set of all projections in $\mathcal{M}$.

Strictly semifinite weights were introduced by Combes in [**Com71**]. Strongly semifinite weights were defined by Trunov [**Tru78**], who called them *locally finite*, and Gardner [**Gar79**], more or less at the same time. They were called *densely semifinite* in [**GP15**]. We decided to use the name *strongly semifinite*, since the paper of Trunov is in Russian and not easily accessible. The name *orthogonally semifinite* was coined in [**GP15**], but main results on such weights had been obtained earlier in [**STS02**] and [**HKZ91**]. This notion may in fact become redundant — it follows from the paper [**HKZ91**] of Halpern, Kaftal and Zsidó that for σ-finite algebras there is no difference between semifinite and orthogonally semifinite.

This material will not be used in future chapters. Nevertheless, the reader may profit from a thorough understanding of those special families of weights. The importance of the notion of locally measurable operators was mentioned at the beginning of Section 2.3. In Section 3.3 we shall see the importance of Radon-Nikodym-type theorems. In particular, we are going to show that each normal weight φ on a semifinite algebra $\mathcal{M}$ with a f.n.s. trace τ possesses a *density* with respect to the trace (see Theorems 3.23 and 3.24). The properties of the weights correspond to appropriate properties of the densities. Theorem 3.30 states that the property of a weight which corresponds to the local measurability of its density is exactly strong semifiniteness.

LEMMA 3.10. *A weight is strongly semifinite if and only if it is orthogonally semifinite on each reduced von Neumann algebra $\mathcal{M}_e$ with $e \in \mathbb{P}(\mathcal{M})$.*

PROOF. "$\Rightarrow$" Let $\{e_i\}$ be a maximal family of mutually orthogonal non-zero subprojections of e of finite weight (use Zorn's lemma and strong semifiniteness to show its existence). If $\sum e_i < e$, then we could enlarge the family by adding to it a non-zero projection subprojection of $e - \sum e_i$. Hence $\sum e_i = e$ and $\varphi \upharpoonright \mathcal{M}_e$ is orthogonally semifinite.

"$\Leftarrow$" Let $0 \neq e \in \mathbb{P}(\mathcal{M})$. If φ is orthogonally semifinite on $\mathcal{M}_e$, then $e = \sum e_i$ with $e_i \in \mathbb{P}(\mathcal{M})$, $\varphi(e_i) < \infty$, and at least one of the e_i's must be non-zero. $\qquad\square$

THEOREM 3.11.(1) *Every finite weight is both strongly and strictly semifinite.*

(2) *Every strongly semifinite or strictly semifinite weight is orthogonally semifinite.*

(3) *Every orthogonally semifinite weight is semifinite.*

PROOF. (1) is obvious by definition.

(2) If φ is strictly semifinite, then $\varphi = \sum \varphi_i$ with $\varphi_i \in \mathcal{M}_*^+$. It is enough to take $e_i := \operatorname{supp} \varphi_i$. If φ is strongly semifinite, then it is orthogonally semifinite by Lemma 3.10.

(3) Assume $\{e_i\}_{i \in I}$ is the family from the definition of orthogonal semifiniteness. For $J \subseteq I$, J finite, put $f_J = \sum_{i \in J} e_i$. We are going to show that for any $a \in \mathcal{M}$, $f_J a f_J \in \mathfrak{m}_\varphi$ and $f_J a f_J \to a$ strongly, which shows that φ is semifinite. In fact, for any $i, j \in I$ we have $e_i, e_j \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and $\mathfrak{n}_\varphi$ is a left ideal, so that $e_i a e_j \in \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi \subseteq \mathfrak{m}_\varphi$. Now, $f_J \to \mathbb{1}$ strongly and, for any $\xi \in H$,

$$\|(f_J a f_J - a)\xi\| \leq \|f_J a (1 - f_J)\xi\| + \|(1 - f_J)a\xi\|$$
$$\leq \|a\|\|(1 - f_J)\xi\| + \|(1 - f_J)a\xi\| \to 0,$$

which ends the proof of (3). $\qquad\square$

Hence, for an arbitrary algebra and a normal weight we have

strictly semifinite

finite

strongly semifinite

orthogonally semifinite $\to$ semifinite
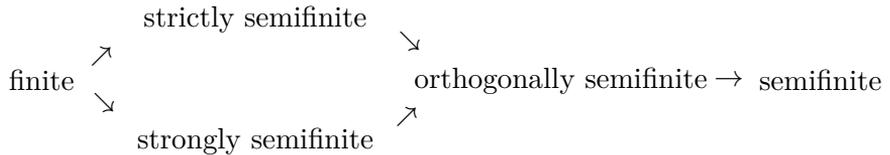
PROPOSITION 3.12. *Let φ be a faithful orthogonally semifinite weight on $\mathcal{M}$. Then there exists a countable family $\{e_n\}_{n \in \mathbb{N}}$ such that $e_n \in \mathbb{P}(\mathcal{M})$ and $\varphi(e_n) < \infty$ for each $n \in \mathbb{N}$, with $\sum_{n \in \mathbb{N}} e_n = \mathbb{1}$, if and only if the algebra $\mathcal{M}$ is σ-finite.*

PROOF. "$\Rightarrow$" Let $\{f_i\}_{i \in I}$ such that for all $i \in I$, $0 \neq f_i \in \mathbb{P}(\mathcal{M})$, $\sum_{i \in I} f_i = \mathbb{1}$ and $\#I > \aleph_0$. Suppose that there is a countable family $\{e_n\}$ of mutually orthogonal projections from $\mathcal{M}$ such that $\varphi(e_n) < \infty$ for all n and $\sum e_n = \mathbb{1}$. As in the proof of part (3) "$\Rightarrow$" of Theorem 3.11, $e_n f_i e_n \in \mathfrak{m}_\varphi$ for any i, n. By normality, $\sum_{i \in I} \varphi(e_n f_i e_n) = \varphi(e_n (\sum_{i \in I} f_i) e_n = \varphi(e_n) < \infty$, so that for each n there is a set I_n such that $\#I \setminus I_n \leq \aleph_0$ and $\varphi(e_n f_i e_n) = 0$ for $i \in I_n$. Thus the intersection of I_n's is non-empty and there is $i_0 \in I$ such that $\varphi(e_n f_{i_0} e_n) = 0$ for all n. We have $|\varphi(e_n f_{i_0} e_m)| \leq \varphi(e_n f_{i_0} e_n)^{1/2} \varphi(e_m)^{1/2} = 0$ for all n, m. Hence $\varphi(f_{i_0}) = \sum_{n,m} \varphi(e_n f_{i_0} e_m) = 0$, which contradicts the faithfulness of φ.

"$\Leftarrow$" is obvious from the definitions of σ-finiteness of the algebra. $\square$

REMARK 3.13. For a normal trace $\varphi = \tau$ on $\mathcal{M}$ the four notions of semifiniteness from Definition 3.9 are equivalent. This is exactly the content of Proposition 2.13.

PROPOSITION 3.14. *If φ is a normal weight on $\mathcal{M}$, then there exist a unique projection $q \in \mathcal{M}$ such that the weight $q\varphi q := \varphi(q \cdot q)$ is semifinite with $\varphi(a) = \infty$ for every $a \in \mathcal{M}_+$ with $(\mathbb{1} - q)a(\mathbb{1} - q) \neq 0$.*

PROOF. By Proposition 3.3(2) and Proposition 1.61(5), there is a projection $q \in \mathcal{M}$ such that $\overline{\mathfrak{n}_\varphi}^{\sigma-\mathrm{w}} = \mathcal{M}q$. The same argument as was used in the first part of the proof of Proposition 2.16, can now be used to show that $\overline{\mathrm{span}(\mathfrak{p}_\tau)} = q\mathcal{M}q$. On setting $q.\varphi.q = \varphi(q \cdot q)$, it is now an exercise to see that $\mathfrak{n}_{q\varphi q} = \mathfrak{n}_\varphi \oplus \mathcal{M}(\mathbb{1} - q)$ and hence that $\overline{\mathfrak{n}_{q\varphi q}}^{\sigma-\mathrm{w}} = \mathcal{M}$. This in turn ensures that $\overline{\mathrm{span}(\mathfrak{p}_{q\varphi q})} = \overline{\mathrm{span}(\mathfrak{n}_{q\varphi q}^* \mathfrak{n}_{q\varphi q})} = \mathcal{M}$, and hence that $q\varphi q$ is semifinite.

If on the other hand $(\mathbb{1} - q)a(\mathbb{1} - q) \neq 0$ for some $a \in \mathcal{M}_+$, then $a^{1/2}$ clearly does not belong to $\overline{\mathfrak{n}_\varphi}^{\sigma-\mathrm{w}} = \mathcal{M}q$, which ensures that $\varphi(a) = \infty$ as required. We leave the proof of uniqueness as an exercise. $\square$

DEFINITION 3.15. For a normal weight φ, the orthogonal projection q from the above proposition above is called the *semifinite projection* and is denoted by $e_\infty(\varphi)$.

It is clear that φ is semifinite if and only if $e_\infty(\varphi) = \mathbb{1}$. Obviously, $e_0(\varphi) \leq e_\infty(\varphi)$.

NOTATION 3.16. For a weight on $\mathcal{M}$ *f.n.s.* abbreviates faithful normal semifinite.

DEFINITION 3.17. *Representation (π_φ, H_φ) induced* by φ is defined as follows: $H_\varphi = (\mathfrak{n}_\varphi/N_\varphi)^\sim$, where the completion is with respect to the scalar product given by $\langle \eta_\varphi(a), \eta_\varphi(b) \rangle_\varphi = \varphi(b^*a)$, where $\eta_\varphi : \mathfrak{n}_\varphi \to \mathfrak{n}_\varphi/N_\varphi$ is the quotient map, and $\pi_\varphi \colon \mathcal{M} \to B(H_\varphi)$ is given by $\pi_\varphi(a)\eta_\varphi(b) = \eta_\varphi(ab)$ for $a \in \mathcal{M}, b \in \mathfrak{n}_\varphi$.

Note that the representation π_φ is normal if φ is normal, faithful if π_φ is faithful and non-degenerate if π_φ is semifinite.

PROPOSITION 3.18. *If φ is f.n.s., then $\pi_\varphi \colon \mathcal{M} \to B(H_\varphi)$ is a $*$-isomorphism of $\mathcal{M}$ onto $\pi_\varphi(M)$.*

3.2 Extensions of weights and traces

THEOREM 3.19. *Any normal weight φ on $\mathcal{M}$ has a unique extension to $\widehat{\mathcal{M}}_+$ (denoted also by φ) such that*

(1) $\varphi(\lambda m) = \lambda\varphi(m)$;
(2) $\varphi(m + n) = \varphi(m) + \varphi(n)$;
(3) *if $m_i \nearrow m$, then $\varphi(m_i) \nearrow \varphi(m)$.*

PROOF. Take any normal weight φ on $\mathcal{M}$ and any $m \in \widehat{\mathcal{M}}_+$. Then by Theorems 3.4 and 1.133, $\varphi = \sum_{i \in I} \omega_i$ with $\omega_i \in \mathcal{M}_*^+$, and $m = \int_0^\infty \lambda de_\lambda + \infty \cdot p$, so that

$$\varphi(m) := \lim_{n \to \infty} \sum_{i \in I} \omega_i(x_n) = \sum_{i \in I} \omega_i(m) \text{ for } x_n = \int_0^n \lambda de_\lambda + np.$$

(1), (2), (3) are easily checked; if φ' is another extension of φ to $\widehat{\mathcal{M}}_+$, then

$$\varphi'(m) = \sum_{i \in I} \omega_i(m) = \varphi(m).$$

$\square$

DEFINITION 3.20. Let $\mathcal{M}$ be semifinite and let τ be a f.n.s. trace on $\mathcal{M}$. For $a, b \in \mathcal{M}_+$,

$$a \cdot b := a^{1/2}ba^{1/2}.$$

THEOREM 3.21. *The map $(a, b) \mapsto \tau(a \cdot b)$ has a unique extension $(m, n) \mapsto \tau(m \,\hat{\cdot}\, n)$ to $\widehat{\mathcal{M}}_+ \times \widehat{\mathcal{M}}_+$ such that*

(1) $\tau(m \,\hat{\cdot}\, n) = \tau(n \,\hat{\cdot}\, m)$ *for $m, n \in \widehat{\mathcal{M}}_+$;*

(2) τ is homogeneous and additive with respect to both m and n;

(3) if $m_\alpha \nearrow m$ and $n_\beta \nearrow n$, then $\tau(m_\alpha \hat{\cdot} n_\beta) \nearrow \tau(m \hat{\cdot} n)$.
This extension satisfies

(4) $\tau((ama^*) \hat{\cdot} n) = \tau(m \hat{\cdot} (a^*na))$ for $m, n \in \widehat{\mathcal{M}}_+, a \in \mathcal{M}$.

PROOF. For $m \in \widehat{\mathcal{M}}_+$ as in Theorem 1.133, put $m_n := \int_0^n \lambda de_\lambda + np$. For $m, n \in \widehat{\mathcal{M}}_+$, put $\tau(m \hat{\cdot} n) := \sup_{n,k} \tau(m_n \cdot n_k)$. Evidently, τ satisfies (1). For $m \in \widehat{\mathcal{M}}_+$, put $\tau_m(a) := \tau(m \cdot a)$ for $a \in \mathcal{M}_+$. Since $\tau_m(a) = \sup_n \tau(m_n^{1/2} a m_n^{1/2})$, it follows that τ_m is a normal weight. By Theorem 3.19, the extension of τ_m to $\widehat{\mathcal{M}}_+$ is given by

$$\tau_m(a) = \sup_k \tau_m(a_k) = \sup_{n,k} \tau(m_n \cdot a_k) = \tau(m \cdot a).$$

By Theorem 3.19, $n \mapsto \tau(m \cdot n)$ is homogeneous, additive and normal. As $\tau(m \cdot n) = \tau(n \cdot m)$, (2) and (3) are satisfied. Uniqueness follows from the uniqueness in Theorem 3.19. If $m, n \in L^2(\mathcal{M}, \tau)$, then $\tau((ama^*) \cdot n) = \tau(m \cdot (a^*na))$. Any $m \in \mathcal{M}_+$ is the limit of an increasing sequence from $L^2(\mathcal{M}, \tau)$. (To see this let $e_n \in \mathcal{M}$ be such that $e_n \nearrow \mathbb{1}$ and $\tau(e_n) < \infty$, and put $m_n = m^{1/2}e_n m^{1/2} \nearrow m$. Then $\tau(m_n^2) = \tau(e_n m e_n m e_n) \le \|m\|^2 \tau(e_n) < \infty$.) But this implies the validity of the formula for $m, n \in \mathcal{M}_+$. By 1.137 we have (4). $\square$

PROPOSITION 3.22. If $a, b \in LS(\mathcal{M})_+$, then

$$\tau(m_a \hat{\cdot} m_b) = \tau(a^{1/2} \bar{\cdot} b \bar{\cdot} a^{1/2}) = \tau(b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2}).$$

PROOF. Let $u \in \mathcal{M}$ be such that $u^*u = s_r(a^{1/2} \bar{\cdot} b^{1/2})$. Using the Spectral Theorem 1.113, we can form an increasing sequence (d_n) from $\mathcal{M}_+$ such that $d_n \nearrow b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2}$ and that $s(d_n) \le s_r(b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2}) = s_r(|a^{1/2} \bar{\cdot} b^{1/2}|^2) = s_r(|a^{1/2} \bar{\cdot} b^{1/2}|) = s_r(a^{1/2} \bar{\cdot} b^{1/2})$. Now $ud_nu^* \nearrow u(b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2})u^*$ by Proposition 2.63. We have, by the last statement of Theorem 1.117,

$$\tau(a^{1/2} \bar{\cdot} b \bar{\cdot} a^{1/2}) = \tau((a^{1/2} \bar{\cdot} b^{1/2})(a^{1/2} \bar{\cdot} b^{1/2})^*)$$
$$= \tau(u(a^{1/2} \bar{\cdot} b^{1/2})^*(a^{1/2} \bar{\cdot} b^{1/2})u^*)$$
$$= \tau(u(b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2})u^*) = \lim_{n \to \infty} \tau(ud_nu^*)$$
$$= \lim_{n \to \infty} \tau(d_n^{1/2}u^*ud_n^{1/2}) = \lim_{n \to \infty} \tau(d_n)$$
$$= \tau(b^{1/2} \bar{\cdot} a \bar{\cdot} b^{1/2}).$$

Next, let (a_n) and (b_m) be increasing sequences from $\mathcal{M}_+$ such that $a_n \nearrow a$ and $b_m \nearrow b$. Then, by the first part of the proof and Proposition 2.63,

$$\begin{aligned}
\tau(m_a \hat{\bullet} m_b) &= \lim_{m,n\to\infty} \tau(a_n \bullet b_m)\\
&= \lim_{m\to\infty} \lim_{n\to\infty} \tau(b_m^{1/2} a_n b_m^{1/2})\\
&= \lim_{m\to\infty} \tau(b_m^{1/2} a b_m^{1/2})\\
&= \lim_{m\to\infty} \tau(a^{1/2} \bar{\bullet} b_m \bar{\bullet} a^{1/2})\\
&= \tau(a^{1/2} \bar{\bullet} b \bar{\bullet} a^{1/2}),
\end{aligned}$$

which ends the proof. $\square$

From now on, we shall write $\tau(x \bullet y)$ instead of $\tau(x \hat{\bullet} y)$ for $x, y \in \widehat{\mathcal{M}}_+$, and $\tau(a \bullet b)$ instead of $\tau(b^{1/2} \bar{\bullet} a \bar{\bullet} b^{1/2})$ for $a, b \in LS(\mathcal{M})_+$. Note that now $\tau(x \bullet \mathbb{1}) = \tau(x)$, as expected.

3.3   Density of weights with respect to a trace

The next result is a version of the Radon-Nikodym theorem that is most useful in applications. It is an easy corollary from the already powerful Pedersen-Takesaki theorem [**PT73**] on the Radon-Nikodym theorem for two weights (further generalized by [**Vae01**]), a difficult theorem using the full apparatus of modular theory. We felt that the tracial version of the theorem should be proved in an "elementary" way.

THEOREM 3.23. *Let $\mathcal{M}$ be a semifinite algebra equipped with a f.n.s. trace τ. Then for any normal state ω there is a τ-measurable operator g_ω such that $\omega(a) = \tau(g_\omega \bullet a)$ for all $a \in \mathcal{M}_+$.*

PROOF. We claim that the space of functionals $\{\tau(f\cdot) : f \in \mathfrak{m}_\tau\}$ is norm dense in $\mathcal{M}_*$. To see this suppose that we are given $a \in \mathcal{M}$ such that $\tau(fa) = 0$ for all $f \in \mathfrak{m}_\tau$. So for any f of the form $f = xx^*u^*$ where u is the partial isometry in the polar decomposition $a = u|a|$ and $x \in \mathfrak{n}_\tau$, we have that $0 = \tau(fa) = \tau(x^*|a|x)$. In other words $x^*|a|x = 0$ for all $x \in \mathfrak{n}_\tau$, which can only be the case if $a = 0$. This is enough to ensure the norm density of $\{\tau(f\cdot) : f \in \mathfrak{m}_\tau\}$ in $\mathcal{M}_*$.

We may therefore select $(g_n) \subset \mathfrak{m}_\tau$ such that $(\tau(g_n\cdot))$ converges to ω in norm. Since $(\tau(g_n^*\cdot))$ will then also converge to ω, we may replace each g_n with its real part if necessary, and assume that $g_n = g_n^*$ for each n. Now let $e_n = \chi_{[0,\|g_n\|]}(g_n)$ for every n. By passing to a subsequence if necessary

we may assume that (e_n) converges σ-weakly to some e_0. Since for every $x \in \mathcal{M}$ we have that

$$|\tau(g_n e_n x) - \omega(e_0 x)| \leq |\tau(g_n e_n x) - \omega(e_n x)| + |\omega(e_n x) - \omega(e_0 x)| \quad (3.1)$$
$$\leq \|\tau(g_n \cdot) - \omega\| \cdot \|x\| + |\omega(e_n x) - \omega(e_0 x)|,$$

it clearly follows that $(\tau(g_n e_n \cdot))$ converges weakly to $\omega(e_0 \cdot)$ in $\mathcal{M}_*$. But that ensures that $\tau(g_n(\mathbb{1} - e_n)) \to \omega(\mathbb{1} - e_0)$. Since by construction $\tau(g_n(\mathbb{1} - e_n)) \leq 0$ for every n with $\omega(\mathbb{1} - e_0) \geq 0$, this can only be the case if in fact $|\tau(g_n(\mathbb{1} - e_n))| \to 0$. But by construction each $a \mapsto \tau(g_n(e_n - \mathbb{1})a)$ is a positive functional with norm $\tau(g_n(e_n - \mathbb{1})) = |\tau(g_n(\mathbb{1} - e_n))|$. So in fact $(\tau(g_n(\mathbb{1} - e_n) \cdot))$ converges to 0 in norm in $\mathcal{M}_*$, ensuring that $(\tau(g_n e_n \cdot))$ actually converges in norm to ω. We have therefore shown that we may assume that $(g_n) \subset \mathfrak{m}_\tau^+$. It is a fairly straightforward exercise to see that $\tau(|g_n - g_m|) \leq \|\tau((g_n - g_m) \cdot)\|$ for each pair (m, n). Given $\epsilon > 0$ and selecting $N \in \mathbb{N}$ so that $\|\tau((g_n - g_m) \cdot)\| \leq \epsilon^2$ for every $n, m \geq N$, we may now use the Borel functional calculus to see that

$$\tau(\chi_{(\epsilon, \infty)}(|g_n - g_m|)) \leq \epsilon^{-1} \tau(|g_n - g_m|) \leq \epsilon^{-1} \|\tau((g_n - g_m) \cdot)\| \leq \epsilon$$

for every $n, m \geq N$. In other words $(g_n - g_m) \in \mathcal{N}(\epsilon, \epsilon)$ for all $n, m \geq N$. So (g_n) is Cauchy in $\widetilde{\mathcal{M}}$, and must therefore converge in measure to some $g \in \widetilde{\mathcal{M}}_+$.

Now let $m \in \mathbb{N}$ be given and let $p_m = \chi_{[0,m]}(g)$. We clearly have $g_n p_m \to g p_m$ in measure, and hence given $\epsilon > 0$, there exists some $N \in \mathbb{N}$ such that $(g p_m - g_n p_m) \in \mathcal{N}(\epsilon, \epsilon)$ for all $n \geq N$. This in turn ensures that we can find projections $(q_n)_{n \in \mathbb{N}}$ so that $\|(g p_m - g_n p_m) q_n\| \leq \epsilon$ and $\tau(\mathbb{1} - q_n) \leq \epsilon$ for all $n \geq N$. For any $x \in \mathfrak{m}_\tau$ we then have

$$|\tau((g p_m - g_n p_m) q_n x)| \leq \|(g p_m - g_n p_m) q_n\| \tau(|x|) \leq \epsilon \tau(|x|) \text{ for all } n \geq N.$$

Thus $\lim_{n \to \infty} \tau((g p_m - g_n p_m) q_n x) = 0$. However, we also have

$$|\tau(g p_m(\mathbb{1} - q_n) x))| = |\tau(x p_m g(\mathbb{1} - q_n)))| \leq \|x\| \tau(g(\mathbb{1} - q_n)) \to 0$$

as $n \to \infty$, which ensures that $\lim_{n \to \infty} \tau(g_n p_m q_n x) = \tau(g p_m x)$. By passing to a subsequence if necessary we may assume that (q_n) converges σ-weakly to some q_0 as $n \to \infty$. For any $y \in \mathfrak{n}_\tau$ we then have that $\tau(y^*(\mathbb{1} - q_n)y) \to \tau(y^*(\mathbb{1} - q_0)y)$ as $m \to \infty$ as $n \to \infty$. But we also have that $|\tau(y^*(\mathbb{1} - q_n)y)| \leq \|y\|^2 \tau(\mathbb{1} - q_n)$ and hence that $\tau(y^*(\mathbb{1} - q_n)y) \to 0$ as $n \to \infty$. The faithfulness of τ now guarantees that $y^*(\mathbb{1} - q_0)y = 0$ for all $y \in \mathfrak{n}_\tau$, and hence that $q_0 = \mathbb{1}$. It is clear that $\tau(g_n p_m \cdot) \to \omega(p_m \cdot)$ in norm. So on arguing as in equation (3.1), it follows that $\tau(g_n p_m q_n \cdot) \to \omega(p_m \cdot)$ weakly

in $\mathcal{M}_*$. Thus for $x \in \mathfrak{m}_\tau$ we have as before that $\lim_{n\to\infty} \tau(g_n p_m q_n x) \to \omega(e_m x)$, and hence that $\tau(g p_m x) = \omega(p_m x)$. That ensures that for any $x \in \mathfrak{m}_\tau^+$ we have that $\tau(g.x) = \sup_{m\geq 1} \tau((g p_m).x) = \lim_{m\geq 1} \omega(g p_m x) = \omega(x)$, which is enough to ensure that $g = g_\omega$. $\qquad\square$

THEOREM 3.24. *Let $\mathcal{M}$ be semifinite and let τ be a f.n.s. trace on $\mathcal{M}$.*

(1) *The map $m \mapsto \boldsymbol{\tau}_m$, where $\boldsymbol{\tau}_m(a) := \tau(m \cdot a)$ for $a \in \mathcal{M}_+$, is a homogeneous and additive bijection of $\widehat{\mathcal{M}}_+$ onto the set of normal weights on $\mathcal{M}$. Moreover,*

$$\boldsymbol{\tau}_{ama^*} = a^* \boldsymbol{\tau}_m a$$
$$m \leq n \text{ iff } \boldsymbol{\tau}_m \leq \boldsymbol{\tau}_n$$
$$m_i \nearrow m \text{ iff } \boldsymbol{\tau}_{m_i} \nearrow \boldsymbol{\tau}_m.$$

(2) *Let $m = \int \lambda de_\lambda + \infty \cdot p \in \widehat{\mathcal{M}}_+$. Then $e_0(\boldsymbol{\tau}_m) = e_0$ and $e_\infty(\boldsymbol{\tau}_m) = p^\perp$, so that $\boldsymbol{\tau}_m$ is faithful iff $e_0 = 0$ and $\boldsymbol{\tau}_m$ is semifinite iff $p = 0$.*

PROOF. (1) All but bijectivity follows from Theorem 3.21. Surjectivity follows from Theorem 3.23: If $\omega \in \mathcal{M}_*^+$, then $\omega = \tau(m \cdot)$ for some $m \in \widehat{\mathcal{M}}_+$. Let φ be an arbitrary normal weight. Then by Theorem 3.4(4), $\varphi = \sum \omega_{i\in I}$ for some $\omega_i \in \mathcal{M}_*^+$. Now $\omega_i = \tau(m_i \cdot)$ so that $\varphi = \tau(m \cdot)$, where $m = \sum_{i\in I} m_i \in \widehat{\mathcal{M}}_+$. For injectivity, assume that $\boldsymbol{\tau}_m = \boldsymbol{\tau}_n$ for some $m, n \in \widehat{\mathcal{M}}_+$. Then $\boldsymbol{\tau}_m(a) = \boldsymbol{\tau}_n(a)$ for all $a \in \mathcal{M}_+$, so that $\tau_a(m) = \tau_a(n)$ for all $a \in \mathcal{M}_+$. By Theorem 3.23, $\omega(m) = \omega(n)$ for all $\omega \in \mathcal{M}_*^+$, which implies $m = n$. (Similarly, if $\boldsymbol{\tau}_m \leq \boldsymbol{\tau}_n$, then $m \leq n$; if $\boldsymbol{\tau}_{m_i} \nearrow \boldsymbol{\tau}_m$, then $m_i \nearrow m$).

(2) $e_0(\boldsymbol{\tau}_m) = e_0$: Put $m_n = \int_0^n \lambda de_\lambda + np$. Then $\boldsymbol{\tau}_{m_n} \nearrow \boldsymbol{\tau}_m$. For a projection $q \in \mathcal{M}$, we have $\boldsymbol{\tau}_m(q) = 0$ iff for all n one has $\boldsymbol{\tau}_{m_n}(q) = 0$ iff $\tau(q m_n q) = 0$ for all n iff $\mathfrak{s}(m_n) \perp q$ for all $n \in \mathbb{N}$ iff $q \leq e_0$. Hence $e_0(\boldsymbol{\tau}_m) = e_0$.

$e_\infty(\boldsymbol{\tau}_m) = p^\perp$: Define $e_n := e_{[0,n]}(m)$, and let $k_j \in \mathcal{M}$, $j \in J$ be such that $k_j \in \mathfrak{n}_\tau$ and $k_j \to \mathbb{1}$ σ-strongly. Then $k_j e_n \in \mathfrak{n}_{\boldsymbol{\tau}_m}$ for all $j \in J$, $n \in \mathbb{N}$, so that $e_n = \lim_{j\in J} k_j e_n$ belongs to the σ-strong closure of $\mathfrak{n}_{\boldsymbol{\tau}_m}$, which implies that $p^\perp$ is in the σ-strong closure of $\mathfrak{n}_{\boldsymbol{\tau}_m}$. Thus, there is a net $(d_i)_{i\in I}$ such that $d_i \in \mathfrak{n}_{\boldsymbol{\tau}_m}$ for all $i \in I$ and $d_i \to p^\perp$ σ-strongly. By Kaplansky's density theorem 1.60, we can choose the d_i's so that $\|d_i\| \leq 1$. For any $a \in \mathcal{M}$, $d_i a d_i \in \mathfrak{n}_{\boldsymbol{\tau}_m}$, so that $d_i a d_i \to p^\perp a p^\perp$ σ-strongly. Hence $p^\perp \leq e_\infty(\boldsymbol{\tau}_m)$.

Put now $q := e_\infty(\boldsymbol{\tau}_m) - p^\perp$. Again, there exists a net $(d_i)_{i\in I}$ from $\mathfrak{n}_{\boldsymbol{\tau}_m}$ such that $d_i \to q$ σ-strongly. Since $m \geq qmq$ (in $\widehat{\mathcal{M}}_+$), we have $\boldsymbol{\tau}_m(qd_i^* d_i q) = \tau(m \cdot qd_i^* d_i q) = \tau(qmq \cdot d_i^* d_i) \leq \tau(m \cdot d_i^* d_i) = \boldsymbol{\tau}_m(d_i^* d_i) < \infty$. On the other hand, if $q \neq 0$, then for sufficiently large $i \in I$ one has $d_i q \neq 0$, hence

$$\boldsymbol{\tau}_m(qd_i^* d_i q) = \tau(m \cdot qd_i^* d_i q) = \lim_{m\to\infty} \tau(m_m \cdot qd_i^* d_i q)$$

$$= \lim_{m\to\infty} \tau(qm_m q \cdot qd_i^* d_i q) = \lim_{m\to\infty} m\tau(qd_i^* d_i q) = \infty.$$

Thus $q = 0$ and $e_\infty(\boldsymbol{\tau}_m) = p^\perp$. $\qquad\square$

Fix a faithful normal weight φ on $\mathcal{M}$.

DEFINITION 3.25. If $m \in \widehat{\mathcal{M}}_+$ is such that $\varphi = \boldsymbol{\tau}_m$, then m is called the *Radon-Nikodym derivative of φ with respect to τ* and we write $d\varphi/d\tau := m$.

At this point we need to sound a warning. Theorem 3.24 is a theorem about elements of $\widehat{\mathcal{M}}_+$ which may without loss of generality be regarded as positive quadratic forms. Hence when claiming that $\boldsymbol{\tau}_{ama^*} = a^* \boldsymbol{\tau}_m a$ and that $\boldsymbol{\tau}_{m_1+m_2} = \boldsymbol{\tau}_{m_1} + \boldsymbol{\tau}_{m_2}$, the product ama^* and the sum $m_1 + m_2$ is a statement about quadratic forms not operators, and should therefore be interpreted in the sense of the definitions made at the start of section 1.6. It therefore behoves us to see how in the case where h, h_1 and h_2 are self-adjoint positive operators affiliated to $\mathcal{M}$, the operator versions of these statements compare to the quadratic form versions. For this we need the technology of form sums. We will here only summarise the essentials. Readers who wish to see a more complete treatment, are referred to [**Tar13**]. Each of h_1 and h_2 induce closed sesquilinear forms $\mathfrak{t}_i(\xi,\zeta) = \langle h_i^{1/2}\xi, h_i^{1/2}\zeta \rangle$ $(i = 1,2)$ on their respective domains, where ξ and ζ are vectors in the underlying Hilbert space. When speaking of representing such operators as elements of $\widehat{\mathcal{M}}_+$, it is more properly these sesquilinear forms that we have in mind. These forms have an obvious action on vector states $\rho_{\xi,\xi} : \mathcal{M} \to \mathbb{R} : a \mapsto \langle a\xi, \xi \rangle$ with such a $\rho_{\xi,\xi}$ being mapped to $\mathfrak{t}_i(\xi,\xi)$ if $\xi \in \text{dom}(h_i)$, and ∞ if not. This action may then in a natural way be extended to an action on the cone of positive normal functionals. The sum $\mathfrak{t}_1 + \mathfrak{t}_2$ is again a sesquilinear form, and in the particular case where $\text{dom}(h_1^{1/2}) \cap \text{dom}(h_2^{1/2})$ is dense,

there exists a unique densely defined positive self-adjoint operator g characterised by the requirements that $\operatorname{dom}(g) = \operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$, and that $\langle g\xi, \zeta \rangle = \langle h_1^{1/2}\xi, h_1^{1/2}\zeta \rangle + \langle h_2^{1/2}\xi, h_2^{1/2}\zeta \rangle$ for all $\xi \in \operatorname{dom}(g)$ and $\zeta \in \operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$. This operator g is called the form sum of h_1 and h_2, and will be denoted by $g = h_1 \,\hat{+}\, h_2$. The natural domain of $h_1 \,\hat{+}\, h_2$ is moreover precisely $\operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$ (see [**Tar13**, Theorem 3.5]). Thus in the case where the terms we add in Theorem 3.24 correspond to positive self-adjoint operators h_1 and h_2 with $\operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$ dense, the sum will correspond to the operator $h_1 \,\hat{+}\, h_2$. Even in the case where $\operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$ is not dense, it is still possible to give expression to $h_1 \,\hat{+}\, h_2$, but in that case $h_1 \,\hat{+}\, h_2$ turns out to be a linear relation (a multi-valued map), rather than an operator. Similar comments apply to the expression $a^* \boldsymbol{\tau}_h a$ in Theorem 3.24, where we actually have what may be called a *form product*. Given a bounded operator a and a closed positive definite sesquilinear form $\mathfrak{t}$, we may define an associated sesquilinear form $a.\mathfrak{t}.a^*$ by the formal prescription $a.\mathfrak{t}.a^*(\xi, \zeta) = \mathfrak{t}(a^*\xi, a^*\zeta)$. If then $\mathfrak{t}$ corresponds to the densely defined self-adjoint positive operator h, we may write $a\,\hat{\cdot}\,h\,\hat{\cdot}\,a^*$ for the unique, possibly not densely defined, self-adjoint positive operator corresponding to the closed positive definite sesquilinear form $a.\mathfrak{t}.a^*$. The relevance of this for Theorem 3.21, lies in the fact that if $a, h \in \mathcal{M}_+$, then $a^{1/2} \boldsymbol{\tau}_h a^{1/2}$ actually equals $\boldsymbol{\tau}_{a \bullet h}$.

PROPOSITION 3.26.(1) *If h_1 and h_2 are self-adjoint positive operators for which $\operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$ is dense in H and $h_1 + h_2$ is essentially self-adjoint, then the unique self-adjoint extension of $h_1 + h_2$ is precisely $h_1 \,\hat{+}\, h_2$.*

    (2) *If h is a self-adjoint positive operator and a a bounded operator such that $\operatorname{dom}(h^{1/2}a^*)$ is dense and aha^* essentially self-adjoint, then the unique self-adjoint extension of aha^* is precisely $a\cdot h\cdot a^*$.*

PROOF. Let h_1, h_2 be self-adjoint positive operators affiliated to $\mathcal{M}$ for which $\operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$ is dense in H and $h_1 + h_2$ is essentially self-adjoint. Now let $\xi \in \operatorname{dom}(h_1) \cap \operatorname{dom}(h_2)$ and $\zeta \in \operatorname{dom}(h_1 \,\hat{+}\, h_2) = \operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2})$. By the Borel functional calculus we will then also have that $\xi \in \operatorname{dom}(h_1^{1/2}) \cap \operatorname{dom}(h_2^{1/2}) = \operatorname{dom}(h_1 \,\hat{+}\, h_2)$. It then clearly

follows that

$$
\begin{aligned}
\langle (h_1 + h_2)\xi, \zeta \rangle &= \langle h_1 \xi, \zeta \rangle + \langle h_2 \xi, \zeta \rangle \\
&= \langle h_1^{1/2}\xi, h_1^{1/2}\zeta \rangle + \langle h_2^{1/2}\xi, h_2^{1/2}\zeta \rangle \\
&= \langle (h_1 \,\hat{+}\, h_2)^{1/2}\xi, (h_1 \,\hat{+}\, h_2)^{1/2}\zeta \rangle \\
&= \langle \xi, (h_1 \,\hat{+}\, h_2)\zeta \rangle.
\end{aligned}
$$

It follows that $h_1 + h_2 \subseteq (h_1 \,\hat{+}\, h_2)^* = h_1 \,\hat{+}\, h_2$. But since by assumption $h_1 + h_2$ is essentially self-adjoint, it must have a unique self-adjoint extension. This extension must then clearly be $h_1 \,\hat{+}\, h_2$.

Next let h be a self-adjoint positive operator and a a bounded operator for which $\mathrm{dom}(h^{1/2}a^*)$ is dense and aha^* essentially self-adjoint. Let h correspond to the sesquilinear form $\mathfrak{t}(\xi, \zeta) = \langle h^{1/2}\xi, h^{1/2}\zeta \rangle$. Then by definition $a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*$ will correspond to the sesquilinear form $a.\mathfrak{t}.a^*(\xi, \zeta) = \langle h^{1/2}a^*\xi, h^{1/2}a^*\zeta \rangle$. In other words $a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^* = |h^{1/2}a^*|^2$, in which case $\mathrm{dom}(a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*) = \mathrm{dom}(h^{1/2}a^*)$. Now let $\xi \in \mathrm{dom}(aha^*) = \mathrm{dom}(ha^*)$ and $\zeta \in \mathrm{dom}(a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*) = \mathrm{dom}(h^{1/2}a^*)$ be given. Observe that we will then also have that $\xi \in \mathrm{dom}(h^{1/2}a^*)$. Consequently

$$
\langle aha^*\xi, \zeta \rangle = \langle ha^*\xi, a^*\zeta \rangle = \langle h^{1/2}a^*\xi, h^{1/2}a^*\zeta \rangle = \langle \xi, |h^{1/2}a^*|^2\zeta \rangle.
$$

This proves that $aha^* \subseteq (|h^{1/2}a^*|^2)^* = (a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*)^* = a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*$. Thus $a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*$ is a self-adjoint extension of aha^*. But since aha^* is essentially self-adjoint, that extension must be unique, and must therefore agree with $a \,\hat{\cdot}\, h \,\hat{\cdot}\, a^*$. $\qquad\square$

The following proposition is useful when comparing different traces on the same von Neumann algebra.

PROPOSITION 3.27. *Let τ be an f.n.s. trace on a semifinite von Neumann algebra $\mathcal{M}$. Given any other semifinite normal trace τ' on $\mathcal{M}$, we have that*

(1) *$\tau + \tau'$ is an f.n.s. trace;*
(2) *there exists a central element $0 \le a \le \mathbb{1}$ such that $\tau(x) = (\tau + \tau')(a \cdot x)$ and $\tau'(x) = (\tau + \tau')((\mathbb{1} - a) \cdot x)$ for all $x \in \mathcal{M}_+$, with $\mathfrak{s}(a) = \mathbb{1}$ and $\mathfrak{s}(\mathbb{1} - a) = \mathrm{supp}\,\tau'$.*

PROOF. The functional $\tau_0 := \tau + \tau'$ is clearly a faithful normal trace. Now observe that $\mathfrak{m}_\tau \cdot \mathfrak{m}_{\tau'} \subseteq \mathfrak{m}_\tau \cap \mathfrak{m}_{\tau'} \subseteq \mathfrak{m}_{\tau_0}$ with $\mathfrak{m}_\tau \cdot \mathfrak{m}_{\tau'}$ σ-weakly dense in $\mathcal{M}$. Thus $\mathfrak{m}_{\tau_0}$ is σ-weakly dense, hence τ_0 is semifinite. Now set $a = \frac{d\tau}{d\tau_0} \in$

$\widehat{\mathcal{M}}_+$ (see Definition 3.25). Since $\tau \leq \tau_0$, the Radon-Nikodym Theorem for f.n.s. traces (see Theorem 3.24) yields $0 \leq a \leq \mathbb{1}$. The inequality $\tau \leq \tau_0$ implies that $\mathfrak{n}_{\tau_0} \subseteq \mathfrak{n}_\tau$. Using this fact a suitable polarisation identity now shows that for all $x, y \in \mathfrak{n}_{\tau_0}$ and any $u \in \mathcal{M}$, we have that

$$\langle \pi_{\tau_0}(u)\pi_{\tau_0}(a)\eta(x), \eta(y) \rangle_{\tau_0} = \langle \pi_{\tau_0}(a)\eta(x), \eta(u^*y) \rangle_{\tau_0}$$
$$= \tau(y^*ux) = \langle \pi_{\tau_0}(a)\eta(ux), \eta(y) \rangle_{\tau_0} = \langle \pi_{\tau_0}(au)\eta(x), \eta(x) \rangle_{\tau_0}.$$

This clearly ensures that $\pi_{\tau_0}(a) \in \pi_{\tau_0}(\mathcal{M})'$ and hence that $a \in \mathcal{M}'$, since the representation π_{τ_0} is faithful.

We have thus far proven that

$$\tau_0(ay^*x) = \tau(y^*x) \text{ for every } x, y \in \mathfrak{n}_{\tau_0}. \tag{3.2}$$

For any $b \in \mathcal{M}_+$ we may select a net $(b_i)_{i \in I}$ in $\mathfrak{n}_{\tau_0}$ increasing to b. When combined with what we have just noted, the normality of the traces involved now shows that $\tau(b) = \sup_{i \in I} \tau(b_i) = \sup_{i \in I} \tau_0(a \cdot b_i) = \tau_0(a \cdot b)$. This proves that $\tau = (\tau + \tau')(a \cdot \cdot)$. The fact that $\mathfrak{s}(a) = \mathbb{1}$ is a trivial consequence of the statement regarding supports in the preceding theorem.

Note that it follows trivially from equation (3.2) that $\tau_0((\mathbb{1}-a) \cdot y^*x) = \tau'(y^*x)$ for every $x, y \in \mathfrak{n}_{\tau_0}$. Thus a similar proof to the one above shows that $\tau'(\cdot) = (\tau + \tau')((\mathbb{1} - a) \cdot \cdot)$. The element $\mathbb{1} - a$ clearly plays the role of a Radon-Nikodym derivative. The uniqueness of the Radon-Nikodym derivative therefore ensures that $\mathbb{1}-a = \frac{d\tau'}{d\tau_0}$. The claim about the supports therefore follows from the preceding theorem. $\qquad\square$

PROPOSITION 3.28. *Let τ be a f.n.s. trace on $\mathcal{M}$. For any normal trace τ' on $\mathcal{M}$ we have that $\frac{d\tau'}{d\tau} \in \widehat{\mathcal{Z}(\mathcal{M})}_+$.*

PROOF. First let τ' be semifinite and let $\tau_0 := \tau + \tau'$ and $a := \frac{d\tau_0}{d\tau}$. We claim that $a^{-1}(\mathbb{1} - a) = \frac{d\tau'}{d\tau}$, which by the fact that $a \in \mathcal{Z}(\mathcal{M})$ will then show that $\frac{d\tau'}{d\tau} \in \overline{\mathcal{Z}(\mathcal{M})}$ in this case (cf. Proposition 1.106(5)). To see this, notice that for any $b \in \mathcal{M}_+$ we have

$$\tau([a^{-1}(\mathbb{1} - a)] \cdot b) = \tau_0(a \cdot b^{1/2}[a^{-1}(\mathbb{1} - a)]b^{1/2}) = \tau_0((\mathbb{1} - a) \cdot b) = \tau'(b).$$

Now let τ' be an arbitrary normal trace. Then $e_\infty = e_\infty(\tau') \in \overline{\mathcal{Z}(\mathcal{M})}$. Since $e_\infty \tau' e_\infty$ is semifinite, we know from the first part of the proof that $\frac{d(e_\infty \tau' e_\infty)}{d\tau} \in \mathcal{Z}(\mathcal{M})$. For simplicity of notation we will write h_∞ for $\frac{d(e_\infty \tau' e_\infty)}{d\tau}$.

Suppose that $h_\infty = \int \lambda de_\lambda$. Since for any $b \in \mathcal{M}e_\infty^\perp$ we have that $\tau(h_\infty b) = (e_\infty \tau' e_\infty)(b) = 0$, it is clear that $\mathfrak{s}(h_\infty) \leq e_\infty$. Hence the

prescription $h_{\tau'} = \int \lambda de_\lambda + \infty \cdot (\mathbb{1} - e_\infty)$ yields a well defined element of $\widehat{\mathcal{Z}(\mathcal{M})}_+$. It is now an exercise to see that for any $b \in \mathcal{M}_+$ we have that $\tau(h_{\tau'} \cdot b) = \tau(h_\infty \cdot b) + \tau([\infty \cdot (\mathbb{1} - e_\infty)] \cdot b) = (e_\infty \tau' e_\infty)(b) + \tau([\infty \cdot (\mathbb{1} - e_\infty)] \cdot b) = \tau'(b)$. It is therefore clear that $\frac{d\tau'}{d\tau} = h_{\tau'}$, which proves the claim. $\qquad\square$

We finish the section with showing how the density of a weight with respect to a trace depends on the properties of the weight. Theorem 3.30 below has been obtained by Trunov [**Tru82**], using modular theory. Again, the proof presented here is elementary, and is based on the proof from [**GP15**].

LEMMA 3.29. *Let $\mathcal{M}$ be a semifinite properly infinite von Neumann algebra endowed with a f.n.s. trace τ, and let $h \in \overline{\mathcal{M}}_+$. Assume there exist: an increasing sequence of natural numbers $1 = n_1 < n_2 < \dots$ and $e \in \mathbb{P}(\mathcal{M})$ such that $\chi_{[n_k, n_{k+1})}(h) \succsim e$ for each k. Then the weight $\boldsymbol{\tau}_h$ is not strongly semifinite.*

PROOF. First, find projections $e_k \sim e$ such that

$$e_k \leq \chi_{[n_k, n_{k+1})}(h) \text{ for all } k \in \mathbb{N}.$$

Choose $u_k \in \mathcal{M}$ so that $u_k^* u_k = e_1, u_k u_k^* = e_k$. Let (α_k) be a sequence of positive reals such that $\sum_{k=1}^\infty \alpha_k^2 = 1$ and $\sum_{k=1}^\infty \alpha_k^2 n_k = \infty$; this is easy, using $n_k \geq k$. For any $\epsilon > 0$ there is an m_0 such that for each $m \geq m_0$

$$\| \sum_{k=m}^\infty \alpha_k u_k \|^2 = \| \sum_{k=m}^\infty \alpha_k^2 u_k^* u_k \| = \| \sum_{k=m}^\infty \alpha_k^2 e_1 \| \leq \epsilon.$$

Hence $v := \sum_{k=1}^\infty \alpha_k u_k$ exists (with convergence in norm topology) and belongs to $\mathcal{M}$. Put $p := vv^*$. Since $v^*v = e_1$, p is a non-zero projection. We will show that if $0 \neq q \leq p$, then $\varphi(q) = \infty$. Put $g := v^*qv$. Then $g \leq v^*v = e_1$ and $q = vgv^*$. From spectral theorem we easily get $h \geq$

$\sum_{m=1}^{\infty} n_m e_m$. We calculate:

$$\varphi(q) = \tau(h \cdot q) = \tau(h \cdot pqp) = \tau(h \cdot vv^*qvv^*)$$

$$= \tau(h \cdot vgv^*) = \tau(v^*hv \cdot g) = \tau((\sum_{j=1}^{\infty} \alpha_j u_j^*) h (\sum_{k=1}^{\infty} \alpha_k u_k^*) \cdot g)$$

$$\geq \tau((\sum_{j=1}^{\infty} \alpha_j u_j^*)(\sum_{m=1}^{\infty} n_m e_m)(\sum_{k=1}^{\infty} \alpha_k u_k^*) \cdot g)$$

$$\geq \tau((\sum_{j,k,m=1}^{\infty} \alpha_j \alpha_k n_m u_j^* e_m u_k) \cdot g) = \tau((\sum_{m=1}^{\infty} \alpha_m^2 n_m u_m^* e_m u_m) \cdot g)$$

$$= \sum_{m=1}^{\infty} \alpha_m^2 n_m \tau(e_1 \cdot g) = \sum_{m=1}^{\infty} \alpha_m^2 n_m \tau(g) = \infty.$$

$$\square$$

THEOREM 3.30. *Let $\mathcal{M}$ be a semifinite von Neumann algebra endowed with a f.n.s. trace τ. Assume φ is a f.n.s. weight on $\mathcal{M}$. Then φ is strongly semifinite iff $d\varphi/d\tau$ is locally measurable.*

PROOF. "$\Rightarrow$" Assume $h := d\varphi/d\tau$ is not locally measurable. We are going to show that $\boldsymbol{\tau}_h$ is not strongly semifinite. By assumption, there exists a (largest) projection $z \in \mathcal{Z}(\mathcal{M})$ such that for any projection $z' \in \mathcal{Z}(\mathcal{M})$, if $0 \neq z' \leq z$, then hz' is not measurable. We can assume that $z = \mathbb{1}$. In fact, if φ is strongly semifinite, then it is strongly semifinite on each $\mathcal{M}z$, with $z \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$.

By the structure theorem for semifinite von Neumann algebras (1.86, $\mathcal{M} = \sum_{\alpha}^{\oplus} \mathcal{M}z_{\alpha}$ with $z_{\alpha} \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$, $\sum_{\alpha} z_{\alpha} = \mathbb{1}$, where α runs over infinite cardinals $\leq \#\mathcal{M}$ (with some of the z_{α} possibly zero), and each $\mathcal{M}z_{\alpha} \simeq \mathcal{N}_{\alpha} \overline{\otimes} \mathcal{F}_{\alpha}$ with $\mathcal{N}_{\alpha}$ finite and $\mathcal{F}_{\alpha}$ a factor of type I_{α}. As above, we can assume that $\mathcal{M} = \mathcal{N} \overline{\otimes} \mathcal{F}$, where $\mathcal{N}$ is finite and $\mathcal{F}$ is an infinite-dimensional type I factor. Note that $\mathcal{Z}(\mathcal{M}) = \mathcal{Z}(\mathcal{N}) \otimes \mathbb{C}\mathbb{1}_{\mathcal{F}}$. Let $\tau_{\mathcal{N}}$ be a normalized trace on $\mathcal{N}$ (i.e. $\tau(\mathbb{1}_{\mathcal{N}}) = 1$) and Tr a normalized trace on $\mathcal{F}$ (i.e. the trace of any minimal projection from $\mathcal{F}$ is 1). Let f be a minimal projection from $\mathcal{F}$ and $e := \mathbb{1}_{\mathcal{N}} \otimes f$. It is clear that e is finite.

Let us fix $n \in \mathbb{N}$. Then, for each $m \geq n$, there is a largest central projection $w_m = u_m \otimes \mathbb{1}_{\mathcal{F}}$ such that

$$\chi_{[n,m)}(h) w_m \succsim e w_m.$$

As m increases, w_m has to increase to some w. Suppose $w \neq 1$. Then, for each $m \geq n$,

$$\chi_{[n,m)}(h)w^\perp \precsim ew^\perp.$$

By [**Tak02**, Lemma V.2.2],

$$\chi_{[n,\infty)}(h)w^\perp = \bigvee_{m=n}^{\infty} \chi_{[n,m)}(h)w^\perp \precsim ew^\perp,$$

so that $\chi_{[n,\infty)}(h)w^\perp$ is finite. This implies that $hw^\perp$ is measurable, which contradicts the first paragraph of the proof. Hence $w = 1$. That means that we can successively choose $1 = n_1 < n_2 < \ldots$ and projections $z_k = z_\mathcal{N}^{(k)} \otimes \mathbb{1}_\mathcal{F}$ in such a way that

$$\chi_{[n_k,n_{k+1})}(h)z_k \succsim ez_k \text{ with } \tau_\mathcal{N}(z_\mathcal{N}^{(k)}) \geq 1 - 1/2^{k+1}.$$

Now we put $z_\mathcal{N} = \bigwedge_{k=1}^{\infty} z_\mathcal{N}^{(k)}$ and $z = z_\mathcal{N} \otimes \mathbb{1}_\mathcal{F}$. We have $\tau_\mathcal{N}(z_\mathcal{N}) \geq 1/2$, whence $z \neq 0$. Moreover,

$$\chi_{[n_k,n_{k+1})}(h)z_k \succsim ez,$$

which implies

$$\chi_{[n_k,n_{k+1})}(h)z = (\chi_{[n_k,n_{k+1})}(h)z_k)z \succsim ez,$$

Now, as in the first paragraph of the proof, we can assume that $z = 1$ and then use Lemma 3.29 to show that φ is not strongly semifinite.

"$\Leftarrow$" Let h be locally measurable. Choose non-zero $p \in \mathbb{P}(\mathcal{M})$. Since τ is strongly semifinite (see Remark 3.13), there is a non-zero $q \in \mathbb{P}(\mathcal{M})$ with $q \leq p$ and $\tau(q) < \infty$. From the definition of local measurability of h we get the existence of a $z \in \mathbb{P}(\mathcal{Z}(\mathcal{M}))$ such that $hz \in S(\mathcal{M})$ and $qz \neq 0$. Put $e_n := \chi_{[0,n]}(hz)$. Note that $e_n^\perp$ is finite for sufficiently large n. If $qz \precsim e_n^\perp$ for each n, then applying the centre-valued trace $\mathcal{T}$ (on the reduced algebra $\mathcal{M}_{e_1^\perp}$) to both sides, we get $qz = 0$, a contradiction. Hence, for some n_0, $r := qz \wedge \chi_{[0,n_0]}(hz) \neq 0$ (see Lemma 2.32). Obviously, $r \leq q \leq p$. Moreover, using Theorem 3.21(4) and Proposition 3.22,

$$\varphi(r) = \tau(h \cdot (qz \wedge e_{n_0})) = \tau(hz \cdot (qz \wedge e_{n_0})) \leq n_0\tau(qz \wedge e_{n_0}) \leq n_0\tau(q) < \infty.$$

$\square$

PROPOSITION 3.31. *If $\mathcal{M}$ is finite, then any semifinite weight φ on $\mathcal{M}$ is both strictly and strongly semifinite.*

PROOF. Let τ be a normal faithful semifinite trace on $\mathcal{M}$. By Proposition 2.44, $d\varphi/d\tau \in \overline{\mathcal{M}} = LS(\mathcal{M})$, hence by Theorem 3.30, φ is strongly semifinite.

Assume now additionally that $\mathcal{M}$ is σ-finite, and choose τ to be finite. Put $e_n = \chi_{[n,n+1)}(h)$. Then $\sum_{n=0}^{\infty} e_n = \mathbb{1}$, and $\varphi(e_n) = \tau(he_n) \leq (n+1)\tau(e_n) < \infty$. Thus $e_n \varphi e_n$ are positive normal functionals on $\mathcal{M}$ with mutually orthogonal supports. We shall show that $\varphi = \sum e_n \varphi e_n$. First of all, for a fixed $a \in \mathcal{M}$, $e_n a e_n \leq \|a\| e_n$, so that $e_n a e_n \in \mathfrak{m}_\varphi^+$. Hence $a^{1/2} e_n \in \mathfrak{n}_\varphi$ for all n, so that $e_m a e_n \in \mathfrak{m}_\varphi$ for all m, n. If $m \neq n$,

$$\varphi(e_m a x e_n) = \tau(h^{1/2} e_m a e_n h^{1/2}) = \tau(a^{1/2} e_n h^{1/2} h^{1/2} e_m a^{1/2})$$
$$= \tau(a^{1/2} e_n h^{1/2} e_n e_m h^{1/2} e_m a^{1/2}) = 0.$$

Consequently, $\varphi(a) = \sum (e_n \varphi e_n)(a)$, which shows that φ is strictly semifinite.

Let finally $\mathcal{M}$ be an arbitrary finite von Neumann algebra. Then there exists a family $\{z_i\}$ of non-zero central projections in $\mathcal{M}$ such that $\sum z_i = \mathbb{1}$ and each of the reduced von Neumann algebras $\mathcal{M}_{z_i}$ is σ-finite. Note that $\mathfrak{p}_{\varphi \upharpoonright (\mathcal{M}_{z_i})_+} = \mathfrak{p}_\varphi z_i$. Hence, for any i, $\varphi \upharpoonright (\mathcal{M}_{z_i})_+$ is semifinite, thus also strictly semifinite. It follows directly from the definition of strict semifiniteness that φ is also strictly semifinite. $\qquad\square$

CHAPTER 4

A basic theory of decreasing rearrangements

In this chapter we develop a noncommutative theory of decreasing rearrangements. This will prove to be an indispensable tool for developing a theory of L^p and Orlicz spaces for not just tracial von Neumann algebras, but also general von Neumann algebras.

4.1 Distributions and reduction to subalgebras

We start by introducing the concept of decreasing rearrangements for τ-measurable operators.

DEFINITION 4.1. Given $f \in \widetilde{\mathcal{M}}$, we define the so-called decreasing rearrangement of f, to be the function $\mathbf{m}_f : [0, \infty) \to [0, \infty] : t \mapsto \mathbf{m}_f(t)$ given by $\mathbf{m}_f(t) = \inf\{\|fe\|_\infty : e \in \mathbb{P}(\mathcal{M}), \tau(\mathbb{1} - e) \le t\}$, and the distribution function of f by $[0, \infty) \to [0, \infty] : s \mapsto d_f(s)$ where $d_f(s) = \tau(\chi_{(s,\infty)}(|f|))$.

We note that what we call the decreasing rearrangement of $f \in \mathcal{M}$, is referred to as the *generalised singular value function* by many authors. However in order to emphasise the analogy of the theory developed with the classical theory of decreasing rearrangements and Banach function spaces, we prefer the term *decreasing rearrangement*. We collate the basic properties of distributions of elements of $\widetilde{\mathcal{M}}$, and then pause to illustrate an interesting application of this theory, before proceeding with the development of the theory of decreasing rearrangements. For this task we need the following lemma. A stronger version of this fact was first proved for bounded operators by Akemann, Anderson and Pedersen [**AAP82**]. However the validity of the weaker version below, extends to the class of τ-measurable operators. The proof we present is due to Kosaki.

LEMMA 4.2 ([**Kos84b**]). *Let $f, g \in \widetilde{\mathcal{M}}$ be given. Then*
- *there exists a partial isometry $w \in \mathcal{M}$ so that $\operatorname{Re}(f)_+ \le w|f|w^*$, where $\operatorname{Re}(f)_+$ is the positive part of $\operatorname{Re}(f)$*

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- *there exist partial isometries $u, v \in \mathcal{M}$ such that $|f+g| \leq u|f|u^* + v|g|v^*$.*

PROOF. We prove the first claim. Let $f = u|f|$ be the polar decomposition of f with e the support projection of $\mathrm{Re}(f)_+$. Now set $a = \frac{1}{2}e(\mathbb{1}+u)$, and let $a|f|^{1/2} = w|a|f|^{1/2}|$ be the polar decomposition of $a|f|^{1/2}$. Each of e, u, a and w of course belongs to $\mathcal{M}$, with $a^*a = \frac{1}{4}(\mathbb{1} + u^*)e(\mathbb{1} + u) \leq \mathbb{1}$. We note that in addition

$$a|f|a^* = w|a|f|^{1/2}|^2 w^* = w|f|^{1/2}a^*a|f|^{1/2}w^* \leq w|f|w^*.$$

Here we made use of the fact that if $g = u_g|g|$ is the polar decomposition of g, then $gg^* = u_g g^* g u_g^*$, from which it follows that $|g^*| = u_g|g|u_g^*$. In view of the fact that $\mathrm{Re}(f)_+ = e\mathrm{Re}(f)e = \frac{1}{2}e(|f|u^* + u|f|)e$, we therefore have that

$$
\begin{aligned}
w|f|w^* - \mathrm{Re}(f)_+ &\geq a|f|a^* - \frac{1}{2}e(|f|u^* + u|f|)e \\
&= \frac{1}{4}e(\mathbb{1} + u)|f|(\mathbb{1} + u^*)e - \frac{1}{2}e(|f|u^* + u|f|)e \\
&= \frac{1}{4}e[(\mathbb{1} + u)|f|(\mathbb{1} + u^*) - 2(|f|u^* + u|f|)]e \\
&= \frac{1}{4}e(\mathbb{1} - u)|f|(\mathbb{1} - u^*)e \\
&\geq 0.
\end{aligned}
$$

as required.

We pass to proving the second claim. Let $(f + g) = v|f + g|$ be the polar decomposition of $(f+g)$. Given that $|f+g| = v^*(f+g) = (f^*+g^*)v$, we have that

$$
\begin{aligned}
|f + g| &= \frac{1}{2}(v^*(f + g) + (f^* + g^*)v) \\
&= \mathrm{Re}(v^*f) + \mathrm{Re}(v^*g) \\
&\leq \mathrm{Re}(v^*f)_+ + \mathrm{Re}(v^*g)_+.
\end{aligned}
$$

By the first part we may now select partial isometries w_1 and w_2 in $\mathcal{M}$ so that $\mathrm{Re}(v^*f)_+ \leq w_1|v^*f|w_1^* \leq w_1|f|w_1^*$ and $\mathrm{Re}(v^*g)_+ \leq w_2|v^*g|w_2^* \leq w_2|g|w_2^*$. Therefore

$$|f + g| \leq \mathrm{Re}(v^*f)_+ + \mathrm{Re}(v^*g)_+ \leq w_1|f|w_1^* + w_2|g|w_2^*$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will shortly see that there is a close link between decreasing rearrangements and the distribution function. We therefore pause to investigate the properties of the distribution function.

PROPOSITION 4.3. *For any $f \in \widetilde{\mathcal{M}}$, the function $d : \mathbb{R} \to [0, \infty] : s \mapsto d_f(s)$ is non-increasing and right-continuous. This function satisfies the following properties:*

(i) *For any $f \in \widetilde{\mathcal{M}}$ and any non-zero scalar α, we have that $d_f(s) = d_{f^*}(s) = d_{|f|}(s)$ and that $d_{\alpha f}(s) = d_f(s/|\alpha|)$ for all $s \geq 0$*

(ii) *Given $a, b \in \widetilde{\mathcal{M}}$ with $a \geq b \geq 0$, we then have that $d_a(s) \geq d_b(s)$ for any $s \geq 0$. Moreover if $(a_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ is a net increasing to a in $\widetilde{\mathcal{M}}$, then $\sup_\alpha d_{a_\alpha}(s) = d_a(s)$ for any $s \geq 0$.*

(iii) *For any $t, s \geq 0$ and any $a, b \in \widetilde{\mathcal{M}}$, $d_{a+b}(t + s) \leq d_a(t) + d_b(s)$.*

(iv) *For any $f \in \widetilde{\mathcal{M}}$ and any $a, b \in \mathcal{M}$, $d_{afb} \leq d_{\|a\|\|b\|f}$.*

PROOF. Assume that $\mathcal{M}$ acts on the Hilbert space H. It is clear from the definition that $\mathbb{R} \to [0, \infty] : s \mapsto d_s(f)$ is non-increasing. Now let $s_n, s \in (0, \infty)$ be given such that $s_n \searrow s$. It then follows from the spectral theorem that $\chi_{(s_n, \infty)}(|f|) \nearrow \chi_{(s, \infty)}(|f|)$ strongly as $n \to \infty$. The normality of the trace τ, then ensures that $d_{s_n}(f) \nearrow d_s(f)$ as $n \to \infty$. Hence $\mathbb{R} \to [0, \infty] : s \mapsto d_s(f)$ is right-continuous.

Claim (i): The fact that $d_f = d_{|f|}$, is clear from the definition. The further fact that $\chi_{(s,\infty)}(|\alpha f|) = \chi_{(s/|\alpha|,\infty)}(|f|)$, and hence that $d_{\alpha f}(s) = d_f(s/|\alpha|)$ for any $s \geq 0$, follows from the Borel functional calculus.

Next let $f = u|f|$ be the polar decomposition of $|f|$. As noted in the previous proof, we then have that $|f^*| = u|f|u^*$. By the uniqueness of the spectral decomposition we must then have that $\chi_{(s,\infty)}(|f^*|) = u\chi_{(s,\infty)}(|f|)u^*$ for all $s > 0$. Consequently $d_{f^*}(s) = \tau(\chi_{(s,\infty)}(|f^*|)) = \tau(u\chi_{(s,\infty)}(|f|)u^*) = \tau(\chi_{(s,\infty)}(|f|)) = d_f(s)$ for all $s > 0$. This then proves the claim.

Claim (ii): First suppose that $a \geq b \geq 0$. If $\chi_{[0,s]}(a)(H) \cap \chi_{(s,\infty)}(b)(H)$ contained non-zero elements, then for any $\xi \in \chi_{[0,s]}(a)(H) \cap \chi_{(s,\infty)}(b)(H)$ with $\|\xi\| = 1$, it would follow that $\langle b\xi, \xi \rangle > s$, and that $\langle a\xi, \xi \rangle \leq s$. But this is impossible since the inequality $a \geq b$ demands that $\langle b\xi, \xi \rangle \leq \langle a\xi, \xi \rangle$. Thus we must have that $\chi_{[0,s]}(a)(H) \cap \chi_{(s,\infty)}(b)(H) = \{0\}$. Equivalently

$\chi_{[0,s]}(a) \wedge \chi_{(s,\infty)}(b) = 0$. This then enables us to conclude that

$$
\begin{aligned}
\chi_{(s,\infty)}(b) &= \chi_{(s,\infty)}(b) - \chi_{[0,s]}(a) \wedge \chi_{(s,\infty)}(b) \\
&\sim \chi_{[0,s]}(a) \vee \chi_{(s,\infty)}(b) - \chi_{[0,s]}(a) \\
&\leq \chi_{(s,\infty)}(a).
\end{aligned}
$$

Consequently $d_b(s) = \tau(\chi_{(s,\infty)}(b)) \leq \tau(\chi_{(s,\infty)}(a)) \leq d_a(s)$ for all $s \geq 0$.

Next suppose that $(a_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ is a net increasing to a in $\widetilde{\mathcal{M}}$. From what we have already proved, it is clear that $d_a(s) \geq \sup_\alpha d_{a_\alpha}(s)$.

To prove the converse inequality, we first show that

$$
\chi_{(s,\infty)}(a) \wedge (\wedge_\alpha \chi_{[0,s]}(a_\alpha)) = 0.
$$

If $\chi_{(s,\infty)}(a)(H) \cap (\cap_\alpha \chi_{[0,s]}(a_\alpha)(H))$ contained non-zero elements, then for any $\xi \in \chi_{(s,\infty)}(a)(H) \cap (\cap_\alpha \chi_{[0,s]}(a_\alpha)(H))$ with $\|\xi\| = 1$, it would follow that $\|a\xi\|^2 = \langle a^2\xi, \xi \rangle > s^2$, with in addition $\|a_\alpha\xi\|^2 = \langle a_\alpha^2\xi, \xi \rangle \leq s^2$. But since $a_\alpha \nearrow a$ strongly, we then have that $\|a\xi\| = \lim_\alpha \|a_\alpha\xi\| \leq s$, which is a clear contradiction. Thus $\chi_{(s,\infty)}(a) \wedge (\wedge_\alpha \chi_{[0,s]}(a_\alpha)) = 0$ as claimed. But in that case we have that

$$
\begin{aligned}
\chi_{(s,\infty)}(a) &= \chi_{(s,\infty)}(a) - \chi_{(s,\infty)}(a) \wedge (\wedge_\alpha \chi_{[0,s]}(a_\alpha)) \\
&\sim \chi_{(s,\infty)}(a) \vee (\wedge_\alpha \chi_{[0,s]}(a_\alpha)) - \wedge_\alpha \chi_{[0,s]}(a_\alpha) \\
&\leq \mathbb{1} - \wedge_\alpha \chi_{[0,s]}(a_\alpha) \\
&= \vee_\alpha \chi_{(s,\infty)}(a_\alpha).
\end{aligned}
$$

The normality of the trace now ensures that

$$
d_a(s) = \tau(\chi_{(s,\infty)}(a)) \leq \tau(\vee_\alpha \chi_{(s,\infty)}(a_\alpha)) = \sup_\alpha \tau(\chi_{(s,\infty)}(a_\alpha)) = \sup_\alpha d_{a_\alpha}(s).
$$

Claim (iii): Given $a, b \in \widetilde{\mathcal{M}}$, we use the preceding lemma to select partial isometries $u, v \in \mathcal{M}$ such that $|a + b| \leq u|a|u^* + v|b|v^*$. Then of course $d_{a+b} \leq d_{u|a|u^*+v|b|v^*}$. Let $t, s > 0$ be given. For any norm one element ξ of $\chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*)(H)$, we then have that $\|(u|a|u^* + v|b|v^*)(\xi)\| > t+s$. But then at least one of the inequalities $\|u|a|u^*(\xi)\| > t$ or $\|v|b|v^*(\xi)\| > s$ must hold, or else we will by the triangle inequality have that $\|(u|a|u^* + v|b|v^*)(\xi)\| \leq t + s$. In other words $\xi \notin \chi_{[0,t]}(u|a|u^*)(H) \cap \chi_{[0,s]}(v|b|v^*)(H)$, or equivalently $\{0\} = \chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*)(H) \cap \chi_{[0,t]}(u|a|u^*)(H) \cap \chi_{[0,s]}(v|b|v^*)(H)$. So by definition

$$
\chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*) \wedge \chi_{[0,t]}(u|a|u^*) \wedge \chi_{[0,s]}(v|b|v^*) = 0.
$$

This in turn ensures that

$$\chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*)$$
$$= \chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*)$$
$$\qquad - \chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*) \wedge \chi_{[0,t]}(u|a|u^*) \wedge \chi_{[0,s]}(v|b|v^*)$$
$$\sim \chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*) \vee (\chi_{[0,t]}(u|a|u^*) \wedge \chi_{[0,s]}(v|b|v^*))$$
$$\qquad - (\chi_{[0,t]}(u|a|u^*) \wedge \chi_{[0,s]}(v|b|v^*))$$
$$\leq \mathbb{1} - (\chi_{[0,t]}(u|a|u^*) \wedge \chi_{[0,s]}(v|b|v^*))$$
$$= \chi_{(t,\infty)}(u|a|u^*) \vee \chi_{(s,\infty)}(v|b|v^*).$$

But then

$$
\begin{aligned}
d_{a+b}(t+s) &\leq d_{u|a|u^*+v|b|v^*}(t+s) \\
&= \tau(\chi_{(t+s,\infty)}(u|a|u^* + v|b|v^*)) \\
&\leq \tau(\chi_{(t,\infty)}(u|a|u^*) \vee \chi_{(s,\infty)}(v|b|v^*)) \\
&\leq \tau(\chi_{(t,\infty)}(u|a|u^*)) + \tau(\chi_{(s,\infty)}(v|b|v^*)) \\
&= d_{u|a|u^*}(t) + d_{v|b|v*}(s)
\end{aligned}
$$

Recall that for example $|u|a|u^*|^2 \leq |au^*|^2$ and hence that $|u|a|u^*| \leq |au^*|$. Similarly $|ua^*| \leq |a^*|$. We may therefore use part (i) to see that $d_{u|a|u^*} \leq d_{au^*} = d_{ua^*} \leq d_{a^*} = d_a$ and similarly that $d_{v|b|v*} \leq d_b$. The claim now follows.

Claim (iv): Given $f \in \widetilde{\mathcal{M}}$ and $a \in \mathcal{M}$, we may use parts (i) and (ii) to see that $d_{af} = d_{|af|} = d_{|\,|a|f|} \leq d_{\|a\|\,|f|} = d_{\|a\|f}$. It then follows from this inequality and part (i) that $d_{fa} = d_{a^*f^*} \leq d_{\|a\|f^*} = d_{\|a\|f}$. $\qquad\square$

The properties of the distribution function allow us to realise $\widetilde{\mathcal{M}}$ as an F-normed space.

DEFINITION 4.4. Let X be a complex vector space.
- A functional $\|\cdot\| : X \to [0, \infty)$ is called an F-norm on X if it satisfies the following criteria for $x, y \in X$ and $\alpha \in \mathbb{C}$:
  - $\|0\| = 0$ and if $\|x\| = 0$, then $x = 0$.
  - $\|e^{it}x\| = \|x\|$ for any $t \in \mathbb{R}$.
  - $\|x + y\| \leq \|x\| + \|y\|$.
  - Given sequences $(x_k) \subseteq X$ and $(\alpha_k) \subseteq \mathbb{C}$ such that $\|x - x_k\| \to 0$ and $|\alpha - \alpha_k| \to 0$ as $k \to \infty$, we will then have that $\|\alpha x - \alpha_k x_k\| \to 0$ as $k \to \infty$.

- A functional $\rho : X \to [0, \infty]$ is called a semimodular (alt. modular) on X if it satisfies the following criteria for $x, y \in X$:
  - (a) $\rho(0) = 0$ and $x = 0$ whenever $\rho(\epsilon x) = 0$ for all $\epsilon > 0$ (alt. $x = 0$ whenever $\rho(x) = 0$).
  - (b) $\rho(e^{it} x) = \rho(x)$ for any $t \in \mathbb{R}$.
  - (c) For any $\epsilon \in [0, 1]$ we have that $\rho(\epsilon x + (1 - \epsilon)y) \le \rho(x) + \rho(y)$.

    We say that ρ is a convex modular if instead of (c) we have that $\rho(\epsilon x + (1 - \epsilon)y) \le \epsilon \rho(x) + (1 - \epsilon)\rho(y)$ for any $\epsilon \in [0, 1]$ and any $x, y \in X$.

    If the (semi-)modular on X satisfies $\lim_{\epsilon \to 0} \rho(\epsilon x) = 0$ for each $x \in X$, X is said to be a modular space .

REMARK 4.5. There is a very close link between F-normed spaces and modular spaces. Any F-norm on a vector space X for which $\epsilon \mapsto \|\epsilon x\|$ is a non-decreasing function of ϵ on $[0, \infty)$ for every x, is a modular. Conversely if X is equipped with a semimodular with respect to which it is a modular space, then the prescription $\|x\| = \inf\{\epsilon > 0 : \rho(\epsilon^{-1}x) \le \epsilon\}$ yields an F-norm on X such that for any sequence $(x_k) \subseteq X$, the claim that $\lim_{k \to \infty} \|x - x_k \cdot \| = 0$ is equivalent to the claim that $\lim_{k \to \infty} \rho(\epsilon(x - x_k)) = 0$ for each $\epsilon > 0$. The interested reader may find proofs of these facts in section I.1 of [**Mus83**].

With the above as background, we are now able to make the following conclusion.

THEOREM 4.6. *The functional* $\rho : \widetilde{\mathcal{M}} \to [0, \infty]$ *defined by* $\rho(f) = d_f(1)$, *is a semimodular on* $\widetilde{\mathcal{M}}$ *with respect to which* $\widetilde{\mathcal{M}}$ *is a modular space. Moreover the prescription* $\|f\| = \inf\{\epsilon > 0 : d_f(\epsilon) \le \epsilon\}$ *defines an* F-*norm on* $\widetilde{\mathcal{M}}$. *The topology induced by this* F-*norm, is exactly the topology of convergence in measure.*

PROOF. We clearly have that $\rho(0) = d_0(1) = \tau(\chi_{(1,\infty)}(0)) = 0$. In addition if for some $f \in \widetilde{\mathcal{M}}$ we have that $0 = \rho(\epsilon f) = d_{\epsilon f}(1) = d_f(\epsilon^{-1}) = \tau(\chi_{(\epsilon^{-1},\infty)}(|f|))$ for all $\epsilon > 0$ (hence $\chi_{(\epsilon^{-1},\infty)}(|f|) = 0$ for all $\epsilon > 0$), we must clearly have that $|f| = 0$ (and hence $f = 0$).

It also follows fairly immediately from Proposition 4.3 that $\rho(e^{it}x) = \rho(x)$ for any $t \in \mathbb{R}$, and that for any $\epsilon \in (0, 1)$ and any $f, g \in \widetilde{\mathcal{M}}$, we have that $\rho(\epsilon f + (1 - \epsilon)g) = d_{(\epsilon f + (1-\epsilon)g)}(1) \le d_{\epsilon f}(\epsilon) + d_{(1-\epsilon)g}(1 - \epsilon) = d_f(1) + d_g(1) = \rho(f) + \rho(g)$. Thus ρ is a semimodular as claimed.

Next note that by definition any element f of $\widetilde{\mathcal{M}}$ must satisfy the property that $\lim_{\gamma \to \infty} d_f(\gamma) = 0$. But since for $\gamma > 0$ we have that $d_f(\gamma) = d_{\gamma^{-1}f}(1)$, this corresponds to the claim that $\lim_{\epsilon \to 0} \rho(\epsilon f) = 0$ for each $f \in \widetilde{\mathcal{M}}$. Thus $\widetilde{\mathcal{M}}$ is a modular space. The fact that the prescription $\|f\| = \inf\{\epsilon > 0 \colon d_f(\epsilon) \leq \epsilon\}$ defines an F-norm on $\widetilde{\mathcal{M}}$, now follows from the preceding remark. It therefore remains to show that the F-norm topology agrees with the topology of convergence in measure. Both topologies are metric topologies, and hence the equivalence will follow if we show that convergence of sequences in the one is equivalent to convergence of sequences in the other. In this regard recall that convergence of a sequence $(f_n) \subseteq \widetilde{\mathcal{M}}$ to $f \in \widetilde{\mathcal{M}}$ in the F-norm topology, is equivalent to the claim that $\lim_{n \to \infty} \rho(\epsilon(f - f_n)) = 0$ for each $\epsilon > 0$. In other words, given $\epsilon > 0$, and some $\delta > 0$, there must exist $N \in \mathbb{N}$ so that $\tau(\chi_{(\epsilon^{-1}, \infty)}(|f - f_n|)) = d_{f - f_n}(\epsilon^{-1}) = d_{\epsilon(f - f_n)}(1) = \rho(\epsilon(f - f_n)) \leq \delta$ for all $n \geq N$. But this is exactly convergence in measure. Hence the result follows. $\qquad \square$

The following result establishes the link between the distribution function and decreasing rearrangements, hinted at earlier.

PROPOSITION 4.7. *For any* $f \in \widetilde{\mathcal{M}}$, *we have that* $\mathbf{m}_f(t) = \inf\{s \geq 0 \colon d_f(s) \leq t\}$. *Moreover the infimum is attained and* $d_f(\mathbf{m}_f(t)) \leq t$ *for all* $t \geq 0$.

PROOF. Assume that $\mathcal{M}$ acts on the Hilbert space H. The second claim follows from the right continuity of $s \mapsto d_f(s)$. For the first let a be the point where the infimum is attained. Then the inequality $d_f(a) \leq t$ amounts to the claim that $\tau(\mathbb{1} - \chi_{[0,a]}(|f|)) \leq t$. In view of the fact that $\|f\chi_{[0,a]}(|f|)\|_\infty = \| \, |f|\chi_{[0,a]}(|f|)\|_\infty \leq a$, it clearly follows that $\mathbf{m}_f(t) \leq a$. It remains to prove the converse.

Let $\epsilon > 0$ be given and select $e \in \mathbb{P}(\mathcal{M})$ such that $\tau(\mathbb{1} - e) \leq t$ with $\|fe\|_\infty < \mathbf{m}_f(t) + \epsilon$. For the sake of simplicity we will write α_ϵ for $\mathbf{m}_f(t) + \epsilon$. Our task is of course to show that $a \leq \alpha_\epsilon$.

For any $\xi \in e(H) \cap \chi_{(\alpha_\epsilon, \infty)}(|f|)(H)$ with $\|\xi\| = 1$, the fact that $\chi_{(\alpha_\epsilon, \infty)}(|f|) = \chi_{(\alpha_\epsilon^2, \infty)}(f^*f)$ ensures that $\langle f^*f\xi, \xi \rangle \geq \alpha_\epsilon^2$. But on the other hand the fact that $\|fe\| < \alpha_\epsilon$ ensures that $\langle f^*f\xi, \xi \rangle < \alpha_\epsilon$. This cannot be,

and hence we must have that $e \wedge \chi_{(\alpha_\epsilon, \infty)}(|f|) = 0$. But in that case

$$
\begin{aligned}
\chi_{(\alpha_\epsilon, \infty)}(|f|) &= \chi_{(\alpha_\epsilon, \infty)}(|f|) - e \wedge \chi_{(\alpha_\epsilon, \infty)}(|f|) \\
&\sim \quad e \vee \chi_{(\alpha_\epsilon, \infty)}(|f|) - e \\
&\leq \quad \mathbb{1} - e.
\end{aligned}
$$

Consequently $d_f(\alpha_\epsilon) = \tau(\chi_{(\alpha_\epsilon, \infty)}(|f|)) \leq \tau(\mathbb{1} - e) \leq t$, which amounts to the claim that $a \leq \alpha_\epsilon = \mathbf{m}_f(t) + \epsilon$, as required. $\qquad\square$

The following remark captures an important consequence of the above result — the fact that decreasing rearrangements are in some sense commutatively realised.

REMARK 4.8. ● Let $f \in \widetilde{\mathcal{M}}$ be given. For any von Neumann subalgebra $\mathcal{M}_0$ of $\mathcal{M}$ containing both f and the spectral projections of $|f|$, the above result actually shows that $\mathbf{m}_f(t) = \inf\{\|fe\|_\infty : e \in \mathbb{P}(\mathcal{M}_0), \tau(\mathbb{1} - e) \leq t\}$

  ● The preceding proposition also shows that the quantity is a faithful extension of the classical concept of a decreasing rearrangement. To see this observe that in the case $\mathcal{M} = L^\infty(X, \Sigma, \nu)$ with $\tau = \int \cdot \, d\nu$, the formula in the preceding proposition corresponds to the claim that $\mathbf{m}_f(t) = \inf\{s > 0 : \nu(\{x \in X : |f(x)| > s\}) \leq t\}$.

We pause to present a particularly elegant application of the above facts, which proves to be a useful tool in lifting the classical theory to the noncommutative context.

PROPOSITION 4.9. *Let $\mathcal{M}$ be a von Neumann algebra with no minimal projections. Then the following holds:*

(a) *For any non-zero projection $e \in \mathcal{M}$, any maximal abelian von Neumann subalgebra $\mathcal{M}_0$ of $e\mathcal{M}e$, also has no minimal projections.*

(b) *Whenever $e\mathcal{M}e$ admits a faithful normal state ω, then any maximal abelian subalgebra $\mathcal{M}_0$ of $e\mathcal{M}e$, corresponds to a classical $L^\infty(\Omega, \Sigma, \nu_\omega)$, where $(\Omega, \Sigma, \nu_\omega)$ is a nonatomic probability space, with the measure ν_ω defined by $\nu_\omega(E) = \omega(\chi_E)$ for each $E \in \Sigma$. In particular if $\mathcal{M}$ is a finite algebra equipped with a finite faithful normal trace τ, then given any $f \in \widetilde{\mathcal{M}}_+$, we may select a maximal abelian subalgebra $\mathcal{M}_0$ containing all the spectral projections of f. The element f corresponds to a Borel function on $(\Omega, \Sigma, \nu_\tau)$, and the classical decreasing rearrangement of f as Borel function*

corresponds exactly to the decreasing rearrangement of f as an element of $\widetilde{\mathcal{M}}_+$.

(c) *Suppose that $\mathcal{M}$ admits a faithful normal semifinite trace τ. For any non-zero projection $e \in \mathcal{M}$, $e\mathcal{M}e$ admits an abelian subalgebra $\mathcal{M}_0$, which has no minimal projections, and on which the restriction of τ is still semifinite. The algebra $\mathcal{M}_0$ corresponds to a classical $L^\infty(\Omega, \Sigma, \nu_\tau)$, where $(\Omega, \Sigma, \nu_\tau)$ is a nonatomic measure space, with the measure ν_τ defined by $\nu_\tau(E) = \tau(\chi_E)$ for each $E \in \Sigma$.*

PROOF. *Part(a):* If $\mathcal{M}$ has no minimal projections, then the same is true of $e\mathcal{M}e$. Hence we may for the sake of simplicity replace $\mathcal{M}$ with $e\mathcal{M}e$ throughout. Let $\mathcal{M}_0$ be a commutative von Neumann subalgebra of $\mathcal{M}$. Suppose that e_0 is a minimal projection in $\mathcal{M}_0$. By hypothesis, there must exist a projection $f_0 \in \mathcal{M} \setminus \mathcal{M}_0$ with $0 < f_0 < e_0$. Now given any other projection e in $\mathcal{M}_0$, we have by commutativity that $e_0 e \in \mathcal{M}_0$ is a subprojection of e_0. So by minimality

either $e_0 e = 0$ (i.e. $e_0 \perp e$)   or $e_0 e = e_0$ (i.e. $e_0 \le e$).

Thus since $f_0 < e_0$ we also have that

$$\text{either} \quad f_0 \perp e \qquad \text{or} \quad f_0 < e$$

for any projection e in $\mathcal{M}_0$. But this means that f_0 commutes with all the projections in $\mathcal{M}_0$. Since the span of these projections is dense in $\mathcal{M}_0$, f_0 commutes with $\mathcal{M}_0$. Therefore $\mathcal{M}_0$ cannot be maximal abelian, since $\{f_0, \mathcal{M}_0\}$ generates a commutative subalgebra which is strictly larger than $\mathcal{M}_0$.

Part(b): The first part of (b) follows from the fact that any commutative von Neumann subalgebra $\mathcal{M}_0$ will correspond to some $L^\infty(\Omega, \Sigma, \rho)$ (see Theorem 1.100). In particular given a faithful normal state ω on $\mathcal{M}$, it is an exercise to show that the restriction of ω to $\mathcal{M}_0 = L^\infty(\Omega, \Sigma, \rho)$ defines a probability measure $\nu_\omega = \nu$ on (Ω, Σ) (with the same sets of measure zero as ν) by means of the prescription $\nu(E) = \omega(\chi_E)$ $E \in \Sigma$. Replacing ρ by ν if necessary, all that remains is to note that the subalgebra $\mathcal{M}_0 = L^\infty(\Omega, \Sigma, \nu)$ has no minimal projections precisely when (Ω, Σ, ν) is nonatomic.

The final part of (b) follows by combining what we have just observed with the preceding remarks and the Borel functional calculus.

Part (c): To prove (c) we firstly note that the semifiniteness of the trace τ on $\mathcal{M}$, ensures that each projection f in $\mathcal{M}$, admits a subprojection with finite trace. To see this recall that the semifiniteness of τ, ensures that there exists $0 \leq x \leq f$ with $0 < \tau(x) < \infty$. For $\epsilon > 0$ small enough, $e_\epsilon = \chi_{(\epsilon,\infty)}(x)$ will be a non-zero projection with $\epsilon e_\epsilon \leq x \leq f$. But then $\tau(e_\epsilon) \leq \epsilon^{-1}\tau(x) < \infty$, with in addition $e_\epsilon = \lim_{n\to\infty} \epsilon^{2^{-n}} e_\epsilon \leq f$. Here we used the fact that the square root preserves order. Given a projection e, we may now apply Zorn's lemma to obtain a maximal family $\{e_\alpha\}$ of mutually orthogonal subprojections of e, each with finite trace. It must then hold that $e = \sum_\alpha e_\alpha$, since if $e - \sum_\alpha e_\alpha \neq 0$, we would be able to select a subprojection e_0 of $e - \sum_\alpha e_\alpha$ with finite trace, which would contradict the maximality of $\{e_\alpha\}$. For each α, we then select a maximal abelian subalgebra $\mathcal{M}_\alpha$ of $e_\alpha \mathcal{M} e_\alpha$. By part (a), there are no minimal projections in any of the $\mathcal{M}_\alpha$'s. The algebra we seek is then given by $\mathcal{M}_0 = \oplus_\alpha \mathcal{M}_\alpha$. Since τ is finite-valued on each $\mathcal{M}_\alpha$, it is an exercise to see that the restriction of τ to $\mathcal{M}_0$, is semifinite. The final part of (c) may now be proven using the same sort of argument as in part (b). $\qquad\square$

4.2   **Algebraic properties of decreasing rearrangements**

As before $\mathcal{M}$ is a von Neumann algebra equipped with a faithful normal semifinite trace τ. To start off with we present yet another way of realising the decreasing rearrangements described earlier.

LEMMA 4.10. *For each $t \geq 0$, we let R_t be the set of all τ-measurable operators f satisfying $\tau(\mathrm{s}(|f|)) \leq t$, where $\mathrm{s}(|f|)$ is the support projection of $|f|$. For any $g \in \widetilde{\mathcal{M}}$, we then have that*

$$\mathbf{m}_g(t) = \inf\{\|g - f\| \colon f \in R_t\}.$$

PROOF. Let $t \geq 0$ be given and let $g = u|g|$ be the polar decomposition of g. With $\lambda \mapsto e_\lambda$ denoting the spectral resolution of $|g|$, we set

$$f = u \int_{(\alpha,\infty)} \lambda \, de_\lambda$$

where $\alpha = \mathbf{m}_g(t)$. Then by construction $\|g - f\| \leq \alpha = \mathbf{m}_g(t)$, with $\tau(\mathrm{s}(|f|)) = d_g(\alpha) = d_g(\mathbf{m}_g(t)) \leq t$ by Proposition 4.7. Therefore $\inf\{\|g - f\| \colon f \in R_t\} \leq \mathbf{m}_g(t)$.

For the converse let $f \in R_t$ be given, and set $e = \mathbb{1} - \mathrm{s}(|f|)$. Then $\|ge\| = \|(g-f)e\| \leq \|g-f\|$. In view of the fact that $\tau(\mathbb{1}-e) \leq t$, it follows

that $\mathbf{m}_g(t) \leq \|ge\| \leq \|g - f\|$, and hence that $\mathbf{m}_g(t) \leq \inf\{\|g - f\| \colon f \in R_t\}$. □

COROLLARY 4.11. *For any $f \in \widetilde{\mathcal{M}}$ and any projection $e \in \mathcal{M}$, we have that $\mathbf{m}_{fe}(t) = 0$ whenever $t \geq \tau(e)$.*

The following Proposition collates several important properties of decreasing rearrangements. The reader is encouraged to take careful note of these properties, as they will repeatedly be used without comment from here on.

PROPOSITION 4.12. *Let f, a and b be τ-measurable operators.*

  (i) *The map $\mathbf{m}_f : (0, \infty) \to \mathbb{R} : t \mapsto \mathbf{m}_f(t)$ is non-increasing and right-continuous. Moreover $\lim_{t \searrow 0} \mathbf{m}_f(t) = \|f\|_\infty$.*
 (ii) *For any $t > 0$ and any $\alpha \in \mathbb{C}$, $\mathbf{m}_f(t) = \mathbf{m}_{f^*}(t) = \mathbf{m}_{|f|}(t)$, and $\mathbf{m}_{\alpha f}(t) = |\alpha| \mathbf{m}_f(t)$.*
(iii) *If $a \geq b \geq 0$, then $\mathbf{m}_a(t) \geq \mathbf{m}_b(t)$ for any $t > 0$. Moreover if $(a_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ is a net increasing to a in $\widetilde{\mathcal{M}}$, then $\sup_\alpha \mathbf{m}_{a_\alpha}(t) = \mathbf{m}_a(t)$.*
 (iv) *For any $s, t > 0$, $\mathbf{m}_{a+b}(t + s) \leq \mathbf{m}_a(t) + \mathbf{m}_b(s)$.*
  (v) *For any $s, t > 0$, $\mathbf{m}_{ab}(t + s) \leq \mathbf{m}_a(t)\mathbf{m}_b(s)$. We have, in particular, $\mathbf{m}_{afb}(t) \leq \|a\|\|b\|\mathbf{m}_f(t)$.*
 (vi) *Let $\Phi : [0, \infty) \to [0, \infty]$ be a non-decreasing function which is continuous on $[0, b_\Phi]$ where $b_\Phi = \sup\{t \in [0, \infty) \colon \Phi(t) < \infty\}$. If $\Phi(|f|)$ is again τ-measurable, then $\Phi(\mathbf{m}_f(t)) = \mathbf{m}_{\Phi(|f|)}(t)$ for any $t \geq 0$.*

PROOF. *Claim (i):* The fact that $\mathbf{m}_f$ is non-increasing follows by definition. Next suppose that at some point $t_0 > 0$, $\mathbf{m}_f$ is not right-continuous. There must then exist some $\alpha > 0$ so that $\mathbf{m}_f(t_0) > \alpha \geq \mathbf{m}_f(t_0 + \epsilon)$ for all $\epsilon > 0$. We may then combine the fact that $s \mapsto d_f(s)$ is non-increasing with Proposition 4.7, to conclude from this that $d_f(\alpha) \leq d_f(\mathbf{m}_f(t_0 + \epsilon)) \leq t_0 + \epsilon$ for any $\epsilon > 0$, that is $d_f(\alpha) \leq t_0$. But if that were the case we ought to have that $\mathbf{m}_f(t_0) \leq \alpha$ – a clear contradiction. Hence the claim regarding right-continuity holds.

It remains to prove the claim regarding the left limit at 0. It is clear from the definition that $\|f\| \geq \mathbf{m}_f(t)$ for all $t > 0$. Suppose that for all $\epsilon > 0$ we have that $\|f\| > \alpha \geq \mathbf{m}_f(\epsilon)$. A similar argument to the one used above then leads to the conclusion that $d_f(\alpha) \leq \epsilon$ for any $\epsilon > 0$, and hence

that $d_f(\alpha) = 0$. But if that were the case we ought to have that $\|f\| \leq \alpha$ – a clear contradiction. Hence the remaining claim follows.

Claim (ii): The fact that $\mathbf{m}_f(t) = \mathbf{m}_{|f|}(t) = \mathbf{m}_{f^*}(t)$ for all $t > 0$, is a clear consequence of part (i) of Proposition 4.3 considered alongside Proposition 4.7.

Claim (iii): First suppose that $a \geq b \geq 0$. A simple application of Proposition 4.7 to part (ii) of Proposition 4.3, then yields the first conclusion.

Next suppose that $(a_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ is a net increasing to a in $\widetilde{\mathcal{M}}$. Given that by Proposition 4.3 $d_a(s) = \sup_\alpha d_{a_\alpha}(s)$ for all $s \geq 0$, the claim similarly follows from Proposition 4.7.

Claim (iv): Let $\epsilon > 0$ and $s, t > 0$ be given. By Lemma 4.10, we may find $a_0, b_0 \in \widetilde{\mathcal{M}}$ such that

$$\|a - a_0\| \leq \mathbf{m}_a(t) + \epsilon, \quad \tau(\mathbb{s}(\|a_0\|)) \leq t$$
$$\|b - b_0\| \leq \mathbf{m}_b(s) + \epsilon, \quad \tau(\mathbb{s}(\|b_0\|)) \leq s$$

We have that

$$\|(a + b) - (a_0 + b_0)\| \leq \|a - a_0\| + \|b - b_0\| \leq \mathbf{m}_a(t) + \mathbf{m}_b(s) + 2\epsilon.$$

In addition we also have that

$$\begin{aligned}
\tau(\mathbb{s}(|a + b|)) &= \tau(\mathbb{s}(|a|) \vee \mathbb{s}(|b|)) \\
&\leq \tau(\mathbb{s}(|a|)) + \tau(\mathbb{s}(|b|)) \\
&\leq t + s
\end{aligned}$$

It follows that $\mathbf{m}_{a+b}(t+s) \leq \mathbf{m}_a(t) + \mathbf{m}_b(s) + 2\epsilon$. Given that ϵ was arbitrary, it is clear that $\mathbf{m}_{a+b}(t + s) \leq \mathbf{m}_a(t) + \mathbf{m}_b(s)$ as required.

Claim (v): Let $\epsilon > 0$ and $s, t > 0$ be given, and select a_0 and b_0 as in the proof of the previous claim. Now set $c = (a - a_0)b_0 + a_0b$. We then have that

$$\begin{aligned}
\|ab - c\| &= \|ab - (a - a_0)b_0 - a_0b\| \\
&= \|(a - a_0)(b - b_0)\| \\
&\leq \|(a - a_0)\|.\|(b - b_0)\| \\
&\leq (\mathbf{m}_a(t) + \epsilon)(\mathbf{m}_b(s) + \epsilon).
\end{aligned}$$

Moreover we also have that

- $\mathbb{s}(|c|) \leq \mathbb{s}(|(a - a_0)b_0|) \vee \mathbb{s}(|a_0b|)$,

- $\mathfrak{s}(|(a - a_0)b_0|) \leq \mathfrak{s}(|b_0|)$,
- $\mathfrak{s}(|b^* a_0^*|) \leq \mathfrak{s}(|a_0^*|)$ together with $\mathfrak{s}(|a_0 b|) \sim \mathfrak{s}(|b^* a_0^*|)$ and $\mathfrak{s}(|a_0^*|)$ $\sim \mathfrak{s}(|a_0|)$.

It follows from this that $\tau(\mathfrak{s}(|c|)) \leq \tau(\mathfrak{s}(|b_0|)) + \tau(\mathfrak{s}(|a_0|)) \leq t + s$. Therefore $\mathbf{m}_{ab}(t + s) \leq (\mathbf{m}_a(t) + \epsilon)(\mathbf{m}_b(s) + \epsilon)$, whence $\mathbf{m}_{ab}(t + s) \leq \mathbf{m}_a(t).\mathbf{m}_b(s)$.

Claim (vi): We may replace $\mathcal{M}$ by a maximal abelian von Neumann subalgebra $\mathcal{M}_0$ to which both $|f|$ and $\Phi(|f|)$ are affiliated (see Remark 4.8). Let e be any projection in this subalgebra. Now notice that $\mathrm{sp}(|f|) \subseteq [0, \infty)$.

First suppose that Φ is bounded on $\mathrm{sp}(|f|e)$. (By the Borel functional calculus $\Phi(|f|e)$ will then of course be bounded.) Since $\lim_{u \to \infty} \Phi(u) = \infty$, we must then have that $\mathrm{sp}(|f|e)$ itself is a bounded subset of $[0, \infty)$. Thus $|f|e$ must be bounded. By the spectral theory for positive elements, we now have that $\| |f|e \| = \max\{\lambda \colon \lambda \in \mathrm{sp}(|f|e)\}$. Since Φ is increasing and non-negative on $[0, \infty)$, the Borel functional calculus also ensures that

$$\Phi(\| |f|e \|) = \max\{\Phi(\lambda) \colon \lambda \in \mathrm{sp}(|f|e)\} = \|\Phi(|f|e)\|.$$

If Φ is not bounded on $\mathrm{sp}(|f|e)$, then

$$\|\Phi(|f|e)\| = \sup\{\Phi(\lambda) \colon \lambda \in \mathrm{sp}(|f|e)\} = \infty.$$

We proceed to show that then $\Phi(\| |f|e \|) = \infty$. If now $\mathrm{sp}(|f|e)$ was an unbounded subset of $[0, \infty)$, we would already have $\| |f|e \| = \infty$, and hence $\Phi(\| |f|e \|) = \infty$ as required. Thus let $\mathrm{sp}(|f|e)$ be a bounded subset of $[0, \infty)$. As noted previously, this forces $\| |f|e \| = \max\{\lambda \colon \lambda \in \mathrm{sp}(|f|e)\}$. Since Φ is increasing on $[0, \infty]$ with $\Phi(0) = 0$, we must have $\Phi(\| |f|e \|) \geq \Phi(\lambda) \geq 0$ for any $\lambda \in \mathrm{sp}(|f|e)$. The fact that Φ is unbounded on $\mathrm{sp}(|f|e)$ therefore forces $\Phi(\| |f|e \|) = \infty$ as required.

The above observations clearly show that

$$\inf\{\Phi(\| |f|e \|) \colon e \in \mathbb{P}(\mathcal{M}_0) \text{ with } \tau(\mathbb{1} - e) \leq t\}$$

$$= \inf\{\|\Phi(|f|e)\| \colon e \in \mathbb{P}(\mathcal{M}_0) \text{ with } \tau(\mathbb{1} - e) \leq t\}.$$

But since Φ is increasing and continuous on $[0, b_\Phi]$, we also have that

$$\inf\{\Phi(\| |f|e \|) \colon e \in \mathbb{P}(\mathcal{M}_0) \text{ with } \tau(\mathbb{1} - e) \leq t\}$$

$$= \Phi(\inf\{\| |f|e \| \colon e \in \mathbb{P}(\mathcal{M}_0) \text{ with } \tau(\mathbb{1} - e) \leq t\}).$$

When combined with the observation in Remark 4.8, this yields the conclusion that $\mathbf{m}_{\Phi(|f|)}(t) = \Phi(\mathbf{m}_{|f|}(t)) = \Phi(\mathbf{m}_f(t))$. $\square$

4.3   **Decreasing rearrangements and the trace**

We saw in Theorem 3.21 that the trace may be extended to the extended positive cone $\widehat{\mathcal{M}}_+$ of $\mathcal{M}$ by means of the following prescription: Given $m \in \widehat{M}_+$ with spectral resolution $m = \int_0^\infty \lambda \, de_\lambda + p.\infty$, we define the trace of m to be $\tau(m) = \sup_{n \in \mathbb{N}} \tau(m_n)$ where the m_n's are the bounded elements of $\mathcal{M}$ given by $m_n = \int_0^n \lambda \, de_\lambda + p.n$. Our goal in this section is to show that on $\widetilde{\mathcal{M}}$, this extension exhibits more subtle behaviour. This first step in realising that goal, is the following proposition.

PROPOSITION 4.13. *For any $f \in \widetilde{\mathcal{M}}_+$,*

$$\tau(f) = \int_0^\infty \mathbf{m}_f(t) \, dt.$$

The above proposition provides us with a powerful alternative for defining the trace on $\widetilde{\mathcal{M}}_+$. On considering this result alongside for example Proposition 4.12, we may then fairly directly deduce highly refined properties of the trace on $\widetilde{\mathcal{M}}_+$ from matching properties of the decreasing rearrangement. We will briefly address this issue in Proposition 4.17, and then revisit this topic in chapter 5.

PROOF. Let $f \in \widetilde{\mathcal{M}}_+$ be given. For each $n \in \mathbb{N}$, let $f_n = f\chi_{[0,n]}(f)$. By definition we then have that $\tau(f) = \sup_{n \in \mathbb{N}} \tau(f_n)$. It is clear from the Borel functional calculus that the f_n's increase to f. So by part (iii) of Proposition 4.12 considered alongside the monotone convergence theorem, we also have that $\int_0^\infty \mathbf{m}_f(t) \, dt = \sup_{n \in \mathbb{N}} \int_0^\infty \mathbf{m}_{f_n}(t) \, dt$. It is therefore clear that the proposition will follow for $\widetilde{\mathcal{M}}_+$, if we are able to prove it for $\mathcal{M}_+$. Hence assume that $f \in \mathcal{M}_+$.

It is well-known that any positive measurable function may be written as the increasing limit of a sequence of positive simple functions. If we combine this fact with the Borel functional calculus, it is clear that f may written as an increasing limit of a sequence of operators (f_N) of the form $\sum_{k=1}^n \alpha_k e_k$ where the α_k's are positive real numbers, and the e_k's mutually orthogonal projections. By the normality of the trace on $\mathcal{M}_+$, $\tau(f) = \sup_N \tau(f_N)$. On once again considering part (iii) of Proposition 4.12 alongside the monotone convergence theorem, we have that $\int_0^\infty \mathbf{m}_f(t) \, dt = \sup_N \int_0^\infty \mathbf{m}_{f_N}(t) \, dt$. The result will therefore follow for $\mathcal{M}_+$, if we are able to verify its validity in the case where $f = \sum_{k=1}^n \alpha_k e_k$. We proceed with the verification of this case. We may assume without

loss of generality that $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. It then follows from the Borel functional calculus that

$$\chi_{(s,\infty)}(f) = \begin{cases} 0 & \text{if } s \geq \alpha_n \\ \sum_{k=m+1}^{n} e_k & \text{if } \alpha_m \leq s < \alpha_{m+1} \\ \sum_{k=1}^{n} e_k & \text{if } s < \alpha_1 \end{cases}$$

and hence that

$$d_f(s) = \begin{cases} 0 & \text{if } s \geq \alpha_n \\ \sum_{k=m+1}^{n} \tau(e_k) & \text{if } \alpha_m \leq s < \alpha_{m+1} \\ \sum_{k=1}^{n} \tau(e_k) & \text{if } s < \alpha_1 \end{cases}$$

Now consider the case where each $\tau(e_k)$ is finite. It then follows from Proposition 4.7 that

$$\mathbf{m}_f(t) = \begin{cases} \alpha_n & \text{if } \tau(e_n) > t \geq 0 \\ \alpha_m & \text{if } \sum_{k=m}^{n} \tau(e_k) > t \geq \sum_{k=m+1}^{n} \tau(e_k) \\ 0 & \text{if } t \geq \sum_{k=1}^{n} \tau(e_k) \end{cases} .$$

It is now clear that then $\int_0^\infty \mathbf{m}_f(t)\,dt = \sum_{k=1}^{n} \alpha_k \tau(e_k) = \tau(f)$ as required.

Now suppose that some of the $\tau(e_k)$'s are infinite-valued, and let m_0 be the largest index for which $\tau(e_{m_0}) = \infty$. Suppose for the sake of the argument that $n > m_0 \geq 1$. (The proof for the case where $m_0 = n$ is similar.) In this case we of course have that $\sum_{k=m}^{n} \tau(e_k) = \infty$ whenever $m \leq m_0$, and hence the formula for $\mathbf{m}_f(t)$ becomes

$$\mathbf{m}_f(t) = \begin{cases} \alpha_n & \text{if } \tau(e_n) > t \geq 0 \\ \alpha_m & \text{if } \sum_{k=m}^{n} \tau(e_k) > t \geq \sum_{k=m+1}^{n} \tau(e_k), \quad m > m_0 \\ \alpha_{m_0} & \text{if } \infty > t \geq \sum_{k=m_0+1}^{n} \tau(e_k) \end{cases} .$$

But then $\int_0^\infty \mathbf{m}_f(t)\,dt = \infty$, which accords with the fact that $\tau(f) = \sum_{k=1}^{n} \alpha_k \tau(e_k) = \infty$. $\qquad\square$

COROLLARY 4.14. *Let $\Phi : [0,\infty) \to [0,\infty]$ be a non-decreasing function which is continuous on $[0, b_\Phi]$ where $b_\Phi = \sup\{t \in [0,\infty) : \Phi(t) < \infty\}$, and let $f \in \widetilde{\mathcal{M}}$ be given such that $\Phi(|f|)$ is again τ-measurable. Then $\tau(\Phi(|f|)) = \int_0^\infty \Phi(\mathbf{m}_t(|f|))\,\mathrm{d}t$.*

PROOF. Apply Proposition 4.13 to part (vi) of Proposition 4.12. $\qquad\square$

We briefly comment on the use of the above corollary.

REMARK 4.15. Let $\mathcal{M}$ be a semifinite algebra with a faithful normal semifinite trace τ. Any element a of $\widehat{\mathcal{M}}_+$ is of course of the form $a = \int_0^\infty \lambda\,de(\lambda) + \infty.p$ for some spectral resolution $e(\lambda)$ and projection p. For

the extension of the trace to $\widehat{\mathcal{M}}_+$, we will by the discussion at the start of this section clearly have that $\tau(a) = \infty$ if $p \neq 0$. So if $\tau(a) < \infty$, we must have that $p = 0$, and hence that a is in fact an operator affiliated to $\mathcal{M}$. But more is true in this case. For any $n, m \in \mathbb{N}$ with $n < m$, we will then have that

$$\tau(e_{(n,m]}) \leq \frac{1}{n}\tau(ae_{(n,m]}) \leq \frac{1}{n}\tau(a) < \infty.$$

If now we let $m \to \infty$, we get that $\tau(e_{(n,\infty)}) \leq \frac{1}{n}\tau(a) < \infty$. Thus if $\tau(a) < \infty$, a must in fact correspond to a τ-measurable element of $\widetilde{\mathcal{M}}$. So given $a \in \widehat{\mathcal{M}}_+$, it follows that $\tau(a) < \infty$ if and only if a corresponds to a positive element of $\widetilde{\mathcal{M}}$ with finite trace.

Now given an element a of $\widetilde{\mathcal{M}}$ and a function $\Phi : [0, \infty) \to [0, \infty]$ of the type described in the preceding result, $\Phi(|a|)$ will in general not be a member of $\widetilde{\mathcal{M}}$ (unless of course $\mathrm{sp}(|a|) \subseteq [0, b_\Phi)$). However we are able to give meaning to $\Phi(|a|)$ as an element of $\widehat{\mathcal{M}}_+$. If we apply the observation made previously to this setting, then given $a \in \widetilde{\mathcal{M}}$, it follows that $\tau(\Phi(|a|)) < \infty$ if and only if $\Phi(|a|)$ corresponds to a positive element of $\widetilde{\mathcal{M}}$ with finite trace.

COROLLARY 4.16. *Let $f, g \in \widetilde{\mathcal{M}}_+$ be given. Then the following are equivalent:*

(i) *$\mathbf{m}_f(t) \leq \mathbf{m}_g(t)$ for all $t > 0$;*
(ii) *$d_f(s) \leq d_g(s)$ for all $s > 0$;*
(iii) *for any non-decreasing function $\Phi : [0, \infty) \to [0, \infty]$ which is continuous on $[0, b_\Phi]$ where $b_\Phi = \sup\{t \in [0, \infty): \Phi(t) < \infty\}$, we have that $\tau(\Phi(f)) \leq \tau(\Phi(g))$.*

PROOF. The implication (ii)$\Rightarrow$(i) is a direct consequence of Proposition 4.7, with (i)$\Rightarrow$(iii) following from the preceding Corollary and Remark. To prove the implication (iii)$\Rightarrow$(ii), we first select a sequence (v_n) of continuous functions on $[0, \infty)$ which increase monotonically to $\chi_{(s,\infty)}$. (In fact given any $\epsilon > 0$, we may select this sequence so that each v_n agrees with $\chi_{(s,\infty)}$ on $[0, s] \cup [s + \epsilon, \infty)$.) Then by the Borel functional calculus, $v_n(f) \nearrow \chi_{(s,\infty)}(f)$ and $v_n(g) \nearrow \chi_{(s,\infty)}(g)$ as $n \to \infty$. The normality of the trace therefore ensures that $\tau(v_n(f)) \nearrow d_f(s)$ and $\tau(v_n(g)) \nearrow d_g(s)$ as $n \to \infty$. The validity of (ii), now follows from the fact that by assumption $\tau(v_n(f)) \leq \tau(v_n(g))$ for every n. $\qquad\square$

As promised we now pause to briefly present a (fairly standard) sampling of the basic properties of the trace on $\widetilde{\mathcal{M}}_+$. The interested reader may easily use the techniques demonstrated in the following proof to deduce further properties.

PROPOSITION 4.17. *The extension of the trace τ to $\widetilde{\mathcal{M}}_+$, is faithful, additive, and positive homogeneous. For any $g \in \widetilde{\mathcal{M}}$ we have that $\tau(g^*g) = \tau(gg^*)$. In addition τ is normal on $\widetilde{\mathcal{M}}_+$, in the sense that if $(f_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ is a net increasing to $f \in \widetilde{\mathcal{M}}_+$, then $\tau(f) = \sup_\alpha \tau(f_\alpha)$.*

PROOF. The proof of each of these properties relies on a repeated use of Proposition 4.13. To see that τ is faithful, observe that if $0 = \tau(f) = \int_0^\infty \mathbf{m}_f(s)\,ds$, then $\mathbf{m}_f = 0$ almost everywhere. But since $\mathbf{m}_f$ is non-increasing, we have that $\mathbf{m}_f(s) = 0$ for all $s \geq s_0$ whenever $\mathbf{m}_f(s_0) = 0$. Thus $\mathbf{m}_f = 0$ on $(0, \infty)$, with the right continuity of $\mathbf{m}_f$, now ensuring that in addition $\|f\|_\infty = \lim_{s \searrow 0} \mathbf{m}_f(s) = 0$.

The positive homogeneity of the trace is a direct consequence of part (ii) of Proposition 4.12.

The fact that $\tau(g^*g) = \tau(gg^*)$ for any $g \in \widetilde{\mathcal{M}}$, is similarly an easy consequence of the fact that part (ii) of Proposition 4.12 ensures that $\mathbf{m}_{g^*g} = \mathbf{m}_{|g|}^2 = \mathbf{m}_{|g^*|}^2 = \mathbf{m}_{gg^*}$.

Next let $(f_\alpha) \subseteq \widetilde{\mathcal{M}}_+$ be a net increasing to $f \in \widetilde{\mathcal{M}}_+$. For each $n \in \mathbb{N}$, let $e_n = \chi_{[0,n]}(f)$. We know from Proposition 2.63 that $(e_n f_\alpha e_n)$ then increases to fe_n. But fe_n belongs to $\mathcal{M}_+$, and hence the same must be true of $\{e_n f_\alpha e_n\}$. It then follows from the normality of the trace on $\mathcal{M}_+$, that $\tau(fe_n) = \sup_\alpha \tau(e_n f_\alpha e_n)$. We next use part (v) of Proposition 4.12 to see that $\tau(e_n f_\alpha e_n) = \int_0^\infty \mathbf{m}_{e_n f_\alpha e_n}(s)\,ds \leq \int_0^\infty \mathbf{m}_{f_\alpha}(s)\,ds = \tau(f_\alpha)$ for each n and each α. But then $\tau(fe_n) \leq \sup_\alpha \tau(f_\alpha)$. By the definition of $\tau(f)$, we then have that $\tau(f) \leq \sup_\alpha \tau(f_\alpha)$. But by part (iii) of Proposition 4.12, we must have that $\tau(f_\alpha) \leq \tau(f)$ for each α. Hence we must have that $\tau(f) = \sup_\alpha \tau(f_\alpha)$.

We now prove the additivity of the extension of the trace. Given $f, g \in \widetilde{\mathcal{M}}_+$, the Borel functional calculus ensures that the sequences $(f_n), (g_n) \subseteq \mathcal{M}_+$ defined by $f_n = f\chi_{[0,n]}(f)$ and $g_n = g\chi_{[0,n]}(g)$ respectively, increase to f and g. Then $(f_n + g_n)$ will increase to $f + g$. So by what has been proved we have that $\tau(f + g) = \sup_{n \in \mathbb{N}} \tau(f_n + g_n) = \sup_{n \in \mathbb{N}}(\tau(f_n) + \tau(g_n)) = \tau(f) + \tau(g)$. $\qquad\square$

LEMMA 4.18. *Let $f \in \widetilde{\mathcal{M}}$ be given. Given any $\alpha > 0$ with $\alpha = d_f(s)$ for some $s > 0$, let $s_\alpha = \inf\{s \colon d_f(s) \leq \alpha\}$. Denoting $f\chi_{(s_\alpha, \infty)}(|f|)$ by f_α, we then have that $\mathbf{m}_{f_\alpha} = \mathbf{m}_f \chi_{[0, \alpha]}$.*

PROOF. Let α, s_α and f_α be as in the hypothesis. From the right-continuity of $s \mapsto d_f(s)$, we have that $d_f(s_\alpha) = \alpha$, with by construction $d_f(s) > \alpha$ whenever $s < s_\alpha$. It is clear that in computing the decreasing rearrangements we may pass to the abelian von Neumann subalgebra generated by the spectral projections of f. This then allows us access to the Borel functional calculus. Using this calculus, it is now an easy exercise to show that

$$\chi_{(s, \infty)}(f_\alpha) = \begin{cases} \chi_{(s, \infty)}(f) & \text{if } s \geq s_\alpha \\ \chi_{(s_\alpha, \infty)}(f) & \text{if } 0 < s < s_\alpha \end{cases}.$$

Therefore

$$d_{f_\alpha}(s) = \begin{cases} d_f(s) & \text{if } s \geq s_\alpha \\ d_f(s_\alpha) & \text{if } 0 < s < s_\alpha \end{cases}.$$

From these formulae it now clearly follows that if $t < \alpha$, then $\mathbf{m}_t(f_\alpha) = \inf\{s \colon d_{f_\alpha}(s) \leq t\} = \inf\{s \colon d_f(s) \leq t\} = \mathbf{m}_t(f)$, and similarly that if $t \geq \alpha$, then $\mathbf{m}_t(f_\alpha) = \inf\{s \colon d_{f_\alpha}(s) \leq t\} = 0$. □

The following result presents a very elegant way of realising what is sometimes called the "second" decreasing rearrangement in the context of "non-atomic" von Neumann algebras.

PROPOSITION 4.19. *Let $\mathcal{M}$ have no minimal projections. For any $f \in \widetilde{\mathcal{M}}$ and any $t > 0$, we then have that $\int_0^t \mathbf{m}_f(s)\, ds = \sup\{\tau(e|f|e) \colon e \in \mathbb{P}(\mathcal{M}), \tau(e) \leq t\}$*

PROOF. We may clearly assume that $f \geq 0$. For any $e \in \mathbb{P}(\mathcal{M})$ with $\tau(e) \leq t$, we may next apply Proposition 4.13 and Corollary 4.11 to see that $\tau(efe) = \int_0^t \mathbf{m}_{efe}(s)\, ds \leq \int_0^t \mathbf{m}_f(s)\, ds$. Clearly $\int_0^t \mathbf{m}_f(s)\, ds \geq \sup\{\tau(e|f|e) \colon e \in \mathbb{P}(\mathcal{M}), \tau(e) \leq t\}$.

To prove the converse inequality, we will consider two cases. Firstly let $s_0 = \inf\{s \geq 0 \colon d_f(s) < \infty\}$, and $\alpha_0 = d_f(s_0)$.

Case 1 $(t \geq \alpha_0)$: In this case we clearly have that $\alpha_0 < \infty$. Since $d_f(s) = \infty$ for all $s < s_0$, it is now also clear that

$$\mathbf{m}_f(v) = \inf\{s > 0 \colon d_f(s) \leq v\} = s_0 \text{ for all } v \geq \alpha_0.$$

It then follows that

$$\int_0^t \mathbf{m}_f(v)\,dv = \int_0^{\alpha_0} \mathbf{m}_f(v)\,dv + (t - \alpha_0)s_0.$$

If in fact $t = \alpha_0$ then for $e_0 = \chi_{(s_0,\infty)}(f)$, it follows from the lemma that

$$\tau(fe_0) = \int_0^\infty \mathbf{m}_v(fe_0)\,dv = \int_0^{\alpha_0} \mathbf{m}_v(f)\,dv.$$

Therefore in this case $\int_0^t \mathbf{m}_f(s)\,ds \le \sup\{\tau(e|f|e)\colon e \in \mathbb{P}(\mathcal{M}), \tau(e) \le t\}$ as required.

Next consider the case where $t > \alpha_0$. Let $\epsilon > 0$ be given. By construction $\tau(\chi_{(s_0-\epsilon,\infty)}(f)) = d_f(s_0-\epsilon) = \infty$. Since by assumption $\tau(\chi_{(s_0,\infty)}(f)) = \alpha_0 < \infty$, we must therefore have that $\tau(\chi_{(s_0-\epsilon,s_0]}(f)) = \infty$. The fact that $\mathcal{M}$ has no minimal projections, now ensures that there must exist a subprojection e_t of $\chi_{(s_0-\epsilon,s_0]}(f)$ with $\tau(e_t) = (t - \alpha_0)$. The easiest way to see this is to use Proposition 4.9 to select an abelian subalgebra of $\chi_{(s_0-\epsilon,s_0]}(f)\mathcal{M}\chi_{(s_0-\epsilon,s_0]}(f)$ with no minimal projections. Proposition 4.9 then informs us that this abelian subalgebra corresponds to a classical L^∞-space living on a nonatomic measure space. The existence of a subprojection e_t for which $\tau(e_t) = (t - \alpha_0)$, therefore follows from classical measure theory. By construction e_0 and e_t are mutually orthogonal, with $e_0 + e_t$ therefore a projection with $\tau(e_0 + e_t) = t$, and with

$$
\begin{aligned}
\tau(f(e_0 + e_t)) &= \tau(fe_0) + \tau(fe_t) \\
&\ge \int_0^{\alpha_0} \mathbf{m}_f(v)\,dv + (s_0 - \epsilon)\tau(e_t) \\
&= \int_0^{\alpha_0} \mathbf{m}_f(v)\,dv + (s_0 - \epsilon)(t - \alpha_0) \\
&= \int_0^t \mathbf{m}_f(v)\,dv - \epsilon(t - \alpha_0).
\end{aligned}
$$

Since $\epsilon > 0$ was arbitrary, it is clear that here too

$$\int_0^t \mathbf{m}_f(s)\,ds \le \sup\{\tau(|f|e)\colon e \in \mathbb{P}(\mathcal{M}), \tau(e) \le t\}$$

as required.

Case 2 $(t < \alpha_0)$: Let $s_t = \inf\{s > 0\colon d_f(s) \le t\}$, and set $\alpha_t = d_f(s_t)$. By the right-continuity of $s \mapsto d_f(s)$ we must have $\alpha_t = d_f(s_t) \le t$, with $d_f(s) > t$ for all $s < s_t$ by construction.

If in fact $t = \alpha_t$ then we may set $e_1 = \chi_{(s_t, \infty)}(f)$, and argue as before to see $\tau(fe_1) = \int_0^t \mathbf{m}_v(f)\,dv$, from which it then follows that $\int_0^t \mathbf{m}_f(s)\,ds \le \sup\{\tau(|f|e) : e \in \mathbb{P}(\mathcal{M}), \tau(e) \le t\}$ as required.

Next consider the case where $t > \alpha_t$. Despite the similarity of this proof to the earlier case, we will for the sake of the reader provide relevant details. Since for all $s < s_t$ we have by construction that $d_f(s) > t$, with $d_f(s) \le \alpha_t$ otherwise, it is clear that for any $\alpha_t < v \le t$, we must have that $\mathbf{m}_f(v) = \inf\{s : d_f(s) \le v\} = s_t$. Therefore $\int_0^t \mathbf{m}_f(v)\,dv = \int_0^{\alpha_t} \mathbf{m}_f(v)\,dv + (t - \alpha_t)s_t$. Let $\epsilon > 0$ be given. By construction $\tau(\chi_{(s_t - \epsilon, \infty)}(f)) = d_f(t) > t$, and $\tau(\chi_{(s_t, \infty)}(f)) = \alpha_t$. Hence $\tau(\chi_{(s_t - \epsilon, s_t]}(f)) > t - \alpha_t$. As before, the fact that $\mathcal{M}$ has no minimal projections, ensures that there must exist a subprojection e_t of $\chi_{(s_t - \epsilon, s_t]}(f)$ with $\tau(e_t) = (t - \alpha_t)$. Let $e_1 = \chi_{(s_t, \infty)}(f)$. Arguing as in the former case it now follows that $e_1 + e_t$ is a projection with $\tau(e_1 + e_t) = t$, and with $\tau(f(e_1 + e_t)) \ge \int_0^{\alpha_0} \mathbf{m}_v(f)\,dv + (s_0 - \epsilon)(t - \alpha_0)$. Since $\epsilon > 0$ was arbitrary, it is clear that $\int_0^t \mathbf{m}_f(s)\,ds \le \sup\{\tau(e|f|e) : e \in \mathbb{P}(\mathcal{M}), \tau(e) \le t\}$ as required. $\square$

The applicability of the above Proposition ranges wider than just von Neumann algebras with no minimal projections. This follows from the following observation.

PROPOSITION 4.20. *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with an f.n.s. trace τ. Then the trace $\tau_\infty = \tau \otimes \int_{[0,1]}(\cdot)d\mathbb{m}(t)$ (where $\mathbb{m}$ is Lebesgue measure) is an f.n.s. trace on the von Neumann algebra tensor product $\mathcal{M}\overline{\otimes}L^\infty[0,1]$. The algebra $\mathcal{M}\overline{\otimes}L^\infty[0,1]$ has no minimal projections, and if we use the trace τ_∞ to define a topology of convergence in measure on $\mathcal{M}\overline{\otimes}L^\infty[0,1]$, the $*$-isomorphism $\mathscr{I}_\infty : \mathcal{M} \to \mathcal{M}\overline{\otimes}L^\infty[0,1] : a \mapsto a \otimes 1$ extends to a homeomorphism between $\widetilde{\mathcal{M}}$, and the closure of $\mathscr{I}_\infty(\mathcal{M})$ in the topology of convergence in measure on $\mathcal{M}\overline{\otimes}L^\infty[0,1]$. Specifically for any $a \in \widetilde{\mathcal{M}}$, the simple tensor $a \otimes 1$ will then be τ_∞-measurable, with $\mathbf{m}_a(t) = \mathbf{m}_{a\otimes1}(t)$ for any $t \ge 0$.*

We pause to point out that as far as $\widetilde{\mathcal{M}} \otimes L^\infty[0,1]$ is concerned one may certainly make sense of the algebraic tensor product. Simple tensors of the form $a \otimes 1$ where $a \in \widetilde{\mathcal{M}}$ certainly belong to this algebraic tensor product. However since $\widetilde{\mathcal{M}}$ is in general not locally convex, there is no natural metric or topological structure we can equip this tensor product with. The action of τ_∞ on simple tensors of the form $a \otimes 1$ is therefore derived from the fact that these tensors may also be realised as elements

of $\mathcal{M}\overline{\otimes}\widetilde{L^\infty}[0,1]$. To see this notice that given $(a_n) \subset \mathcal{M}$ such that $a_n \to a$ in $\widetilde{\mathcal{M}}$, we will then have that $(a_n \otimes 1) \to (a \otimes 1)$ in $\mathcal{M}\overline{\otimes}\widetilde{L^\infty}[0,1]$. There is a theory of tensor products of locally convex algebras of unbounded operators (see for example [**FIW14**]), but these algebras are typically not all that closely related to $\widetilde{\mathcal{M}}$.

PROOF. We leave the claim that τ_∞ is a faithful normal semifinite trace as an exercise. The claim that $\mathscr{I}_\infty$ extends to a homeomorphism between $\widetilde{\mathcal{M}}$, and the closure of $\mathscr{I}_\infty(\mathcal{M})$ in the topology of convergence in measure, is a direct consequence of Proposition 2.73 applied to the fact that $\tau_\infty \circ \mathscr{I}_\infty = \tau$. Given $a \in \widetilde{\mathcal{M}}$, the fact that $\mathbf{m}_a(t) = \mathbf{m}_{a\otimes 1}(t)$ for all $t \geq 0$, will easily follow if we are able to show that $d_a(t) = d_{a\otimes 1}(t)$ for all $t \geq 0$. To see this one may check that $|a \otimes 1| = |a| \otimes 1$, and that $p(|a\otimes 1|) = p(|a|)\otimes 1$ for any polynomial in one real variable. Now use the Stone-Weierstrass theorem to see that $f(|a\otimes 1|) = f(|a|)\otimes 1$ holds for non-negative continuous functions f on $[0,\infty)$, and then use that fact to deduce that this equality also holds for non-negative Borel functions on $[0,\infty)$. Given $t \geq 0$ it then trivially follows that $d_{a\otimes 1}(t) = \tau_\infty(\chi_{(t,\infty)}(|a \otimes 1|)) = \tau_\infty(\chi_{(t,\infty)}(|a| \otimes 1)) = \tau_\infty((\chi_{(t,\infty)}(|a|)) \otimes 1) = \tau(\chi_{(t,\infty)}(|a|)) = d_a(t)$. It remains to prove that $\mathcal{M}\overline{\otimes}L^\infty[0,1]$ has no minimal projections. So let $p \in \mathbb{P}(\mathcal{M}\overline{\otimes}L^\infty[0,1])$ be given, and consider the set $\mathcal{P} = \{e \in \mathbb{P}(L^\infty[0,1]) \colon p \leq 1\otimes e\}$. We need to find a non-zero subprojection of p, which is distinct from p. Let $e_m = \wedge\{e \colon e \in \mathcal{P}\}$. The measure space $([0,1], \mathscr{B}([0,1]))$ is of course non-atomic under Lebesgue measure, and hence we may find a non-zero projection $e_0 \in \mathbb{P}(L^\infty[0,1])$ for which we have that $e_0 \neq e_m$ and $e_0 \leq e_m$. It is not difficult to see that $1\otimes e_0$ is a central element of $\mathcal{M}\overline{\otimes}L^\infty[0,1]$. Hence $(1\otimes e_0)p$ is a subprojection of p. Note that we must have that $(1\otimes e_0)p \neq p$, since equality would ensure that $p \leq (1 \otimes e_0)$, which would contradict the minimality of e_m. If now we can show that $(1 \otimes e_0)p \neq 0$, this projection would be the subprojection of p we seek. Assume by way of contradiction that $(1\otimes e_0)p = 0$. That ensures that $p \leq (1\otimes 1) - (1\otimes e_0) = 1\otimes(1 - e_0)$, and hence that $(1 - e_0) \in \mathcal{P}$. The minimality of e_m, then ensures that $e_m \leq (1-e_0)$. Since by assumption $e_0 \leq e_m$, the only way both statements can be true, is if $e_0 = 0$, which contradicts the assumption that $e_0 \neq 0$. Thus as required, $(1 \otimes e_0)p \neq 0$. $\qquad\square$

The next result and the one that follows it, will prove to be very useful when we get to the analysis of Orlicz spaces.

THEOREM 4.21 ([**HLP29**]). *Let f, g be non-negative Borel measurable functions on $[0, \infty)$ which are finite almost everywhere. Then the following are equivalent:*

(i) *$\int_0^t f(s)\, ds \leq \int_0^t g(s)\, ds$ for all $t > 0$;*

(ii) *any non-negative, non-decreasing, convex function $\Psi : [0, \infty) \to [0, \infty]$, which is neither identically zero nor infinite valued on $(0, \infty)$, and which is continuous on $[0, b_\Psi]$, satisfies the inequality $\int_0^t \Psi(f(s))\, ds \leq \int_0^t \Psi(g(s))\, ds$ for all $t > 0$. (Note that Ψ may be infinite-valued at b_Ψ.)*

PROOF. Hardy, Littlewood and Polya [**HLP29**] (see also [**HLP88**, §249]) proved that on a bounded interval $[0, a]$, one has that $\int_0^t f(s)\, ds \leq \int_0^t g(s)\, ds$ for all $t \in [0, a]$ if and only if $\int_0^a \Psi(f(s))\, ds \leq \int_0^a \Psi(g(s))\, ds$ for any convex and continuous function Ψ on $[0, a]$. (They actually used a different criterion to (i) above, but their criterion can be seen to be equivalent to the one stated here. See Exercise 1 of [**BS88**, Chapter 2].) However for our purposes we need to know that the claims in [**HLP29**] hold for functions on the half-line. The thrust of the proof will then be the verification of this extension.

First assume that Ψ is finite-valued on all of $[0, \infty)$. For any fixed $r > 0$, criterion (i) of course holds for any $0 < t \leq r$. So by the Hardy, Littlewood, Polya result applied to the interval $[0, r]$, we have $\int_0^r \Psi(f(s))\, ds \leq \int_0^r \Psi(g(s))\, ds$. Since $r > 0$ was arbitrary, we are done. It now remains to show that the (ii) still holds if Ψ is infinite-valued on part of $(0, \infty)$. We may therefore pass to the case where $b_\Psi < \infty$. Observe that on replacing Ψ with $t \mapsto \Psi(t) - \Psi(0)$, we may without loss of generality assume that $\Psi(0) = 0$. Since Ψ is convex, we know that it is left-differentiable at all points of $(0, b_\Psi)$. In fact this left-derivative ψ turns out to be a left-continuous non-negative non-decreasing function on $[0, b_\Phi)$. On defining ψ to be infinite valued on (b_Ψ, ∞), it follows that $\Phi(t) = \int_0^t \psi(s)\, ds$ for all $t > 0$ (see §1.3 & 1.6 of [**NP06**]). Now select a sequence $(t_n) \subseteq (0, b_\Psi)$ increasing to b_Ψ, and define ψ_n by

$$\psi_n(t) = \begin{cases} \psi(t) & \text{for all } 0 \leq t \leq t_n \\ \psi(t_n) & \text{for all } t_n \leq t \leq b_\Psi \\ \exp(n(t - b_\Psi))\psi(t_n) & \text{for all } t > b_\Psi \end{cases}$$

On setting $\Psi_n(t) = \int_0^t \psi_n(s)\, ds$ it is an exercise to see that each Ψ_n is finite-valued on all of $(0, \infty)$. From what we have already proved, we then

have that $\int_0^t \Psi_n(f(s))\,ds \leq \int_0^t \Psi_n(g(s))\,ds$ for all $t > 0$ and all n. Now observe that for any fixed $t > 0$, the sequence $(\psi_n(t))$ will by construction monotonically increase to $\psi(t)$. We may therefore use the monotone convergence theorem to see that $\lim_{n\to\infty} \Psi_n(t) = \lim_{n\to\infty} \int_0^t \psi_n(s)\,ds = \int_0^t \psi(s)\,ds = \Psi(t)$. Since by construction the sequence (Ψ_n) is itself increasing, we may now once again use the monotone convergence theorem to see that $\int_0^t \Psi(f(s))\,ds = \lim_{n\to\infty} \int_0^t \Psi_n(f(s))\,ds \leq \lim_{n\to\infty} \int_0^t \Psi_n(g(s))\,ds = \int_0^t \Psi(g(s))\,ds$ for any $t > 0$. Thus (ii) holds in general. $\qquad\square$

4.4 Integral inequalities and Monotone Convergence Theorem

The proof of the following result illustrates how the reduction described in the Proposition 4.20 may be used in practice.

THEOREM 4.22. *Let $a, b \in \widetilde{\mathcal{M}}$ be given and let Φ be a convex function Φ of the type described in the preceding theorem. Then the following holds:*

- *$\int_0^t \mathbf{m}_{a+b}(s)\,ds \leq \int_0^t \mathbf{m}_a(s) + \int_0^t \mathbf{m}_b(s)$ for any $t > 0$;*
- *for any convex function Φ of the type described in the preceding theorem, we have that*

$$\int_0^t \Phi(\mathbf{m}_{a+b}(s))\,ds \leq \int_0^t \Phi(\mathbf{m}_a(s) + \mathbf{m}_b(s))\,ds \text{ for all } t \in (0, \infty].$$

PROOF. We observe that for $t \in (0, \infty)$ the second claim will follow from the first on setting $f(s) = \mathbf{m}_{a+b}(s)\chi_{[0,t]}(s)$ and $g(s) = (\mathbf{m}_a(s) + \mathbf{m}_b(s))\chi_{[0,t]}(s)$ in the preceding theorem. The case $t = \infty$ will then follow by letting $t \to \infty$. It therefore remains to prove the first claim. It is clear from Proposition 4.20 that we may assume that $\mathcal{M}$ has no minimal projections. This observation then gives us access to Proposition 4.19. With this fact in mind, let $e \in \mathcal{M}$ be a projection with $\tau(e) \leq t$. Now use Lemma 4.2 to select partial isometries u and v so that $|f + g| \leq u|f|u^* + v|g|v^*$.

We may now apply Proposition 4.19 to see that

$$\begin{aligned}
\tau(e|f+g|e) &\leq \tau(eu|f|u^*e) + \tau(ev|g|v^*e) \\
&\leq \int_0^t \mathbf{m}_{u|f|u^*}(s)\,ds + \int_0^t \mathbf{m}_{v|g|v^*}(s)\,ds \\
&\leq \int_0^t \mathbf{m}_f(s)\,ds + \int_0^t \mathbf{m}_g(s)\,ds.
\end{aligned}$$

To conclude the proof we once again use Proposition 4.19 to see that taking the supremum over all projections e with $\tau(e) \leq t$, yields the required conclusion. $\qquad\square$

We are now ready to describe the topology of convergence in measure in terms of decreasing rearrangements.

PROPOSITION 4.23. *The basic neighbourhoods of 0 of the topology of convergence in measure on* $\widetilde{\mathcal{M}}$, *are given by* $\mathscr{N}(\epsilon, \delta) = \{g \in \widetilde{\mathcal{M}} \colon \mathbf{m}_g(\delta) \leq \epsilon\}$ *where* $\epsilon, \delta > 0$. *Hence given a net* $(f_\alpha) \subseteq \widetilde{\mathcal{M}}$ *and* $f \in \widetilde{\mathcal{M}}$, *we have that* $f_\alpha \to f$ *in the topology of convergence in measure if and only if* $\mathbf{m}_{f - f_\alpha}(t) \to 0$ *for every* $t > 0$.

PROOF. If $g \in \mathscr{N}(\epsilon, \delta)$ then by definition there exists a projection $e \in \mathcal{M}$ such that $\tau(\mathbb{1} - e) \leq \delta$ and $\|fe\| \leq \epsilon$. But then $\mathbf{m}_g(\delta) \leq \epsilon$ by Definition 4.1.

Conversely suppose that $\mathbf{m}_g(\delta) \leq \epsilon$, and consider the projection $e = \chi_{[0, \rho_\delta]}(|g|)$, where $\rho_\delta = \mathbf{m}_g(\delta)$. We then clearly have that $\|ge\| = \|, |g|e\| \leq \mathbf{m}_g(\delta) \leq \epsilon$. It moreover follows from Proposition 4.7 that the projection e satisfies $\tau(\mathbb{1} - e) = \tau(\chi_{(\rho_\delta, \infty)}(|g|)) = d_g(\mathbf{m}_g(\delta)) \leq \delta$. But then $g \in \mathscr{N}(\epsilon, \delta)$ as required. $\qquad\square$

PROPOSITION 4.24. *Let* $f \in \widetilde{\mathcal{M}}$ *be given. Then the following are equivalent:*

   (i) $d_f(s) < \infty$ *for all* $s > 0$;
   (ii) $\mathbf{m}_f(t) \to 0$ *as* $t \to \infty$;
   (iii) *there exists a sequence* $(f_n) \subseteq \widetilde{\mathcal{M}}$ *with* $\tau(\mathbb{s}(f_n)) < \infty$ *for each* n, *converging to* f *in the topology of convergence in measure.*

PROOF. (i) $\Rightarrow$ (iii) : Let $f \in \widetilde{\mathcal{M}}$ be given with $d_f(s) < \infty$ for all $s > 0$, and let $|f| = \int_0^\infty \lambda \, de_\lambda(|f|)$ be the spectral resolution of $|f|$. If $f = u|f|$ is the polar decomposition of f, then clearly $f = u \int_0^\infty \lambda \, de_\lambda(|f|)$. Now set $f_n = u \int_{1/n}^n \lambda \, de_\lambda(|f|)$ for each $n \in \mathbb{N}$. For each n, the left support projection of f_n is just $\chi_{[1/n, n]}(|f|)$. By assumption $\tau(\chi_{(s, \infty)}(|f|)) < \infty$ for any $0 < s < 1/n$, and hence $\tau(\chi_{[1/n, n]}(|f|)) < \infty$ as required.

Next set $e_n = \chi_{(n, \infty)}(|f|)$. The τ-measurability of f ensures that $\tau(e_n) = d_f(n) \to 0$ as $n \to \infty$. In addition

$$\|(f - f_n)(\mathbb{1} - e_n)\| = \left\| u \int_0^{1/n} \lambda \, de_\lambda(|f|) \right\| \leq \frac{1}{n}.$$

Hence as required, (f_n) converges to f in measure.

(**iii**) $\Rightarrow$ (**ii**) : Assume that (iii) holds. Given $\epsilon > 0$, we may then apply Proposition 4.23 to select n_0 so that $\mathbf{m}_{f-f_n}(1) < \epsilon$ for all $n \geq n_0$. Since by Corollary 4.11 $\mathbf{m}_{f_{n_0}}(t) = 0$ for all $t > \tau(\mathbf{s}(f_{n_0}))$, it now follows that

$$\mathbf{m}_f(t) \leq \mathbf{m}_{f_{n_0}}(t-1) + \mathbf{m}_{f-f_{n_0}}(1) \leq \epsilon \text{ for all } t > \tau(\mathbf{s}(f_{n_0})) + 1.$$

Thus (ii) follows.

(**ii**) $\Rightarrow$ (**i**) : Suppose that (ii) holds, and let $\epsilon > 0$ be given. Select $t_0 > 0$ such that $\mathbf{m}_f(t_0) \leq \epsilon$. Using the fact that $s \mapsto d_f(s)$ is non-increasing, we may then conclude from Proposition 4.7 that $d_f(\epsilon) \leq d_f(\mathbf{m}_f(t_0)) \leq t_0 < \infty$. $\qquad\square$

The following useful Fatou-like lemma was proved in [**Kos84b**].

LEMMA 4.25. *Let $(f_n) \subseteq \widetilde{\mathcal{M}}$ be a sequence converging to $f \in \widetilde{\mathcal{M}}$ in the topology of convergence in measure. Then*

(i) $\mathbf{m}_f(t) \leq \liminf_{n\to\infty} \mathbf{m}_{f_n}(t)$ *for all $t > 0$;*

(ii) $\mathbf{m}_f(t) = \lim_{n\to\infty} \mathbf{m}_{f_n}(t)$ *if either $s \mapsto \mathbf{m}_f(s)$ is continuous at $s = t$, or $\mathbf{m}_{f_n} \leq \mathbf{m}_f$ for all n.*

PROOF. (i): Let $\epsilon > 0$ be given. On observing that $\mathbf{m}_f(t + \epsilon) \leq \mathbf{m}_{f_n}(t) + \mathbf{m}_{f-f_n}(\epsilon)$, we may apply Proposition 4.23 to see that $\mathbf{m}_f(t+\epsilon) \leq \liminf_{n\to\infty} \mathbf{m}_{f_n}(t)$. Since this holds for any $\epsilon > 0$, it follows from the right-continuity of $t \mapsto \mathbf{m}_f(t)$ that $\mathbf{m}_f(t) \leq \liminf_{n\to\infty} \mathbf{m}_{f_n}(t)$ as required.

(ii): Given $t > 0$ select $\epsilon > 0$ with $\epsilon < t$. Arguing as before we firstly note that $\mathbf{m}_{f_n}(t) \leq \mathbf{m}_f(t-\epsilon) + \mathbf{m}_{f_n-f}(\epsilon)$, and then use Proposition 4.23 to conclude from this that $\limsup_{n\to\infty} \mathbf{m}_{f_n}(t) \leq \mathbf{m}_f(t-\epsilon)$. If indeed $s \mapsto \mathbf{m}_f(s)$ is continuous at $s = t$, letting ϵ decrease to 0 yields the conclusion that $\limsup_{n\to\infty} \mathbf{m}_{f_n}(t) \leq \mathbf{m}_f(t)$. On combining this inequality with what we have proved in part (i), it follows that $\lim_{n\to\infty} \mathbf{m}_{f_n}(t)$ exists and equals $\mathbf{m}_f(t)$. The second claim easily follows from part (i). $\qquad\square$

The above Fatou-lemma yields the following analogue of the Monotone Convergence theorem.

THEOREM 4.26. *Let $(f_n) \subseteq \widetilde{\mathcal{M}}$ be a sequence of positive operators converging to $f \in \widetilde{\mathcal{M}}$ in the topology of convergence in measure. Then*

(i) *(Fatou's lemma) $\tau(f) \leq \liminf_{n\to\infty} \tau(f_n)$;*

(ii) *(Monotone convergence)* $\tau(f) = \lim_{n\to\infty} \tau(f_n)$ *whenever* $\mathbf{m}_{f_n} \leq \mathbf{m}_f$ *for all* n.

PROOF. (i): We may use the above lemma alongside the usual Fatou's lemma to conclude that

$$
\begin{aligned}
\tau(f) &= \int_0^\infty \mathbf{m}_f(t)\, dt \\
&\leq \int_0^\infty \liminf_{n\to\infty} \mathbf{m}_{f_n}(t)\, dt \\
&\leq \liminf_{n\to\infty} \int_0^\infty \mathbf{m}_{f_n}(t)\, dt \\
&= \liminf_{n\to\infty} \tau(f_n).
\end{aligned}
$$

(ii): The fact that $\tau(f_n) = \int_0^\infty \mathbf{m}_{f_n}(t)\, dt \leq \int_0^\infty \mathbf{m}_f(t)\, dt = \tau(f)$ for each $n \in \mathbb{N}$, ensures that $\limsup_{n\to\infty} \tau(f_n) \leq \tau(f)$. Considered alongside part (i), this inequality in turn ensures that $\lim_{n\to\infty} \tau(f_n)$ exists and equals $\tau(f)$. $\qquad\square$

L^p and Orlicz spaces for semifinite algebras

We will here use the technology of decreasing rearrangement to develop a comprehensive theory of L^p-spaces for tracial semifinite algebras, before finally indicating how that theory may be extended to allow for a theory of noncommutative Orlicz spaces.

5.1 L^p-spaces for von Neumann algebras with a trace

5.1.1 Definition and (p-)normability

We start with

DEFINITION 5.1. Given $0 < p < \infty$, we define the space $L^p(\mathcal{M}, \tau)$ to be the set of all $f \in \widetilde{\mathcal{M}}$ satisfying $\tau(|f|^p) < \infty$. The (p-)norm on such an L^p is defined to be $\|f\|_p = \tau(|f|^p)^{1/p}$. In the case $p = \infty$, we define $L^\infty(\mathcal{M}, \tau)$ to be $\mathcal{M}$ itself.

Our first task is to show that each $L^p(\mathcal{M}, \tau)$ is a well-defined Banach space when $p \geq 1$, and a complete p-normed space when $0 < p < 1$. For the case $p = \infty$, there is of course nothing to prove. We start our analysis by proving a very general noncommutative version of Hölder's inequality. Here the Weyl-type inequality established in part (ii) of the following theorem, proves to be crucial. Given $f \in \widetilde{\mathcal{M}}$, we formally define the quantity $\Delta_t(f)$ by

$$\Delta_t(f) = \exp\left(\int_0^t \log(\mathbf{m}_f(s))\, ds\right) \qquad t > 0.$$

In order to ensure the well-definiteness of this quantity, we will restrict our analysis of this quantity to the class of all $f \in \widetilde{\mathcal{M}}$ for which there exists some $\alpha > 0$ and $C > 0$ for which $\mathbf{m}_f(t) < Ct^{-\alpha}$ for all $t > 0$. We write $\mathscr{L}(\widetilde{\mathcal{M}})$ for this class. It is an interesting exercise to use Proposition 4.12 to prove that this class is closed under finite sums and products. In

addition any $f \in \widetilde{\mathcal{M}}$ for which $\tau(|f|^p) < \infty$ for some $0 < p < \infty$, is in this class. To see this, observe that for such an f we have that

$$\mathbf{m}_f(t) = (\mathbf{m}_{|f|^p}(t))^{1/p} \leq \left(\frac{1}{t} \int_0^t \mathbf{m}_{|f|^p}(s) \, ds \right)^{1/p}$$

$$\leq t^{-1/p} \left(\int_0^\infty \mathbf{m}_{|f|^p}(s) \, ds \right)^{1/p} = t^{-1/p} \tau(|f|^p)^{1/p}$$

for all $t > 0$.

THEOREM 5.2. *Let $f, g \in \widetilde{\mathcal{M}}$ be given.*

(i) *Let $p, q, r > 0$ be given. If $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\|fg\|_r \leq \|f\|_p \|g\|_q$.*

(ii) *If in fact $f, g \in \mathscr{L}(\widetilde{\mathcal{M}})$, we have that $\Delta_t(fg) \leq \Delta_t(f)\Delta_t(g)$ for any $t > 0$.*

(iii) *If $F : [0, \infty) \to \mathbb{R}$ is a function for which $F \circ \exp$ is continuous, convex and non-decreasing, we have that*

$$\int_0^t F(\mathbf{m}_{fg}(s)) \, ds \leq \int_0^t F(\mathbf{m}_f(s)\mathbf{m}_g(s)) \, ds$$

for any $f, g \in \mathscr{L}(\widetilde{\mathcal{M}})$.

We will first prove (ii) for the case where $f, g \in \mathcal{M}$, then show how the general version of (ii) may be deduced from this case, before finally deducing the general versions of (i) and (iii) from (ii). We will need the following technical result regarding the holomorphic functional calculus. A proof may be found in [**FK52**].

LEMMA 5.3. *Let $\mathcal{M}$ be a von Neumann algebra equipped with a finite faithful normal trace. Let F be an analytic function defined on some domain Λ in the complex plane, bounded by a curve Γ, and let $t \mapsto x(t)$ $(0 \leq t \leq 1)$ be a norm-differentiable family of invertible operators in $\mathcal{M}$, such that the spectrum of each $x(t)$ lies in Λ. Then $F(x(t))$ is differentiable with respect to t on $(0, 1)$, and*

$$\tau \left(\frac{d}{dt} F(x(t)) \right) = \tau \left(\frac{dF(x(t))}{dt} \cdot x'(t) \right).$$

We gather some technical facts regarding rearrangements, before starting the actual proof of Theorem 5.2.

LEMMA 5.4. *Let $f \in \widetilde{\mathcal{M}}$ be given.*

(i) *If $f \geq 0$, then $\mathbf{m}_{f+\epsilon}(t) = \mathbf{m}_f(t) + \epsilon$ for any $t < \tau(\mathbb{1})$ and any $\epsilon > 0$.*

(ii) *Given a projection $e \in \mathcal{M}$ with $\tau(e) < \infty$, we have that $\mathbf{m}^e_{|fe|} = \mathbf{m}_{|fe|}$, where $\mathbf{m}^e$ denotes the decreasing rearrangement of $|fe|$ with respect to the von Neumann algebra $e\mathcal{M}e$.*

PROOF. *Part (i):* Given $s \geq 0$, it follows from the Borel functional calculus that $\chi_{(s+\epsilon,\infty)}(f + \epsilon\mathbb{1}) = \chi_{(s,\infty)}(f)$. It then clearly follows that $d_f(s) = d_{(f+\epsilon\mathbb{1})}(s + \epsilon)$ for all $s > 0$, with $d_{(f+\epsilon\mathbb{1})}(r) = \tau(\mathbb{1})$ if $0 < r < \epsilon$. Therefore in the case where $0 < t < \tau(\mathbb{1})$, we will have that

$$
\begin{aligned}
\mathbf{m}_f(t) &= \inf\{s > 0 \colon d_f(s) \leq t\} \\
&= \inf\{s > 0 \colon d_{(f+\epsilon\mathbb{1})}(s + \epsilon) \leq t\} \\
&= \inf\{r > \epsilon \colon d_{(f+\epsilon\mathbb{1})}(r) \leq t\} - \epsilon \qquad (\text{set } r = s + \epsilon) \\
&= \inf\{r > 0 \colon d_{(f+\epsilon\mathbb{1})}(r) \leq t\} - \epsilon \\
&= \mathbf{m}_{(f+\epsilon\mathbb{1})}(t) - \epsilon.
\end{aligned}
$$

Part (ii): Observe that $|fe|$ commutes with e, and that $|fe|(\mathbb{1}-e) = 0$. If now we apply the Borel functional calculus to the commutative von Neumann algebra generated by e and the spectral projections of $|fe|$, it follows that $\chi_{(s,\infty)}(|fe|) \leq e$ for all $s > 0$. In other words each such $\chi_{(s,\infty)}(|fe|)$ belongs to $e\mathcal{M}e$. We therefore clearly have that $d^e_{|fe|}(s) = d_{|fe|}(s)$ for all $s > 0$, where $d^e_{|fe|}$ denotes the distribution computed with respect to the compression $e\mathcal{M}e$. This fact suffices to prove the claim. $\square$

LEMMA 5.5. *Let $f \in \mathscr{L}(\widetilde{\mathcal{M}})$ be given with $f \geq 0$. If $\mathcal{M}$ has no minimal projections, we may select an abelian von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ so that*

- *$\mathcal{M}_1$ contains all the spectral projections of f,*
- *τ is semifinite on $\mathcal{M}_1$, and*
- *$\mathcal{M}_1$ also has no minimal projections.*

PROOF. Let $f \in \mathscr{L}(\widetilde{\mathcal{M}})$ be given with $f \geq 0$. By definition of the class $\mathscr{L}(\widetilde{\mathcal{M}})$ we can find $C > 0$ and $\alpha > 0$ so that $\mathbf{m}_f(t) < Ct^{-\alpha}$ for all $t > 0$. But then each $\chi_{(s,\infty)}(f)$ $(s > 0)$ must have finite trace by Proposition 4.24. Write e_0 for $\chi_{(0,\infty)}(f)$. The trace τ must therefore be semifinite on the commutative subalgebra of $e_0\mathcal{M}e_0$ containing these spectral projections. It is a fairly straighforward exercise to see that the trace will still be semifinite on any randomly selected maximal abelian subalgebra $\mathcal{M}_0$ of

$e_0 \mathcal{M} e_0$, containing the former commutative subalgebra. (This follows from the fact that for any non-zero subprojection e of e_0, there must be some $s > 0$ for which the subprojection $e \wedge \chi_{(s,\infty)}(f)$ of e is non-zero.) In addition $\mathcal{M}_0$ has no minimal projections by Proposition 4.9.

Finally observe that we may use part (c) of Proposition 4.9, to select an abelian subalgebra $\mathcal{M}_c$ of $(\mathbb{1} - e_0)\mathcal{M}(\mathbb{1} - e_0)$ on which τ is still semifinite. The algebra we seek is then $\mathcal{M}_1 = \mathcal{M}_0 \oplus \mathcal{M}_c$. $\qquad\square$

PROOF OF THEOREM 5.2. The first fact we note, is that we may without loss of generality assume that $\mathcal{M}$ has no minimal projections (see Proposition 4.20.)

Phase 1 of the proof of part (ii) (bounded case): Let $f, g \in \mathcal{M}$ be given. *We first consider the case where $f, g \geq 0$, and $\tau(\mathbb{1}) < \infty$.* For each $t \in [0, 1]$ set

$$x(t) = \exp(t \log(g + \epsilon\mathbb{1}))^{\cdot}(f + \epsilon\mathbb{1})^{2 \cdot} \exp(t \log(g + \epsilon\mathbb{1})),$$

where $\epsilon > 0$ is arbitrary. We may then apply Lemma 5.3 to see that

$$
\begin{aligned}
\tau\left(\frac{d}{dt} \log(x(t))\right) &= \tau(x(t)^{-1} x'(t)) \\
&= \tau(x(t)^{-1}(\log(g + \epsilon\mathbb{1})^{\cdot} x(t) + x(t)^{\cdot} \log(g + \epsilon\mathbb{1}))) \\
&= 2\tau(\log(g + \epsilon\mathbb{1}))
\end{aligned}
$$

But then

$$
\begin{aligned}
2\tau(\log(g + \epsilon\mathbb{1})) &= \int_0^1 \tau\left(\frac{d}{dt} \log(x(t))\right) dt \\
&= \tau(\log(x(1))) - \tau(\log(x(0))) \\
&= \tau(\log((g + \epsilon\mathbb{1})(f + \epsilon\mathbb{1})^2(g + \epsilon\mathbb{1}))) - \tau(\log((f + \epsilon\mathbb{1})^2)).
\end{aligned}
$$

We may rewrite this last equality as

$$\tau(\log(|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|)) = \tau(\log(f + \epsilon\mathbb{1})) + \tau(\log(g + \epsilon\mathbb{1})).$$

Since by assumption $\tau(\mathbb{1}) < \infty$, we may use log-rules and Corollary 4.14 to see that

$$
\begin{aligned}
\tau(\log(f + \epsilon\mathbb{1})) &= \tau(\log((f/\epsilon) + \mathbb{1})) + \log(\epsilon)\tau(\mathbb{1}) \\
&= \int_0^{\tau(\mathbb{1})} \log(\mathbf{m}_{(f/\epsilon)}(s) + 1) \, ds + \log(\epsilon)\tau(\mathbb{1}) \\
&= \int_0^{\tau(\mathbb{1})} \log(\mathbf{m}_f(s) + \epsilon) \, ds.
\end{aligned}
$$

A similar claim obviously also holds for $\tau(\log(g + \epsilon\mathbb{1}))$. If we combine this with what we have just shown, it then follows that for any $\epsilon > 0$, we have that

$$\tau(\log(|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|)) = \int_0^{\tau(\mathbb{1})} (\log(\mathbf{m}_f(s) + \epsilon) + \log(\mathbf{m}_g(s) + \epsilon)\, ds$$

Now pass to the general case where $\tau(\mathbb{1}) = \infty$ is allowed. Let $\epsilon > 0$ be given. Notice that $|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|^2 = (g + \epsilon\mathbb{1})(f + \epsilon\mathbb{1})^2(g + \epsilon\mathbb{1}) \geq \epsilon^2(g + \epsilon\mathbb{1})^2 \geq \epsilon^4\mathbb{1}$. Thus $|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|$ is of the form $p + \epsilon^2\mathbb{1}$ for some $p \in \mathcal{M}_+$. It then follows from Proposition 4.19 and Lemma 5.4, that

$$\sup\{\tau(e(\log|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|)e)\colon e \in \mathbb{P}(\mathcal{M}), \tau(e) = t\}$$
$$= \sup\{\tau(e(\log|(p + \epsilon^2\mathbb{1})|)e)\colon e \in \mathbb{P}(\mathcal{M}), \tau(e) = t\}$$
$$= \sup\{\tau(e(\log|(p/\epsilon^2 + \mathbb{1})|)e)\colon e \in \mathbb{P}(\mathcal{M}), \tau(e) = t\} + t\log(\epsilon^2)$$
$$= \int_0^t [\log(\mathbf{m}_p(s)/\epsilon^2 + 1)]\, ds + t\log(\epsilon^2)$$
$$= \int_0^t \log(\mathbf{m}_p(s) + \epsilon^2)\, ds$$
$$= \int_0^t \log(\mathbf{m}_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})}(s))\, ds$$

It is a straightforward exercise to see that $|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|^2 - \epsilon^4 = (g + \epsilon\mathbb{1})(f + \epsilon\mathbb{1})^2(g + \epsilon\mathbb{1}) - \epsilon^4$ is a linear combination of finite products of elements of $\mathscr{L}(\widetilde{\mathcal{M}})$, and hence itself an element of $\mathscr{L}(\widetilde{\mathcal{M}})$. Thus by Remark 4.8 and Lemma 5.5, there exists an abelian subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ containing all the spectral projections, on which τ is still semifinite, and which contains no minimal projections. In terms of the formula for computing $\int_0^t \log(\mathbf{m}_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})}(s))\, ds$ that we have just verified, this means that we may restrict attention to projections e with $\tau(e) = t$, which commute with $|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})|$. Let e be such a projection. Then of course $e(\log|(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})|)e = e(\log|(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})e|)e$. Now by construction $\mathfrak{s}(|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e|) \leq e$. If $(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e = u|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e|$ is the polar decomposition of $(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e$, this means that

$$|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e| = e|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e| = eu^*(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})e.$$

Now let $(g + \epsilon\mathbb{1})e = v|(g + \epsilon\mathbb{1})e|$ be the polar decomposition of $(g + \epsilon\mathbb{1})e$. On similarly using the fact that $\mathfrak{s}(|(g + \epsilon\mathbb{1})e|) \leq e$, we may rewrite the

above equality in the following way:

$$|(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})e| = eu^*(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})e = e[u^*(f + \epsilon \mathbb{1})v]e|(g + \epsilon \mathbb{1})e|.$$

An application of the technology we developed for the case $\tau(\mathbb{1}) < \infty$ to the compression $e\mathcal{M}e$, now shows that

$$\tau(e(\log |(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})|)e)$$
$$= \tau(e(\log |(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})e|)e)$$
$$= \tau(e(\log |e[u^*(f + \epsilon \mathbb{1})v]e|(g + \epsilon \mathbb{1})e|)e)$$
$$= \tau(e(\log |e[u^*(f + \epsilon \mathbb{1})v]e|)e) + \tau(e(\log |(g + \epsilon \mathbb{1})e|)e)$$
$$= \int_0^{\tau(e)} \log(\mathbf{m}^e_{e[u^*(f+\epsilon\mathbb{1})v]e}(s))\, ds + \int_0^{\tau(e)} \log(\mathbf{m}^e_{|(g+\epsilon\mathbb{1})e|}(s)\, ds$$

By Lemma 5.4 this in turn ensures that

$$\tau(e(\log |(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})|)e)$$
$$= \int_0^t \log(\mathbf{m}_{e[u^*(f+\epsilon\mathbb{1})v]e}(s)) \, ds + \int_0^t \log(\mathbf{m}_{|(g+\epsilon\mathbb{1})e|}(s)) \, ds$$
$$\leq \int_0^t \log(\mathbf{m}_f(s) + \epsilon)) \, ds + \int_0^t \log(\mathbf{m}_g(s) + \epsilon) \, ds.$$

If now we take the supremum over all projections e commuting with $|(f + \epsilon \mathbb{1})(g + \epsilon \mathbb{1})|$ for which $\tau(e) = t$, it follows that

$$\int_0^t \log \mathbf{m}_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})}(s)\, ds \leq \int_0^t \log(\mathbf{m}_f(s) + \epsilon)) \, ds + \int_0^t \log(\mathbf{m}_g(s) + \epsilon) \, ds.$$

Observe that for any two bounded positive elements f, g of $\mathcal{M}$, we may use log-rules to write the inequality proven above as the claim that

$$\int_0^t \log \mathbf{m}_{(\epsilon^{-1}f+\mathbb{1})(\epsilon^{-1}g+\mathbb{1})}(s)\, ds$$
$$\leq \int_0^t \log(\epsilon^{-1}\mathbf{m}_f(s) + 1)) \, ds + \int_0^t \log(\epsilon^{-1}\mathbf{m}_g(s) + 1) \, ds$$

for all $\epsilon > 0$. Given $r > 0$, we may then apply Theorem 4.21 to the fact that $t^r \circ \exp$ is convex and increasing on $\mathbb{R}$, to conclude that

$$\int_0^t \mathbf{m}^r_{(\epsilon^{-1}f+\mathbb{1})(\epsilon^{-1}g+\mathbb{1})}(s)\, ds \leq \int_0^t (\epsilon^{-1}\mathbf{m}_f(s) + 1)^r(\epsilon^{-1}\mathbf{m}_g(s) + 1)^r\, ds$$

or equivalently that

$$\int_0^t \mathbf{m}^r_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})}(s)\, ds \leq \int_0^t (\mathbf{m}_f(s) + \epsilon)^r(\mathbf{m}_g(s) + \epsilon)^r\, ds$$

for all $\epsilon > 0$. Observe that $\mathbf{m}_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})} \geq \mathbf{m}_{f(g+\epsilon\mathbb{1})}$ since $|(f + \epsilon\mathbb{1})(g + \epsilon\mathbb{1})| \geq |f(g + \epsilon\mathbb{1})|$. Similarly $\mathbf{m}_{(g+\epsilon\mathbb{1})f} \geq \mathbf{m}_{gf}$. So since $(gf)^* = fg$ and $[(g + \epsilon\mathbb{1})f]^* = f(g + \epsilon\mathbb{1})$, we have that $\mathbf{m}_{f(g+\epsilon\mathbb{1})} \geq \mathbf{m}_{fg}$, and hence that $\mathbf{m}_{(f+\epsilon\mathbb{1})(g+\epsilon\mathbb{1})} \geq \mathbf{m}_{fg}$. It therefore follows that

$$\int_0^t \mathbf{m}_{fg}^r(s)\, ds \leq \int_0^t (\mathbf{m}_f(s) + \epsilon)^r (\mathbf{m}_g(s) + \epsilon)^r\, ds$$

for all $\epsilon > 0$, and hence that

$$\int_0^t \mathbf{m}_{fg}^r(s)\, ds \leq \int_0^t (\mathbf{m}_f(s)\mathbf{m}_g(s))^r\, ds.$$

For general possibly non-positive elements f and g of $\mathcal{M}$, we observe that $|fg| = |fu|g|\,| = |\,|fu|.|g|\,|$, where $g = u|g|$ is the polar decomposition of g. Using what we have just verified, it will then follow that $\int_0^t \mathbf{m}_{fg}^r(s)\, ds = \int_0^t \log \mathbf{m}_{|fu|\,|g|}^r(s)\, ds \leq \int_0^t (\mathbf{m}_{|fu|}(s)\mathbf{m}_{|g|}(s))^r\, ds \leq \int_0^t (\mathbf{m}_f(s)\mathbf{m}_g(s))^r\, ds$.

   Phase 2 of the proof of part (ii) (general case): We proceed with using what we have just verified, to prove the general version of (ii). We now modify the technique used in the proof of Proposition 4.24 to show that claim (ii) follows from the inequality proved above for bounded elements. Let $f, g \in \mathscr{L}(\mathcal{M})$ be given, and let $|f| = \int_0^\infty \lambda\, de_\lambda(|f|)$ be the spectral resolution of $|f|$. If $f = u|f|$ is the polar decomposition of f, then clearly $f = u \int_0^\infty \lambda\, de_\lambda(|f|)$. Now set $f_n = u \int_0^n \lambda\, de_\lambda(|f|)$ for each $n \in \mathbb{N}$. Each f_n will clearly be bounded, with $|f_n| \leq |f|$ for each n (so also $\mathbf{m}_{f_n} \leq \mathbf{m}_f$). We proceed to show that $f_n \to f$ in measure as $n \to \infty$. For each n, the left support projection of f_n is just $\chi_{[0,n]}(|f|)$. On writing $e_n = \chi_{(n,\infty)}(|f|)$, the τ-measurability of f ensures that $\tau(e_n) = d_{|f|}(n) \to 0$ as $n \to \infty$. In addition by construction $\|(f - f_n)(\mathbb{1} - e_n)\| = 0$. Hence as required, (f_n) converges to f in measure. We may similarly construct a sequence $(g_n) \subseteq \mathcal{M}$ such that $g_n \to g$ in measure with $\mathbf{m}_{g_n} \leq \mathbf{m}_g$ for all n. Of course then also $f_n g_m \to fg_m$ in measure as $n \to \infty$, with $\mathbf{m}_{f_n g_m} \leq \mathbf{m}_{fg_m}$ for all n and m. It is now an easy exercise to see that we may then use Theorem 4.26 to conclude that

$$\int_0^t \mathbf{m}_{fg_m}^r(s)\, ds = \lim_{n\to\infty} \int_0^t \mathbf{m}_{f_n g_m}^r(s)\, ds$$

$$\leq \lim_{n\to\infty} \int_0^t \mathbf{m}_{f_n}^r(s)\mathbf{m}_{g_m}^r(s)\, ds$$

$$= \int_0^t \mathbf{m}_f^r(s)\mathbf{m}_{g_m}^r(s)\, ds.$$

Again by Theorem 4.26, now taking the limit as $m \to \infty$, yields $\int_0^t \mathbf{m}_{fg}^r(s)\, ds \leq \int_0^t \mathbf{m}_f^r(s)\mathbf{m}_g^r(s)\, ds$. Using the known fact that

$$\exp(t^{-1} \int_0^t \log|w(s)|\, ds) = \lim_{r \searrow 0} \left[t^{-1} \int_0^t |w(s)|^r\, ds \right]^{1/r}$$

if $\int_0^t |w(s)|^r\, ds < \infty$ for some $r > 0$ (see for example exercise 5 of [**Rud74**, Chapter 3]), it now follows from the above that

$$\Delta_t(fg) = \exp(\int_0^t \log(\mathbf{m}_{fg}(s))\, ds)$$
$$\leq \exp(\int_0^t \log(\mathbf{m}_f(s)\mathbf{m}_g(s))\, ds)$$
$$= \Delta_t(f)\Delta_t(g),$$

as required.

Proof of parts (iii) & (i): By part (ii) we know that $\int_0^t \log(\mathbf{m}_{fg}(s))\, ds \leq \int_0^t \log(\mathbf{m}_f(s)\mathbf{m}_g(s))\, ds$. Thus to see that (iii) holds, we simply need to consider this fact alongside Theorem 4.21. Finally let $p, q, r > 0$ be given. Hölder's inequality clearly holds if one of $\|f\|_p$ or $\|g\|_q$ is 0. On assuming both to be nonzero, the inequality will then similarly hold if one of $\|f\|_p$ or $\|g\|_q$ is infinite. Hence we may assume that both are finite. But then $f, g \in \mathscr{L}(\mathcal{M})$. It then follows from part (iii) that $(\int_0^t \mathbf{m}_{fg}^r(s)\, ds)^{1/r} \leq (\int_0^t (\mathbf{m}_f(s)\mathbf{m}_g(s))^r\, ds)^{1/r}$ for any $t > 0$. On letting $t \to \infty$ and applying the classical Hölder inequality, we have that

$$\|fg\|_r = \left(\int_0^\infty \mathbf{m}_{fg}^r(s)\, ds \right)^{1/r} \leq \left(\int_0^\infty \mathbf{m}_f^p(s)\, ds \right)^{1/p} \cdot \left(\int_0^\infty \mathbf{m}_g^q(s)\, ds \right)^{1/q}$$
$$= \|f\|_p \cdot \|g\|_q,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The proof of a noncommutative Minkowski inequality, is a lot more straightforward than was the case for Hölder's inequality.

THEOREM 5.6. *Let $f, g \in \widetilde{\mathcal{M}}$ be given. For any $1 \leq p < \infty$, we then have that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.*

PROOF. For this result the hard work was done in Theorem 4.22. To see that the stated claim holds, simply apply that theorem to the function $t \mapsto t^p$, to see that $\int_0^t \mathbf{m}_{f+g}(s)^p\, ds \leq \int_0^t (\mathbf{m}_f(s) + \mathbf{m}_g(s))^p\, ds$ for any $t > 0$.

If now we let $t \to \infty$ and apply the usual Minkowski inequality, we will have that

$$\left(\int_0^\infty \mathbf{m}_{f+g}(s)^p \, ds \right)^{1/p} \leq \left(\int_0^\infty (\mathbf{m}_f(s) + \mathbf{m}_g(s))^p \, ds \right)^{1/p}$$
$$\leq \left(\int_0^\infty \mathbf{m}_f(s)^p \, ds \right)^{1/p} + \left(\int_0^\infty \mathbf{m}_g(s)^p \, ds \right)^{1/p}.$$

By Proposition 4.13, this boils down to the statement that $(\tau(|f+g|^p)^{1/p} \leq \tau(|f|^p)^{1/p} + \tau(|g|^p)^{1/p}$. $\qquad\square$

We next start the process of proving that $\|\cdot\|_p$ is a p-norm if $0 < p < 1$. This requires some careful preparation.

LEMMA 5.7. *Let f be a positive element of $\widetilde{\mathcal{M}}$, $u \in \mathcal{M}$ a contraction, and $F: [0, \infty) \to [0, \infty]$ a positive non-decreasing function with $F(0) = 0$, which is continuous on the interval $[0, b_F]$ where $b_F = \sup\{t > 0 : F(t) < \infty\}$. Then*

- *if F is convex we have that $F \circ \mathbf{m}_{ufu^*} = \mathbf{m}_{F(ufu^*)} \leq \mathbf{m}_{uF(f)u^*}$ whenever $F(f)$ is again τ-measurable,*
- *and $F \circ \mathbf{m}_{ufu^*} = \mathbf{m}_{F(ufu^*)} \geq \mathbf{m}_{uF(f)u^*}$ if F is concave.*

Note that it is only in the convex case that b_F can be finite.

The proof relies on the classical fact that a convex function of the type described above, is of the form $F(t) = \sup_{\gamma \in \Gamma}(c_\gamma t + d_\gamma)$ for some collection of affine linear functions where $d_\gamma \leq 0$ for each γ, with a concave function of the above type of the form $F(t) = \inf_{\gamma \in \Gamma}(c_\gamma t + d_\gamma)$ where in this case $d_\gamma \geq 0$ for each γ.

PROOF. We prove only the first claim. The second can be proven by appropriately modifying the current proof. Hence let F be a convex function of the type described above and let $f \in \widetilde{\mathcal{M}}$ be given with $F(f)$ again τ-measurable. Suppose that $\mathcal{M}$ acts on the Hilbert space H. For each unit vector $\xi \in H$ and each $\gamma \in \Gamma$, we then have that

$$\langle uF(f)u^*\xi, \xi \rangle \geq \langle u(c_\gamma f + d_\gamma)u^*\xi, \xi \rangle$$
$$= \langle c_\gamma ufu^*\xi, \xi \rangle + d_\gamma \|u^*\xi\|^2$$
$$\geq \langle c_\gamma ufu^*\xi, \xi \rangle + d_\gamma$$
$$= \langle (c_\gamma ufu^* + d_\gamma \mathbb{1})\xi, \xi \rangle$$

since $d_\gamma \leq 0$. Now take the supremum over γ to see that $\langle uF(f)u^*\xi,\xi\rangle \leq \langle F(ufu^*)\xi,\xi\rangle$ for each $\xi \in H$. This in particular shows that if $uF(f)u^*$ is bounded on a subspace of H, then so is $F(ufu^*)$. Since $uF(f)u^*$ is τ-measurable, the same must therefore be true of $F(ufu^*)$. The fact that $F \circ \mathbf{m}_{ufu^*} = \mathbf{m}_{F(ufu^*)}$ therefore follows from earlier results (Proposition 4.12.)

Now observe that for any $g \in \widetilde{\mathcal{M}}$, the formula for the decreasing rearrangement in Definition 4.1 may be expressed in the form

$$\mathbf{m}_g(t) = \inf_{e\in\mathbb{P}(\mathcal{M}),\tau(\mathbb{1}-e)\leq t}\left[\sup_{\xi\in e(H),\|\xi\|=1}\|g\xi\|\right].$$

In the case where $g \geq 0$, this formula may be rewritten as

$$\mathbf{m}_{\sqrt{g}}(t) = \inf_{e\in\mathbb{P}(\mathcal{M}),\tau(\mathbb{1}-e)\leq t}\left[\sup_{\xi\in e(H),\|\xi\|=1}\sqrt{\langle g\xi,\xi\rangle}\right].$$

If we apply this fact to the above inequality, it is clear that we must then have that $\mathbf{m}_{\sqrt{uF(f)u^*}} \geq \mathbf{m}_{\sqrt{F(ufu^*)}}$. Squaring both sides, now yields the fact that $\mathbf{m}_{uF(f)u^*} \geq \mathbf{m}_{F(ufu^*)}$. as required. Given that $F \circ \mathbf{m}_{ufu^*} = \mathbf{m}_{F(ufu^*)}$, this proves the claim. $\qquad\square$

We are now finally ready to prove that in the case $0 < p < 1$, the $L^p(\mathcal{M},\tau)$-spaces are p-normed spaces. This fact follows from part (i) of the following theorem applied to the function $t \mapsto t^p$.

PROPOSITION 5.8. *Let* $f,g \in \widetilde{\mathcal{M}}$ *be given. Also let* $F : [0,\infty) \to [0,\infty]$ *be a positive non-decreasing function with* $F(0) = 0$ *which is continuous on the interval* $[0,b_F]$ *where* $b_F = \sup\{t > 0: F(t) < \infty\}$, *and let* $a,b \in \mathcal{M}$ *be given with* $a^*a + b^*b \leq \mathbb{1}$.

(i) *Let* F *be concave. If* f,g *are positive, then* $\mathbf{m}_{a^*F(f)a+b^*F(f)b} \leq \mathbf{m}_{F(a^*fa+b^*fb)}$, *so that* $\tau(a^*F(f)a) + \tau(b^*F(g)b) \leq \tau(F(a^*fa + b^*gb))$. *For general* f *and* g, *we have* $\tau(F(|f + g|)) \leq \tau(F(|f|)) + \tau(F(|g|))$.

(ii) *Let* F *be convex. If* f,g *are positive, then* $\mathbf{m}_{a^*F(f)a+b^*F(f)b} \geq \mathbf{m}_{F(a^*fa+b^*fb)}$ *whenever* $F(f)$ *and* $F(g)$ *are again* τ-measurable, *whence* $\tau(a^*F(f)a) + \tau(b^*F(g)b) \leq \tau(F(a^*fa + b^*gb))$. *If* f *and* g *are positive, we have* $\tau(F(f + g)) \geq \tau(F(f)) + \tau(F(g))$.

PROOF. Both proofs run along similar lines, and hence we will only prove part (i). To prove the first claim of part (i), we pass to the von

Neumann algebra $\mathcal{M}\overline{\otimes}M_2(\mathbb{C})$ of 2×2 matrices of elements of $\mathcal{M}$. In this algebra positive elements are of the form $\left[\begin{smallmatrix} a & c \\ c^* & b \end{smallmatrix}\right]$, where $a, b \in \mathcal{M}_+$. It is a semifinite algebra, with the prescription $\widetilde{\tau}\left(\left[\begin{smallmatrix} a & c \\ c^* & b \end{smallmatrix}\right]\right) = \tau(a + b)$ defining a faithful normal semifinite trace. Given $a, b \in \mathcal{M}$ with $a^*a + b^*b \leq \mathbb{1}$, it is now an easy exercise to see that $\left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right]$ is a contraction. Hence for any two positive elements f, g of $\widetilde{\mathcal{M}}$, we may apply Lemma 5.7 to the pair $\left[\begin{smallmatrix} a & 0 \\ b & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} f & 0 \\ 0 & g \end{smallmatrix}\right]$, to see that

$$\mathbf{m}\left(\left[\begin{array}{cc} a^*F(f)a + b^*F(g)b & 0 \\ 0 & 0 \end{array}\right]\right) \leq \mathbf{m}\left(\left[\begin{array}{cc} F(a^*fa + b^*gb) & 0 \\ 0 & 0 \end{array}\right]\right).$$

These decreasing rearrangements are of course with respect to $\widetilde{\tau}$. However for any $h \in \widetilde{\mathcal{M}}_+$, on writing $m(h)$ for $\left[\begin{smallmatrix} h & 0 \\ 0 & 0 \end{smallmatrix}\right]$, it is an easy exercise to see that

$$\begin{aligned}
d_{m(h)}(s) &= \widetilde{\tau}\left(\chi_{(s,\infty)}\left(\left[\begin{array}{cc} h & 0 \\ 0 & 0 \end{array}\right]\right)\right) \\
&= \widetilde{\tau}\left[\begin{array}{cc} \chi_{(s,\infty)}(h) & 0 \\ 0 & 0 \end{array}\right] \\
&= \tau(\chi_{(s,\infty)}(h)) \\
&= d_h(s)
\end{aligned}$$

Thus by Proposition 4.7, the above inequality may be rephrased as

$$\mathbf{m}_{a^*F(f)a+b^*F(g)b} \leq \mathbf{m}_{F(a^*fa+b^*gb)}.$$

We will prove the second claim in two stages. First suppose that $f, g \in \widetilde{\mathcal{M}}$ are positive. It then follows from Proposition 2.65 that there exist contractions $c_f, c_g \in \mathcal{M}_+$ supported on $\mathbb{s}(f + g)$, so that $(f + g)^{1/2}c_f(f + g)^{1/2} = f$ and $(f+g)^{1/2}c_g(f+g)^{1/2} = g$. Since $(f+g)^{1/2}(c_f+c_g)(f+g)^{1/2} = f + g$, we must have that $c_f + c_g = \mathbb{s}(f + g)$. Since $|c_f^{1/2}(f + g)^{1/2}| = f^{1/2}$ and $|c_g^{1/2}(f + g)^{1/2}| = g^{1/2}$, we may select partial isometries u_f and u_g so that for $a = u_f c_f^{1/2}$ and $b = u_g c_g^{1/2}$, we have that $a(f + g)^{1/2} = f^{1/2}$ and $b(f + g)^{1/2} = g^{1/2}$. By construction a and b satisfy $a^*a + b^*b = \mathbb{s}(f + g)$.

We may therefore apply what we have just proven, to see that

$$\tau(F(f)) + \tau(F(g)) \geq \tau(F(a(f+g)a^*)) + \tau(F(b(f+g)b^*))$$
$$\geq \tau(aF(f+g)a^*) + \tau(bF(f+g)b^*)$$
$$= \tau(F(f+g)^{1/2}a^*aF(f+g)^{1/2}) +$$
$$\tau(F(f+g)^{1/2}b^*bF(f+g)^{1/2})$$
$$= \tau(F(f+g)^{1/2}s(f+g)F(f+g)^{1/2})$$
$$= \tau(F(f+g)).$$

To see the validity of the last equality above, observe that on any interval of the form $[0, \beta]$ where $\beta < \infty$, the fact that $F(0) = 0$ ensures that we may use the Stone-Weierstrass theorem to uniformly approximate $\sqrt{F}$ with polynomials with no constant term on that interval. For any such polynomial p we have that $p(f+g) = p((f+g)s(f+g)) = p(f+g)s(f+g)$ on the subspace $\chi_{[0,\beta]}(f+g)(H)$. But then by the continuous functional calculus $F(f+g)^{1/2}s(f+g) = F(f+g)^{1/2}$ on the subspace $\chi_{[0,\beta]}(f+g)(H)$. Hence we have that $F(f+g)s(f+g) = F(f+g)$ as τ-measurable operators.

For general $f, g \in \widetilde{\mathcal{M}}$, we use Lemma 4.2 to select partial isometries $u, v \in \mathcal{M}$ so that $|f+g| \leq u|f|u^* + v|g|v^*$. We may then use what we have just proven, to see that

$$\begin{aligned}
\tau(F(|f+g|)) &= \tau(F(u|f|u^* + v|g|v^*)) \\
&\leq \tau(F(u|f|u^*)) + \tau(F(v|g|v^*)) \\
&= \int_0^\infty \mathbf{m}_{F(u|f|u^*)}(s)\,ds + \int_0^\infty \mathbf{m}_{F(v|g|v^*)}(s)\,ds \\
&= \int_0^\infty F(\mathbf{m}_{u|f|u^*}(s))\,ds + \int_0^\infty F(\mathbf{m}_{v|g|v^*}(s))\,ds \\
&\leq \int_0^\infty F(\mathbf{m}_f(s))\,ds + \int_0^\infty F(\mathbf{m}_g(s))\,ds \\
&\leq \int_0^\infty \mathbf{m}_{F(f)}(s)\,ds + \int_0^\infty \mathbf{m}_{F(g)}(s)\,ds \\
&\leq \tau(F(f)) + \tau(F(g))
\end{aligned}$$

$\square$

We pause to note the following easy corollary before concluding this subsection with a preliminary investigation of the action of the trace on $L^1(\mathcal{M}, \tau)$ and $L^2(\mathcal{M}, \tau)$, and of the bounded elements of $L^p(\mathcal{M}, \tau)$ spaces.

THEOREM 5.9. *Let $f, g \in \widetilde{\mathcal{M}}$ be given. For any non-decreasing continuous concave function F on $[0, \infty)$ with $F(0) = 0$, we have that*

$$\int_0^t F(\mathbf{m}_{f+g}(s)) \, ds \leq \int_0^t F(\mathbf{m}_f(s)) \, ds + \int_0^t F(\mathbf{m}_g(s)) \, ds$$

for all $t > 0$.

PROOF. As we have done many times before, we apply Proposition 4.20 to reduce the proof to the case where $\mathcal{M}$ has no minimal projections, thereby gaining access to Proposition 4.19. We firstly select partial isometries u and v in $\mathcal{M}$ so that $|f + g| \leq u|f|u^* + v|g|v^*$. Next observe that on any interval of the form $[0, \beta]$ where $\beta < \infty$, the fact that $F(0) = 0$ ensures that we may use the Stone-Weierstrass theorem to uniformly approximate F with polynomials with no constant term. For any such polynomial p we have that $p(e|f + g|e) = ep(|f + g|)e$ on the subspace $\chi_{[0,\beta]}(|f + g|)(H)$. But then by the continuous functional calculus $eF(|f + g|)e = F(e|f + g|e)$ on the subspace $\chi_{[0,\beta]}(|f + g|)(H)$. Hence we must have that $eF(|f + g|)e = F(e|f + g|e)$ as τ-measurable operators. Given any projection $e \in \mathcal{M}$ with $\tau(e) = t$, we may then apply part (i) of the preceding theorem, and Corollary 4.11, to see that

$$
\begin{aligned}
\tau(eF(|f+g|)e) \;&=\; \tau(F(e|f+g|e)) \\
&\leq\; \tau(F(eu|f|u^*e + ev|g|v^*e)) \\
&\leq\; \tau(F(eu|f|u^*e)) + \tau(F(ev|g|v^*e)) \\
&=\; \int_0^\infty F(\mathbf{m}_{eu|f|u^*e}(s)) \, ds + \int_0^\infty F(\mathbf{m}_{ev|g|v^*e}(s)) \, ds \\
&=\; \int_0^t F(\mathbf{m}_{eu|f|u^*e}(s)) \, ds + \int_0^t F(\mathbf{m}_{ev|g|v^*e}(s)) \, ds \\
&\leq\; \int_0^t F(\mathbf{m}_{|f|}(s)) \, ds + \int_0^t F(\mathbf{m}_{|g|}(s)) \, ds \\
&=\; \int_0^t F(\mathbf{m}_f(s)) \, ds + \int_0^t F(\mathbf{m}_g(s)) \, ds
\end{aligned}
$$

By Proposition 4.19 taking the supremum over all projections e with $\tau(e) = t$ yields the conclusion. □

REMARK 5.10. Given $p > 0$, it easily follows from Propositions 4.12 & 4.13 that $\|f\|_p = \| \, |f| \, \|_p = \|f^*\|_p$ for any $f \in L^p(\mathcal{M}, \tau)$. In addition for any $f \in L^p(\mathcal{M}, \tau)$, it will in the case $p \geq 1$, follow from Theorem 4.22 and what we have just noted, that $\tau(|\operatorname{Re}(f)|^p)^{1/p} \leq \frac{1}{2}(\|f\|_p + \|f^*\|_p) = \|f\|_p$.

In the case $1 > p > 0$ we may replace Theorem 4.22 with Proposition 5.8 to obtain the conclusion that $\tau(|\operatorname{Re}(f)|^p) \leq 2^{-p}\tau(|f + f^*|^p) \leq 2^{-p}(\tau(|f|^p) + \tau(|f^*|^p\|_p) = 2^{1-p}\tau(|f|^p)$. A similar conclusion clearly holds for $\operatorname{Im}(f)$.

Now let $f \in L^p(\mathcal{M}, \tau)$ be self-adjoint, and consider the elements $f_+ = f\chi_{[0,\infty)}(f)$ and $f_- = -f\chi_{(-\infty,0)}(f)$. Both f_+ and f_- are positive by construction. Moreover since for any $t > 0$, $\mathbf{m}_{f_\pm} \leq \mathbf{m}_f$, it is clear from Proposition 4.13 that $\tau((f_\pm)^p) \leq \tau(|f|^p)$, and hence that $f_\pm \in L^p(\mathcal{M}, \tau)$. In fact since by construction $f_+f_- = 0$, we have that $|f|^p = f_+^p + f_-^p$, and hence that $\|f\|_p^p = \|f_+\|_p^p + \|f_-\|_p^p$.

> PROPOSITION 5.11.   • *The canonical trace τ on $\mathcal{M}_+$ extends to a linear functional on $L^1(\mathcal{M}, \tau)$. In its action on $L^1(\mathcal{M}, \tau)$ we have that $\overline{\tau(a)} = \tau(a^*)$ and $|\tau(a)| \leq \tau(|a|)$ for any $a \in L^1(\mathcal{M}, \tau)$.*
> • *The space $L^2(\mathcal{M}, \tau)$ is an inner product space with the inner product given by $\langle a, b \rangle = \tau(b^*a)$ for all $a, b \in L^2(\mathcal{M}, \tau)$.*

PROOF. We start by proving the first two claims regarding $L^1(\mathcal{M}, \tau)$. Given any self-adjoint element a of $L^1(\mathcal{M}, \tau)$, it is clear that $a = a_+ - a_-$ where $a_+ = a\chi_{[0,\infty)}(a)$ and $a_- = -a\chi_{(\infty,0)}(a)$. Both a_+ and a_- are positive by construction, with $a_\pm \in L^1(\mathcal{M}, \tau)$ by Remark 5.10. Now let p_0 and p_1 be any other two positive elements of $L^1(\mathcal{M}, \tau)$ for which $a = p_0 - p_1$. Since then $a_+ + p_1 = p_0 + a_-$, an application of the trace reveals that $\tau(a_+) + \tau(p_1) = \tau(a_+ + p_1) = \tau(p_0 + a_-) = \tau(p_0) + \tau(a_-)$, and hence that $\tau(a_+) - \tau(a_-) = \tau(p_0) - \tau(p_1)$. Thus for any self-adjoint element $a \in L^1(M, \tau)$, we may uniquely define $\tau(a)$ to be $\tau(p_0) - \tau(p_1)$ where p_0 and p_1 are any two positive elements of $L^1(\mathcal{M}, \tau)$ for which $a = p_0 - p_1$.

Now let b be another self-adjoint element of $L^1(\mathcal{M}, \tau)$ of the form $b = q_0 - q_1$ where q_0 and q_1 are positive elements of $L^1(\mathcal{M}, \tau)$. Then by definition $\tau(a + b) = \tau(p_0 + q_0) - \tau(p_1 + q_1) = [\tau(p_0) + \tau(q_0)] - [\tau(p_1) + \tau(q_1)] = [\tau(p_0) - \tau(p_1)] + [\tau(q_0) - \tau(q_1)] = \tau(a) + \tau(b)$. We may similarly use the fact that $\tau(\alpha f) = \alpha\tau(f)$ for any $\alpha \geq 0$ and any $f \in \widetilde{\mathcal{M}}_+$, to show that for any a in the self-adjoint portion $L^1(\mathcal{M}, \tau)_h$ of $L^1(\mathcal{M}, \tau)$, we have $\tau(\alpha a) = \alpha\tau(a)$ for any $\alpha \in \mathbb{R}$. Thus τ is real-linear on $L^1(\mathcal{M}, \tau)_h$.

Given a general element $a \in L^1(\mathcal{M}, \tau)$, it follows from Remark 5.10 that $\operatorname{Re}(a), \operatorname{Im}(a) \in L^1(\mathcal{M}, \tau)$. We therefore have that

$$L^1(\mathcal{M}, \tau) = L^1(\mathcal{M}, \tau)_h + iL^1(\mathcal{M}, \tau)_h.$$

On defining $\tau(a)$ to be $\tau(a) = \tau(\operatorname{Re}(a)) + i\tau(\operatorname{Im}(a))$, it is now an exercise to see that this definition ensures that τ is complex-linear on $L^1(\mathcal{M}, \tau)$.

From this definition, it is now clear that

$$\overline{\tau(a)} = \overline{\tau(\mathrm{Re}(a)) + i\tau(\mathrm{Im}(a))} = \tau(\mathrm{Re}(a)) - i\tau(\mathrm{Im}(a)) = \tau(a^*).$$

We pass to proving the claims regarding $L^2(\mathcal{M}, \tau)$. It easily follows from Theorem 5.2 that b^*a will belong to $L^1(\mathcal{M}, \tau)$ whenever $a, b \in L^2(\mathcal{M}, \tau)$. Thus $\langle a, b \rangle = \tau(b^*a)$ is well-defined. It follows from what we have already proven that $\overline{\langle a, b \rangle} = \overline{\tau(b^*a)} = \tau((b^*a)^*) = \tau(a^*b) = \langle b, a \rangle$. All the other properties of an inner product are easy to verify, including the fact that $\langle a, a \rangle = \tau(|a|^2) = \|a\|_2^2$.

To prove the final claim regarding $L^1(\mathcal{M}, \tau)$, we will make use of the Cauchy-Schwarz inequality for the inner product on $L^2(\mathcal{M}, \tau)$. Given any $x \in L^1(\mathcal{M}, \tau)$, we may let $a = |x|^{1/2}u^*$ and $b = |x|^{1/2}$, where $x = u|x|$ is the polar form of x. It then clearly follows that $\|b\|_2 = \tau(|x|)^{1/2}$ and that $\|a\|_2 = \|a^*\|_2 = \tau(|x|^{1/2}u^*u|x|^{1/2})^{1/2} = \tau(|x|)^{1/2}$. Thus we have that $a, b \in L^2(\mathcal{M}, \tau)$, with $|\tau(x)| = |\tau(u|x|)| = |\tau(a^*b)| = |\langle b, a \rangle| \leq \|b\|_2\|a\|_2 = \tau(|x|)$, as required. $\qquad\square$

PROPOSITION 5.12. *For any $0 < p < 1$, $(L^p \cap L^\infty)(\mathcal{M}, \tau)$ is dense in $L^p(\mathcal{M}, \tau)$. In the case $1 \leq p < \infty$, $(L^1 \cap L^\infty)(\mathcal{M}, \tau)$ is a norm-dense subspace of $L^p(\mathcal{M}, \tau)$.*

PROOF. We first prove that $(L^p \cap L^\infty)(\mathcal{M}, \tau)$ is dense in $L^p(\mathcal{M}, \tau)$ for any $0 < p < \infty$. To see this, it is enough to show that any positive element of $L^p(\mathcal{M}, \tau)$ is in the closure of $(L^p \cap L^\infty)(\mathcal{M}, \tau)$. Given $f \in L^p(\mathcal{M}, \tau)$ with $f \geq 0$, it follows from the normality of the trace that $\tau(f^p) = \lim_{n \to \infty} \tau(f^p \chi_{[0,n]}(f^p)) = \sup_{n \in \mathbb{N}} \tau(f^p \chi_{[0,n]}(f^p))$. Therefore $\lim_{n \to \infty} \tau((f - f\chi_{[0,n]}(f^p))^p) = \lim_{n \to \infty} \tau(f^p \chi_{(n,\infty)}(f^p)) = 0$. Since $\|f\chi_{[0,n]}(f^p)\|_\infty \leq n^{1/p}$, we have that $(f\chi_{[0,n]}(f^p)) \subseteq (L^p \cap L^\infty)(\mathcal{M}, \tau)$, and hence we are done with the first part.

We now pass to the case $1 \leq p < \infty$. In the case where $\tau(\chi_{(0,\infty)}(f^p)) < \infty$, the sequence (f_n) constructed above actually lies in $(L^1 \cap L^\infty)(\mathcal{M}, \tau)$. To see this note that in this case $\tau(f_n) = \tau(f_n\chi_{(0,\infty)}(f^p))$
$\leq \|f_n\|_\infty \tau(\chi_{(0,\infty)}(f^p)) < \infty$.

Next consider the case $\tau(\chi_{(0,\infty)}(f^p)) = \infty$. It then follows from Proposition 4.24 that $d_{f^p}(s) < \infty$ for each $s > 0$. By the Borel functional calculus we have that $f^p\chi_{(1/n,\infty)}(f^p)$ increases to f^p. So by the normality of the trace, $\lim_{n \to \infty} \tau(f^p\chi_{[0,1/n]}(f^p)) = \tau(f^p) - \lim_{n \to \infty} \tau(f^p\chi_{(1/n,\infty)}(f^p)) = 0$. It follows that the sequence $f\chi_{[0,1/n]}(f^p)$ converges to 0 in L^p-norm. By what we have already shown, the sequence $\widetilde{f_n} = f\chi_{[0,n]}(f^p) - f\chi_{[0,1/n]}(f^p) =$

$f\chi_{(1/n,n]}(f^p)$ $(n \in \mathbb{N})$ must then converge to f in L^p-norm. It remains to show that $(f\chi_{(1/n,n]}(f^p)) \subseteq (L^1 \cap L^\infty)(\mathcal{M}, \tau)$. We clearly have that $\|f\chi_{(1/n,n]}(f^p)\|_\infty \leq n^{1/p}$. To see that each $f\chi_{(1/n,n]}(f^p)$ is in $L^1(\mathcal{M}, \tau)$, recall that for $C_n = n^{(1-(1/p))}$, we have that $\lambda^{1/p} \leq C_n\lambda$ whenever $\lambda \geq \frac{1}{n}$. But then

$$f\chi_{(1/n,n]}(f^p) = \int_{1/n}^{n} \lambda^{1/p}\, de_\lambda(f^p) \leq C_n \int_{1/n}^{n} \lambda\, de_\lambda(f^p) = C_n f^p \chi_{(1/n,n]}(f^p),$$

whence $\tau(f\chi_{(1/n,n]}(f^p)) \leq C_n\tau(f^p\chi_{(1/n,n]}(f^p)) < \infty$, as required. $\square$

5.1.2   Convergence and completeness

Before proceeding with the proof of the completeness of these spaces, we pause to establish a Dominated Convergence Theorem appropriate to the present context.

THEOREM 5.13. *Let f_n $(n \in \mathbb{N})$ be a sequence of τ-measurable opera-tors converging to $f \in \widetilde{\mathcal{M}}$ in the topology of convergence in measure on $\widetilde{\mathcal{M}}$. Suppose there exist operators g_n $(n \in \mathbb{N})$ and g in $L^p(\mathcal{M}, \tau)$ $(0 < p < \infty)$, for which we have that*

   (i) $\mathbf{m}_{f_n} \leq \mathbf{m}_{g_n}$ *for all n,*
   (ii) $\lim_{n\to\infty} \|g_n\|_p = \|g\|_p$,
   (iii) *and $\mathbf{m}_g(t) \leq \liminf_{n\to\infty} \mathbf{m}_{g_n}(t)$ for each $t > 0$.*

Then f_n $(n \in \mathbb{N})$ and f are all in $L^p(\mathcal{M}, \tau)$, with $\lim_{n\to\infty} \|f - f_n\|_p = 0$. In the case $p = 1$, we additionally have that $\tau(f) = \lim_{n\to\infty} \tau(f_n)$.

PROOF. It is palpably clear from (i) that for each n, we have that

$$\tau(|f_n|^p) = \int_0^\infty \mathbf{m}_{f_n}^p(s)\, ds \leq \int_0^\infty \mathbf{m}_{g_n}^p(s)\, ds = \tau(|g_n|^p),$$

and hence that each f_n belongs to $L^p(\mathcal{M}, \tau)$. We may next apply Lemma 4.25 and the usual Fatou lemma to see that

$$\tau(|f|^p) = \int_0^\infty \mathbf{m}_f^p(s)\, ds \leq \int_0^\infty \liminf_{n\to\infty} \mathbf{m}_{f_n}^p(s)\, ds$$

$$\leq \int_0^\infty \liminf_{n\to\infty} \mathbf{m}_{g_n}^p(s)\, ds \leq \liminf_{n\to\infty} \int_0^\infty \mathbf{m}_{g_n}^p(s)\, ds = \liminf_{n\to\infty} \|g_n\|_p^p = \|g\|_p^p.$$

Therefore $f \in L^p(\mathcal{M}, \tau)$.

We proceed to prove that $\lim_{n \to \infty} \|f - f_n\|_p = 0$. Firstly note for any $t > 0$,

$$\mathbf{m}_{f-f_n}(t) \leq \mathbf{m}_f(t/2) + \mathbf{m}_{f_n}(t/2) \leq \mathbf{m}_f(t/2) + \mathbf{m}_{g_n}(t/2).$$

Hence for $C_p = \max\{1, 2^{p-1}\}$, we have that

$$\mathbf{m}_{f-f_n}^p(t) \leq (\mathbf{m}_f(t/2) + \mathbf{m}_{g_n}(t/2))^p \leq C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_{g_n}^p(t/2)].$$

Thus $t \mapsto C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_{g_n}^p(t/2)] - \mathbf{m}_{f-f_n}^p(t)$ is a non-negative function. Now observe that for any $t > 0$ we must have that $\lim_{n \to \infty} \mathbf{m}_{f-f_n}(t) = 0$ by Proposition 4.23. If we combine this with assumption (iii), it now follows that

$$\liminf_{n \to \infty}\{C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_{g_n}^p(t/2)] - \mathbf{m}_{f-f_n}^p(t)\} \geq C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_g^p(t/2)]$$

for each $t > 0$. The standard Fatou lemma now ensures that

$$
\begin{aligned}
2C_p(\|f\|_p^p + \|g\|_p^p) &= C_p \int_0^\infty [\mathbf{m}_f^p(t/2) + \mathbf{m}_g^p(t/2)]\, dt \\
&\leq \int_0^\infty \liminf_{n \to \infty}\{C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_{g_n}^p(t/2)] - \mathbf{m}_{f-f_n}^p(t)\}\, dt \\
&\leq \liminf_{n \to \infty} \int_0^\infty \{C_p[\mathbf{m}_f^p(t/2) + \mathbf{m}_{g_n}^p(t/2)] - \mathbf{m}_{f-f_n}^p(t)\}\, dt \\
&= \liminf_{n \to \infty}[2C_p(\|f\|_p^p + \|g\|_p^p) - \|f - f_n\|_p^p],
\end{aligned}
$$

or equivalently that $0 \leq -\limsup_{n \to \infty} \|f - f_n\|_p^p$. This clearly suffices to prove that $\lim_{n \to \infty} \|f - f_n\|_p = 0$ as required.

To see the claim regarding the case $p = 1$, observe that in this case we may use Proposition 5.11 to see that $|\tau(f) - \tau(f_n)| \leq \tau(|f - f_n|)$. $\square$

LEMMA 5.14. *Each of the spaces $L^p(\mathcal{M}, \tau)$ $(0 < p \leq \infty)$ injects continuously into $\widetilde{\mathcal{M}}$.*

PROOF. In the case $p = \infty$ the claim is obvious. So assume that $0 < p < \infty$, with (f_n) converging to f in $L^p(\mathcal{M}, \tau)$. For any $t > 0$ we may now use the fact that $\mathbf{m}_{f-f_n}$ is non-increasing to see that

$$
\begin{aligned}
t^{1/p}\mathbf{m}_{f-f_n}(t) &\leq \left(\int_0^t \mathbf{m}_{f-f_n}^p(s)\, ds\right)^{1/p} \\
&\leq \left(\int_0^\infty \mathbf{m}_{f-f_n}^p(s)\, ds\right)^{1/p} \\
&= \|f - f_n\|_p.
\end{aligned}
$$

The claim now follows from Proposition 4.23.   □

As a consequence of Theorem 5.13, we obtain the following important description of convergence in $L^p(\mathcal{M}, \tau)$:

THEOREM 5.15. *Let f_n ($n \in \mathbb{N}$), f be elements of $L^p(\mathcal{M}, \tau)$, where $0 < p < \infty$. Then the following are equivalent:*

   (i) $\lim_{n \to \infty} \|f - f_n\|_p = 0$;
   (ii) $\lim_{n \to \infty} \|f_n\|_p = \|f\|_p$, *and $f_n \to f$ in the topology of convergence in measure.*

PROOF. The fact that (i)⇒(ii) easily follows from the lemma, whereas (ii)⇒(i) follows from Theorem 5.13 with $g_n = |f_n|$ and $g = |f|$.   □

We proceed with the proof of the completeness of the L^p-spaces.

THEOREM 5.16. *Each of the spaces $L^p(\mathcal{M}, \tau)$ ($0 < p \leq \infty$) is complete.*

PROOF. In the case $p = \infty$ there is of course nothing to prove, so let $p < \infty$. We prove the case where $p \geq 1$. A similar proof, suitably modified, holds for the case $0 < p < 1$. Let (f_n) be a sequence in $L^p(\mathcal{M}, \tau)$ for which $\sum_{k=1}^{\infty} \|f_k\|_p < \infty$. We need to show that $\sum_{k=1}^{\infty} f_k$ converges in $L^p(\mathcal{M}, \tau)$. By separately considering the series $\sum_{k=1}^{\infty} \mathrm{Re}(f_k)$ and $\sum_{k=1}^{\infty} \mathrm{Im}(f_k)$, we may and do assume that each f_k is self-adjoint. Now let $x_n = \sum_{k=1}^{n} f_k$ and $z_n = \sum_{k=1}^{n} |f_k|$ for each $n \in \mathbb{N}$. Given $n > m$, we have that $\|z_n - z_m\|_p = \|\sum_{k=m+1}^{n} |f_n|\|_p \leq \sum_{k=m+1}^{\infty} \|f_k\|_p$. Since by assumption $\sum_{k=m}^{\infty} \|f_k\|_p \to 0$ as $m \to \infty$, the sequence (z_n) is Cauchy in $L^p(\mathcal{M}, \tau)$. By Lemma 5.14, (z_n) is then Cauchy in $\widetilde{\mathcal{M}}$. But $\widetilde{\mathcal{M}}$ is a complete linear metric space. So there must exist some $z \in \widetilde{\mathcal{M}}$ which is the limit of (z_n) in the topology of convergence in measure. But (z_n) is an increasing sequence of positive operators. So by part (iii) of Proposition 4.12, we have that $\mathbf{m}_z(t) = \lim_{n \to \infty} \mathbf{m}_{z_n}(t) = \sup_{n \in \mathbb{N}} \mathbf{m}_{z_n}(t)$. Equivalently $\mathbf{m}_{z^p}(t) = \mathbf{m}_z^p(t) = \lim_{n \to \infty} \mathbf{m}_{z_n}^p(t) = \lim_{n \to \infty} \mathbf{m}_{z_n^p}(t)$. It therefore follows from Lemma 4.25, that $\|z\|_p = \tau(z^p)^{1/p} = \lim_{n \to \infty} \tau(z_n^p)^{1/p} = \lim_{n \to \infty} \|z_n\|$. But then Theorem 5.15 ensures that in fact $z_n \to z$ in $L^p(\mathcal{M}, \tau)$.

Next observe that the self-adjointness assumption on the f_k's, ensure that $0 \leq z_n + x_n \leq 2z_n$ for each n (and hence that $\mathbf{m}_{z_n+x_n}(t) \leq \mathbf{m}_{2z_n}(t)$ for each n). Using the fact that for $n > m$, we have $\|x_n - x_m\|_p = \|\sum_{k=m+1}^{n} f_n\|_p \leq \sum_{k=m+1}^{\infty} \|f_k\|_p$, we may argue as before, to conclude that the sequence (x_n) is Cauchy in $L^p(\mathcal{M}, \tau)$. Again as before the fact that $L^p(\mathcal{M}, \tau)$ continuously injects into $\widetilde{\mathcal{M}}$, ensures that (x_n) is Cauchy in

$\widetilde{\mathcal{M}}$, and hence must have a limit x in $\widetilde{\mathcal{M}}$. The sequence $(x_n + z_n)$ therefore converges to $x + z$ in the topology of convergence in measure. We may therefore apply Theorem 5.13 to the pair of sequences $(x_n + z_n)$ and $(2z_n)$, to conclude that $(x_n + z_n)$ converges to $x + z$ in $L^p(\mathcal{M}, \tau)$, and hence that (x_n) converges to $x = (x + z) - z$ in $L^p(\mathcal{M}, \tau)$. This then proves the claim. $\qquad\square$

REMARK 5.17. It now follows from Proposition 5.11 and the preceding theorem, that $L^2(\mathcal{M}, \tau)$ is a Hilbert space.

5.1.3 L^p-duality

We now come to the final ingredient in the development of the rudimentary theory of $L^p(\mathcal{M}, \tau)$ spaces, namely duality theory. A more refined understanding of the action of the trace on $L^1(\mathcal{M}, \tau)$ is crucial to this endeavour.

PROPOSITION 5.18. *For any $x \in \mathcal{M}$ and $y \in L^1(\mathcal{M}, \tau)$, we have that $\tau(xy) = \tau(yx)$. Similarly for any $a, b \in L^2(\mathcal{M}, \tau)$, we have that $\tau(ab) = \tau(ba)$.*

PROOF. First let $a, b \in L^2(\mathcal{M}, \tau)$ be given. By the polarization identity

$$4ab = \sum_{k=0}^{3} i^k (a^* + i^k b)^* (a^* + i^k b) \qquad 4ba = \sum_{k=0}^{3} i^k (a^* + i^k b)(a^* + i^k b)^*$$

with each term in each of the sums an element of $L^1(\mathcal{M}, \tau)$. Hence by the linearity of τ on $L^1(\mathcal{M}, \tau)$ and the known action of τ on $\widetilde{\mathcal{M}}_+$, we have that

$$\tau(ab) = \frac{1}{4} \sum_{k=0}^{3} i^k \tau((a^* + i^k b)^* (a^* + i^k b))$$

$$= \frac{1}{4} \sum_{k=0}^{3} i^k \tau((a^* + i^k b)(a^* + i^k b)^*) = \tau(ba).$$

Now let $x \in \mathcal{M}$ and $y \in L^1(\mathcal{M}, \tau)$ be given. With $y = u|y|$ being the polar form of y, it is now an exercise to see that each of $u|y|^{1/2}$ and $|y|^{1/2}$ belong to $L^2(\mathcal{M}, \tau)$. Observe that for any $a \in L^2(\mathcal{M}, \tau)$, Hölder's inequality ensures that $ax, xa \in L^2(\mathcal{M}, \tau)$. If we combine this fact with

what we have already proven regarding $L^2(\mathcal{M}, \tau)$, it follows that

$$\tau(xy) = \tau((xu|y|^{1/2})|y|^{1/2}) = \tau(|y|^{1/2}(xu|y|^{1/2}))$$
$$= \tau((|y|^{1/2}x)u|y|^{1/2}) = \tau(u|y|^{1/2}(|y|^{1/2}x)) = \tau(yx)$$

as required. □

The following Lemma is now an easy consequence of the above result considered alongside Hölder's inequality.

LEMMA 5.19. *For any $x \in L^1(\mathcal{M}, \tau)$, we have that*

$$\|x\|_1 = \sup\{|\tau(yx)|\colon y \in \mathcal{M}, \|y\| \le 1\} = \sup\{|\tau(xy)|\colon y \in \mathcal{M}, \|y\| \le 1\}.$$

PROOF. We prove the first equality. Hölder's inequality combined with the fact that $|\tau(yx)| \le \tau(|yx|)$ for each $x \in L^1(\mathcal{M}, \tau)$, $y \in \mathcal{M}$, ensures that $\sup\{|\tau(yx)|\colon y \in \mathcal{M}, \|y\| \le 1\} \le \|x\|_1$. Since for $y = u^*$, where $x = u|x|$ is the polar decomposition of x, we have that $\tau(yx) = \tau(|x|) = \|x\|_1$, it is clear that equality must hold. □

THEOREM 5.20. *The bilinear form*

$$L^1(\mathcal{M}, \tau) \times \mathcal{M} \to \mathbb{C} : (x, y) \mapsto \tau(yx)$$

defines a dual action of $\mathcal{M}$ on $L^1(\mathcal{M}, \tau)$ with respect to which $\mathcal{M}$ is identified with $(L^1(\mathcal{M}, \tau))^*$. Specifically for each $x \in L^1(\mathcal{M}, \tau)$, the prescription $y \mapsto \tau(yx)$ defines a σ-weakly continuous linear functional ω_x on $\mathcal{M}$. Moreover the mapping $x \mapsto \omega_x$ is a surjective isometry from $L^1(\mathcal{M}, \tau)$ onto $\mathcal{M}_*$.

PROOF. The lemma ensures that the mapping $\iota : L^1(\mathcal{M}, \tau) \to \mathcal{M}_* : x \mapsto \omega_x$, is a linear isometry. We therefore merely need to verify the surjectivity of this map, and that each ω_x is σ-weakly continuous. We saw in Remark 5.10 that $L^1(\mathcal{M}, \tau)$ is spanned by its positive elements. So to prove the claim regarding the σ-weak continuity of the ω_x's, it is enough to do this for the case where $x \ge 0$. Hence let this be the case, and let (y_i) be a net in $\mathcal{M}_+$ increasing to some $y \in \mathcal{M}_+$. Then $(x^{1/2}y_ix^{1/2})$ will of course increase to $x^{1/2}yx^{1/2}$. So by the normality of the trace on $\widetilde{\mathcal{M}}$, we get that

$$\sup_i \omega_x(y_i) = \sup_i \tau(y_ix) = \sup_i \tau(x^{1/2}y_ix^{1/2})$$
$$= \tau(x^{1/2}yx^{1/2}) = \tau(yx) = \omega_x(y).$$

So ω_x is, as required, a positive normal functional.

Since $L^1(\mathcal{M}, \tau)$ is complete and $\iota : L^1(\mathcal{M}, \tau) \to \mathcal{M}_*$ an isometric embedding, $\iota(L^1(\mathcal{M}, \tau)) = \{\omega_x : x \in L^1(\mathcal{M}, \tau)\}$ is a closed subspace of $\mathcal{M}_*$. If therefore we can show that $\iota(L^1(\mathcal{M}, \tau))$ is dense in $\mathcal{M}_*$, we will have that $\iota(L^1(\mathcal{M}, \tau)) = \mathcal{M}_*$, which would conclude the proof. Since $\iota(L^1(\mathcal{M}, \tau))$ is a linear subspace, the norm closure will agree with the $\sigma(\mathcal{M}_*, \mathcal{M})$-closure. By the bipolar theorem $[(\iota(L^1(\mathcal{M}, \tau)))^\circ]_\circ$ in turn corresponds to the $\sigma(\mathcal{M}_*, \mathcal{M})$-closure. (For any $A \subseteq \mathcal{M}_*$ and $B \subseteq \mathcal{M}$, A° denotes the polar of A in $\mathcal{M}$, and $B_\circ$ the polar of B in $\mathcal{M}_*$.) We show that $(\iota(L^1(\mathcal{M}, \tau)))^\circ = \{0\}$ from which the theorem will then follow. For the sake of contradiction suppose that $(\iota(L^1(\mathcal{M}, \tau)))^\circ$ contains a non-zero element a. Let $a = u|a|$ be the polar decomposition of a. By the semifiniteness of the trace, there exists a non-zero subprojection e of $\chi_{(0,\infty)}(|a|)$ with $\tau(e) < \infty$. Then $e|a|e$ must of course be non-zero, and hence $\tau(e|a|e) \neq 0$. Next observe that for $x = eu^*$ we have that $\tau(|x^*|) = \tau(e) < \infty$. Hence x^*, and therefore x, belongs to $L^1(\mathcal{M}, \tau)$. But by the assumption on a we must in that case have that $\omega_x(a) = 0$. Since $\omega_x(a) = \tau(xa) = \tau(e|a|) = \tau(e|a|e)$, this is a clear contradiction. The space $(\iota(L^1(\mathcal{M}, \tau)))^\circ$ can therefore contain no nonzero elements. $\qquad\square$

COROLLARY 5.21. *The space $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is σ-weakly dense in $\mathcal{M}$.*

PROOF. It is enough to show that any $a \in \mathcal{M}_+$ is in the σ-weak closure of $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. Given $a \in \mathcal{M}_+$, the semi-finiteness of the trace ensures that we may select a net of projections (e_α) each with finite trace, increasing to $\chi_{(0,\infty)}(a)$. The existence of such a net is fairly well-known, but for the sake of the reader we pause to indicate how its existence may be verified. Firstly recall that the semifiniteness of the canonical trace τ on $\mathcal{M}$, ensures that each projection e in $\mathcal{M}$, admits a subprojection with finite trace. (This fact was verified in part(c) of the proof of Proposition 4.9.) One may then use Zorn's lemma to select a maximal family $\{f_\alpha\}$ of mutually orthogonal projections each with finite trace. The property just noted ensures that $\mathbb{1} = \sum_\alpha f_\alpha$. The net we seek, consists of finite sums of elements of $\{f_\alpha\}$.

But then (e_α) converges to $\chi_{(0,\infty)}(a)$ in the σ-strong* topology, and hence also in the σ-weak topology. For any $x \in L^1(\mathcal{M}, \tau)$, the duality established in the preceding theorem therefore ensures that

$$\lim_\alpha \tau((ae_\alpha)x) = \lim_\alpha \tau(e_\alpha(xa)) = \tau(\chi_{(0,\infty)}(a)xa)$$
$$= \tau(a\chi_{(0,\infty)}(a)x) = \tau(ax).$$

Since for any α we have that $|ae_\alpha| \leq \|a\|_\infty e_\alpha$, it is clear that $\tau(ae_\alpha) = \tau(e_\alpha ae_\alpha) \leq \|a\|_\infty \tau(e_\alpha) < \infty$ for each α, and hence $(ae_\alpha) \subseteq (L^1(\mathcal{M}, \tau) \cap \mathcal{M})$. This proves the claim. $\qquad\square$

The following theorem significantly sharpens Proposition 5.18, and greatly improves our understanding of the action of the trace on $L^1(\mathcal{M}, \tau)$. The proof presented here of the first statement is due to Brown and Kosaki [**BK90**, Theorem 17], with the proof of the second due to Dodds, Dodds and de Pagter [**DDdP93**, Proposition 3.4].

THEOREM 5.22. *Let $x, y \in \widetilde{\mathcal{M}}$ be given with $xy, yx \in L^1(\mathcal{M}, \tau)$. Then $\tau(xy) = \tau(yx)$. If in addition $x \geq 0$ and $y \geq 0$ with x satisfying the requirement that $\lim_{t\to\infty} \mathbf{m}_x(t) = 0$, then $x^{1/2}yx^{1/2}, y^{1/2}xy^{1/2} \in L^1(\mathcal{M}, \tau)$ with*

$$\tau(xy) = \tau(x^{1/2}yx^{1/2}) = \tau(y^{1/2}xy^{1/2}).$$

PROOF. Let $x, y \in \widetilde{\mathcal{M}}$ be given with $xy, yx \in L^1(\mathcal{M}, \tau)$. For $p := \chi_{(0,\infty)}(|x|)$ and $q := \chi_{(0,\infty)}(|x^*|)$, we define $p_n = \chi_{(1/n,n]}(|x|)$ and $q_n = \chi_{(1/n,n]}(|x^*|)$ for each $n \in \mathbb{N}$. By construction $px = xq$ and $p_n x = xq_n$, with (p_n) and (q_n) respectively increasing to p and q in the σ-strong* topology. Since $p_n \leq n|x|$ for any n, we have that $|p_n y| = (y^* p_n y)^{1/2} \leq n(y^* x^* xy)^{1/2} = |xy|$. In addition since by assumption $xy \in L^1(\mathcal{M}, \tau)$, we have that $(p_n y) \subseteq L^1(\mathcal{M}, \tau)$. The fact that $L^1(\mathcal{M}, \tau)$ is an $\mathcal{M}$-bimodule by Theorem 5.2, additionally ensures that $(p_n yx) \subseteq L^1(\mathcal{M}, \tau)$. Clearly $q_n x, p_n \in \mathcal{M}$ for each n. On repeatedly using Proposition 5.18, it now follows that

$$\tau(q_n xy) = \tau(q_n xp_n y) = \tau(p_n yq_n x) = \tau(p_n yxp_n) = \tau(p_n yx).$$

The fact that both xy and yx are in $L^1(\mathcal{M}, \tau)$, now enables us to conclude from Theorem 5.20 that $\tau(xy) = \lim_{n\to\infty} \tau(q_n xy) = \lim_{n\to\infty} \tau(p_n yx) = \tau(yx)$.

Now pass to the case where $x, y \geq 0$ with in addition $\lim_{t\to\infty} \mathbf{m}_x(t) = 0$. We remind the reader that in this case $d_x(s) < \infty$ for any $s > 0$ (Proposition 4.24). We clearly have that $p = q$ and $p_n = q_n$. Since $\frac{1}{n} p_n \leq x$, we then have that $|p_n x^{1/2} y|^2 = yx^{1/2} p_n x^{1/2} y \leq n|xy|^2$. But then $\tau(|p_n x^{1/2} y|) \leq \sqrt{n}\tau(|xy|) < \infty$, whence $(p_n x^{1/2} y) \in L^1(\mathcal{M}, \tau)$. Since each $p_n x^{1/2}$ is clearly bounded, we may now use Proposition 5.18 to see that

$$\tau(p_n xy) = \tau(p_n x^{1/2} p_n x^{1/2} y) = \tau(p_n x^{1/2} yp_n x^{1/2}) = \tau(p_n x^{1/2} yx^{1/2} p_n).$$

If we are able to show that $x^{1/2}yx^{1/2} \in L^1(\mathcal{M}, \tau)$, we would be able to use Theorem 5.20, to conclude from the above that $\tau(xy) = \tau(pxy) = \lim_{n\to\infty} \tau(p_n xy) = \lim_{n\to\infty} \tau(p_n x^{1/2}yx^{1/2}p_n) = \lim_{n\to\infty} \tau(p_n x^{1/2}yx^{1/2}) = \tau(px^{1/2}yx^{1/2}) = \tau(x^{1/2}yx^{1/2})$. Since in this case

$$\tau(|y^{1/2}x^{1/2}|^2) = \tau(x^{1/2}yx^{1/2}) < \infty,$$

we would then clearly have that $y^{1/2}x^{1/2} \in L^2(\mathcal{M}, \tau)$. We could then use Proposition 5.18 to see that $\tau(x^{1/2}yx^{1/2}) = \tau(y^{1/2}xy^{1/2})$. It therefore remains to prove that $x^{1/2}yx^{1/2} \in L^1(\mathcal{M}, \tau)$. To this aim, notice that an easy modification of the argument used to prove the implication (i)$\Rightarrow$(iii) in Proposition 4.24, shows that $(p_n x^{1/2})$ converges to $x^{1/2}$ in measure. Hence $(p_n x^{1/2}yx^{1/2}p_n)$ converges to $x^{1/2}yx^{1/2}$ in measure. From what we have already proven, it is clear that $\tau(p_n x^{1/2}yx^{1/2}p_n) = \tau(p_n xy) \le \tau(|p_n xy|) \le \tau(|xy|)$. (Here we used Proposition 5.11 and the fact that $xy \in L^1(\mathcal{M}, \tau)$.) This fact combined with Theorem 4.26 would then ensure that

$$\tau(x^{1/2}yx^{1/2}) \le \liminf_{n\to\infty} \tau(p_n x^{1/2}yx^{1/2}p_n) \le \tau(|xy|) < \infty.$$

Thus as required $x^{1/2}yx^{1/2} \in L^1(\mathcal{M}, \tau)$. $\square$

We pass to developing the duality theory for the case $1 < p < \infty$.

LEMMA 5.23. *Suppose $1 < p < \infty$. For any $x \in L^p(\mathcal{M}, \tau)$, we have that*

$$\|x\|_p = \sup\{|\tau(yx)| \colon y \in L^q(\mathcal{M}, \tau), \|y\|_q \le 1\}$$
$$= \sup\{|\tau(xy)| \colon y \in L^q(\mathcal{M}, \tau), \|y\|_q \le 1\}$$

where $1 = \frac{1}{p} + \frac{1}{q}$.

PROOF. We prove the first equality. Hölder's inequality combined with the fact that $|\tau(yx)| \le \tau(|yx|)$ for each $x \in L^p(\mathcal{M}, \tau)$, $y \in L^q(\mathcal{M}, \tau)$ ensures that $\sup\{|\tau(yx)| \colon y \in L^q(\mathcal{M}, \tau), \|y\|_q \le 1\} \le \|x\|_p$.

For the converse let $0 \ne x \in L^p(\mathcal{M}, \tau)$ be given. With $x = u|x|$ the polar decomposition of x, set $y = \|x\|_p^{-p/q}|x|^{p-1}u^*$. By Propositions 4.12 and 4.13, we have that $\tau(|y|^q) = \tau(|y^*|^q) = \|x\|^{-p}\tau(|x|^{qp-q}) = \|x\|_p^{-p}\tau(|x|^p) = 1$, and hence that $y \in L^q(\mathcal{M}, \tau)$ with $\|y\| = 1$. By construction

$$\tau(yx) = \|x\|_p^{-p/q}\tau(|x|^p) = \|x\|_p^{p-pq} = \|x\|_p.$$

Hence equality must hold. $\square$

LEMMA 5.24. *Let $1 \leq p < \infty$, and let $a, b \in L_+^p(\mathcal{M}, \tau)$ be given. Then*

$$2^{1-p}\|a + b\|_p^p \leq \|a\|_p^p + \|b\|_p^p \leq \|a + b\|_p^p.$$

PROOF. We know that $\|a + b\|_p \leq \|a\|_p + \|b\|_p$. Since $1 \leq p < \infty$, $t \mapsto t^p$ is a convex function on $[0, \infty)$, which then ensures that

$$\|a + b\|_p^p = (\|a\|_p + \|b\|_p)^p = 2^p \left(\frac{\|a\|_p}{2} + \frac{\|b\|_p}{2}\right)^p \leq 2^p \left(\frac{1}{2}\left[\|a\|_p^p + \|b\|_p^p\right]\right).$$

This proves the first inequality. The second is an immediate consequence of part (ii) of Proposition 5.8. □

Using the above lemma, we are now able to prove part of the famous Clarkson-McCarthy inequalities.

PROPOSITION 5.25. *Let $2 \leq p < \infty$, and let $a, b \in L^p(\mathcal{M}, \tau)$ be given. Then*

$$\|a + b\|_p^p + \|a - b\|_p^p \leq 2^{p-1}(\|a\|_p^p + \|b\|_p^p).$$

PROOF. On setting $r = p/2$, two applications of the lemma show that

$$
\begin{aligned}
\|a + b\|_p^p + \|a - b\|_p^p &= \| |a + b|^2 \|_r^r + \| |a - b|^2 \|_r^r \\
&\leq \| |a + b|^2 + |a - b|^2 \|_r^r \\
&= 2^r \| |a|^2 + |b|^2 \|_r^r \\
&\leq 2^r 2^{r-1} \left(\| |a|^2 \|_r^r + \| |b|^2 \|_r^r \right) \\
&\leq 2^{p-1} \left(\|a\|_p^p + \|b\|_p^p \right).
\end{aligned}
$$

□

We now introduce the concept of uniformly convex Banach spaces.

DEFINITION 5.26. A Banach space X is said to be uniformly convex if for every $0 < \epsilon \leq 2$, we can find a $\delta > 0$, so that for any two norm 1 vectors $x, y \in X$, the situation $\|x - y\| \geq \epsilon$, ensures that $\frac{\|x+y\|}{2} \leq 1 - \delta$.

It is an easy exercise to see that the Clarkson-McCarthy inequalities verified above, ensure that for each $2 \leq p < \infty$, the space $L^p(\mathcal{M}, \tau)$ is uniformly convex. A crucial step in the development of a duality theory for L^p-spaces is showing that $L^p(\mathcal{M}, \tau)$ is reflexive (first for $2 \leq p < \infty$ then later for $1 \leq p < 2$ as well). For this step the Milman-Pettis theorem, which asserts that all uniformly convex Banach spaces are reflexive,

comes to the rescue. For the sake of a deeper understanding of the under-lying principles, we will give a more self-contained proof of L^p-duality by embedding much of the proof of the Milman-Pettis theorem in that proof.

THEOREM 5.27. *Let $1 < p \leq \infty$ and $1 \leq q < \infty$ be given with $1 = \frac{1}{p} + \frac{1}{q}$. The bilinear form*

$$L^p(\mathcal{M}, \tau) \times L^q(\mathcal{M}, \tau) \to \mathbb{C} : (x, y) \mapsto \tau(yx)$$

defines a dual action of $L^p(\mathcal{M}, \tau)$ on $L^q(\mathcal{M}, \tau)$ with respect to which $L^p(\mathcal{M}, \tau)$ is identified with $(L^q(\mathcal{M}, \tau))^*$. Specifically, for $x \in L^p(\mathcal{M}, \tau)$ the prescription $y \mapsto \tau(yx)$ defines a bounded linear functional ω_x on $L^q(\mathcal{M}, \tau)$. Moreover the mapping $x \mapsto \omega_x$ is a surjective isometry from $L^p(\mathcal{M}, \tau)$ onto $(L^q(\mathcal{M}, \tau))^*$. In addition $\omega_x \geq 0$ if and only if $x \geq 0$.

PROOF. Apart from the final claim, the case where $q = 1$ corresponds to Theorem 5.20. In considering the first two claims, we may therefore assume that $q > 1$. It is clear from Lemma 5.23 that the mapping $\iota : L^p(\mathcal{M}, \tau) \to (L^q(\mathcal{M}, \tau))^* : x \mapsto \omega_x$ is a linear isometry from $L^p(\mathcal{M}, \tau)$ onto a subspace of $(L^q(\mathcal{M}, \tau))^*$. Since $L^p(\mathcal{M}, \tau)$ is complete, $\iota(L^p(\mathcal{M}, \tau))$ must be a closed subspace.

We will next show that $\iota(L^p(\mathcal{M}, \tau))$ is a weak*-dense subspace of $(L^q(\mathcal{M}, \tau))^*$. By the bipolar theorem the $\sigma((L^q)^*, L^q)$-closure $\iota(L^p(\mathcal{M}, \tau))$, is given by $[(\iota(L^1(\mathcal{M}, \tau)))_\circ]^\circ$. (For an $A \subseteq L^q(\mathcal{M}, \tau)$ and $B \subseteq (L^q(\mathcal{M}, \tau))^*$, A° denotes the polar of A in $(L^q(\mathcal{M}, \tau))^*$, and $B_\circ$ the polar of B in $L^q(\mathcal{M}, \tau)$.) We show that $(\iota(L^p(\mathcal{M}, \tau)))_\circ = \{0\}$, from which the claim will then follow. For the sake of contradiction suppose that $(\iota(L^p(\mathcal{M}, \tau)))^\circ$ contains a non-zero element a. Let $a = u|a|$ be the polar decomposition of a. From the proof of Lemma 5.23, we know that then $x = \|a\|_q^{-q/p}|a|^{q-1}u^*$ is a norm 1 element of $L^p(\mathcal{M}, \tau)$. Since $a \in (\iota(L^p(\mathcal{M}, \tau)))_\circ$, we must have $\omega_x(a) = 0$. But on arguing as in the proof of Lemma 5.23, it follows that $\omega_x(a) = \tau(xa) = \|a\|_q \neq 0$. This is a clear contradiction. Hence $(\iota(L^p(\mathcal{M}, \tau)))^\circ = \{0\}$ as claimed.

Case 1 ($q \geq 2$): First consider the case where $q \geq 2$. We start by proving that $L^q(\mathcal{M}, \tau)$ is then reflexive. To see this, identify $L^q(\mathcal{M}, \tau)$ with the image of the natural embedding of $L^q(\mathcal{M}, \tau)$ into $(L^q(\mathcal{M}, \tau))^{**}$, and select an arbitrary norm 1 element $\tilde{x}$ of $(L^q(\mathcal{M}, \tau))^{**}$. By Goldstine's theorem, there must exist a net (x_α) in the closed unit ball of $L^q(\mathcal{M}, \tau)$ converging to $\tilde{x}$ in the weak*-topology on $(L^q(\mathcal{M}, \tau))^{**}$. Given $\delta > 0$, we may select a norm 1 element x^* of $(L^q(\mathcal{M}, \tau))^*$ so that $1 - \delta = \|\tilde{x}\| - \delta <$

$|\widetilde{x}(x^*)| \leq 1$. By the weak* convergence we will then have that $1 - \delta < \lim_\alpha |x^*(x_\alpha)| \leq \liminf_\alpha \|x_\alpha\| \leq \limsup_\alpha \|x_\alpha\|_q \leq 1$. Since $\delta > 0$ was arbitrary, we must have that $\lim_\alpha \|x_\alpha\| = 1$.

Now suppose that (x_α) is not Cauchy in norm, and let x^* be as before. Given $\epsilon > 0$, we may then inductively select an increasing sequence (α_n) of indices such that

$$\|x_{\alpha_{n+1}} - x_{\alpha_n}\| \geq \epsilon, \text{ and } |\widetilde{x}(x^*) - x^*(x_{\alpha_n})| \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Now use the Clarkson-McCarthy inequalities to conclude that

$$\frac{\|x_{\alpha_{n+1}} + x_{\alpha_n}\|^q}{2^q} + \frac{\|x_{\alpha_{n+1}} - x_{\alpha_n}\|^q}{2^q} \leq \frac{\|x_{\alpha_{n+1}}\|^q + \|x_{\alpha_n}\|^q}{2}$$

for all $n \in \mathbb{N}$. By the previously centred inequality, this in turn leads to

$$\frac{\|x_{\alpha_{n+1}} + x_{\alpha_n}\|^q}{2^q} + \frac{\epsilon^q}{2^q} \leq \frac{\|x_{\alpha_{n+1}} + x_{\alpha_n}\|^q}{2^q} + \frac{\|x_{\alpha_{n+1}} - x_{\alpha_n}\|^q}{2^q}$$

$$\leq \frac{\|x_{\alpha_{n+1}}\|^q + \|x_{\alpha_n}\|^q}{2} \leq 1.$$

Note that by construction $(x^*(x_{\alpha_{n+1}} + x_{\alpha_n}))$ will converge to $2\widetilde{x}(x^*)$ in the weak* topology. So on arguing as before, we have that $2 - 2\delta \leq |\widetilde{x}(x^*)| \leq \liminf_n \|x_{\alpha_{n+1}} + x_{\alpha_n}\| \leq 2$. But the previously centred inequality then leads to $(1 - \delta)^q + \frac{\epsilon^q}{2^q} \leq 1$, which is impossible for appropriate choices of ϵ and δ. Thus the assumption that (x_α) is not Cauchy in norm must be false. Being Cauchy, (x_α) must now by completeness converge to some element x of $L^q(\mathcal{M}, \tau)$. The net (x_α) will then also converge to x in the weak* topology on $(L^q(\mathcal{M}, \tau))^{**}$. By uniqueness of limits, we must have that $\widetilde{x} = x \in L^q(\mathcal{M}, \tau)$. This shows that $L^q(\mathcal{M}, \tau)$ is reflexive.

Since $L^q(\mathcal{M}, \tau)$ is reflexive, so is $(L^q(\mathcal{M}, \tau))^*$. But in that case the weak* closure of any subspace of $(L^q(\mathcal{M}, \tau))^*$ will agree with its norm closure. Since $\iota(L^p(\mathcal{M}, \tau))$ is then both norm-dense and closed, we have that $\iota(L^p(\mathcal{M}, \tau)) = (L^q(\mathcal{M}, \tau))^*$ as required.

Case 2 ($1 < q < 2$): If $1 < q < 2$, then $p > 2$. The first part of the proof then shows that $L^q(\mathcal{M}, \tau) = (L^p(\mathcal{M}, \tau))^*$ via the tracial bilinear form. But then $L^q(\mathcal{M}, \tau)$, is also reflexive, which means that the same argument as before shows that $\iota(L^p(\mathcal{M}, \tau)) = (L^q(\mathcal{M}, \tau))^*$.

For the final claim the "if" part follows from the observation that $\tau(xy) = \tau(y^{1/2}xy^{1/2}) \geq 0$ for any positive element y of $L^q(\mathcal{M}, \tau)$. Conversely assume that ω_x is positive. For any positive element y of $L^q(\mathcal{M}, \tau)$, the fact that $\omega_x(y) \geq 0$ then enables us to conclude that $\omega_x(y) = \tau(xy) =$

$\overline{\tau(xy)} = \tau(yx^*) = \omega_{x^*}(y)$. In view of the fact that any element of $L^q(\mathcal{M}, \tau)$ may be written as a linear combination of four positive elements of $L^q(\mathcal{M}, \tau)$, we therefore have that $\omega_x = \omega_{x^*}$, and hence that $x = x^*$ since $x \mapsto \omega_x$ is a bijection. Now consider the operator $|x|^{p-1}$. (In the case $p = 1$ we simply take $\mathbb{1}$ here.) Given that $\tau((|x|^{p-1})^q) = \tau(|x|^p) < \infty$, it is clear that $|x|^{p-1}$ belongs to $L^q(\mathcal{M}, \tau)$. Hence $|x|^{p-1}|x|^{p-1}\chi_{(-\infty,0)}(x)$ is a positive element of $L^q(\mathcal{M}, \tau)$. But for this positive element we have by the Borel functional calculus that $|x|^{p-1}x\chi_{(-\infty,0)}(x) = -|x|^p\chi_{(-\infty,0)}(x)$. So if $\chi_{(-\infty,0)}(x)$, and hence $x\chi_{(-\infty,0)}(x)$ were non-zero, $|-x\chi_{(-\infty,0)}(x)|^p = |x|^p\chi_{(-\infty,0)}(x)$ would be non-zero, in which case we would have that

$$\omega_x(|x|^{p-1}\chi_{(-\infty,0)}(x)) = \tau(x|x|^{p-1}\chi_{(-\infty,0)}(x)) = -\tau(|x|^p\chi_{(-\infty,0)}(x)) < 0,$$

contradicting the positivity of ω_x. Hence we must have that $x \geq 0$. □

5.2 Introduction to Orlicz spaces

The starting point of the theory of Orlicz spaces is the concept of a Young function (often also called an Orlicz function in the literature).

DEFINITION 5.28. We say that a function $\Phi : [0, \infty) \to [0, \infty]$ is a Young function if

- Φ is convex and increasing with $\Phi(0) = 0$,
- Φ is continuous on $[0, b_\Phi]$ where $b_\Phi = \sup\{t \in [0, \infty) : \Phi(t) < \infty\}$,
- Φ is neither identically zero nor infinite-valued on all of $(0, \infty)$. With a_Φ denoting the constant $\inf\{t \in [0, \infty) : \Phi(t) > 0\}$, neither of the situations $a_\Phi = \infty$, nor $b_\Phi = 0$ may therefore occur.

For any such Young function, we define the conjugate Young function to be Φ^* where for each $s > 0$, we set $\Phi^*(s) = \sup_{t>0}(st - \Phi(t))$.

EXERCISE 5.29. Show that Φ^* is a Young function and that $\Phi^{**} = \Phi$.

REMARK 5.30.(a) If $b_\Phi < \infty$, then $\Phi^*(s) \leq b_\Phi s$ for any $s \geq 0$. To see this let $s > 0$ be given, and observe that if $t > b_\Phi$, then $st - \Phi(t) = -\infty$. Hence $\Phi^*(s) = \sup_{b_\Phi \geq t > 0}(st - \Phi(t)) \leq b_\Phi s$ as claimed.

(b) There is an interesting alternative for computing Φ^*. Recall that in the proof of Theorem 4.21 we saw that a Young function Φ is of the form $\Phi(t) = \int_0^t \phi(s)\,ds$ for some non-negative left-continuous non-decreasing function ϕ on $[0, \infty)$, which is infinite-valued on (b_Φ, ∞), but neither identically 0 nor infinite-valued on all of $(0, \infty)$. Now consider the function $\psi(t) = \inf\{s : \phi(s) \geq t\}$.

The function ψ is then also a non-negative, non-decreasing, left-continuous function on $[0, \infty)$, which is finite-valued on some portion $[0, r]$ of $[0, \infty)$, and neither identically 0 nor infinite-valued on all of $(0, \infty)$. Then Ψ defined by $\Psi(t) = \int_0^\infty \psi(s) \, ds$, turns out to be nothing but the Young function Φ^*. The fact that Ψ as defined above agrees with Φ^*, follows from [**BS88**, Theorem IV.8.12] (see also §1.7 of [**NP06**].)

(c) Note that for any Young function Φ, the pair (Φ, Φ^*) will by the definition of Φ^*, satisfy the *Hausdorff-Young inequality*

$$st \leq \Phi(t) + \Phi^*(s) \quad \text{for all } s, t \geq 0.$$

The description of Φ and Φ^* given above, forms the basis for the equality criteria for this inequality, namely that $st = \Phi(t) + \Phi^*(s)$ if and only if either $s = \phi(t)$, or $t = \psi(s)$. For a proof of this fact the reader is referred to [**BS88**, Theorem IV.8.12].

(d) Given a Young function, we shall have occasion to use the "right-continuous inverse" Φ^{-1} of Φ on $[0, \infty)$, given by

$$\Phi^{-1}(t) = \sup\{s \colon \Phi(s) \leq t\}.$$

It is only in the case where $a_\Phi = 0$ and $b_\Phi = \infty$ that this is an inverse in the true sense of the word. To see this note that it is an exercise to see that $\Phi^{-1}(0) = a_\Phi$. Now observe that the fact that Φ is strictly increasing on (a_Φ, b_Φ), ensures that for every $s \in (a_\Phi, b_\Phi)$, $\Phi(s) = t$ if and only if $s = \Phi^{-1}(t)$. If in fact $\Phi(b_\Phi) < \infty$, it is similarly an exercise to see that $\Phi^{-1}(t) = b_\Phi$ for every $t \geq \Phi(b_\Phi)$. So Φ^{-1} is a continuous function for which we have that $\Phi \circ \Phi^{-1}(t) \leq t \leq \Phi^{-1} \circ \Phi(t)$ for all $t \geq 0$.

EXERCISE 5.31. Prove that the conjugate Young function of $\Phi_1(t) = t$ is

$$\Phi_\infty(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ \infty & \text{if } 1 < t \end{cases},$$

and that the conjugate function of $\cosh -1$, is $\int_0^t \sinh^{-1}(s) \, ds = t \log(t + \sqrt{t^2 + 1}) - \sqrt{t^2 + 1} + 1$.

DEFINITION 5.32. For a given Young function Φ, we define the non-commutative Orlicz space $L^\Phi(\mathcal{M}, \tau)$ to be the collection of all $f \in \widetilde{\mathcal{M}}$ for which there exists some $\alpha > 0$ such that $\tau(\Phi(\alpha|f|)) < \infty$. Note that even when Φ is infinite-valued on some part of the half-line, we can give meaning to $\Phi(\alpha|f|)$ as an element of the extended positive part of $\mathcal{M}$. Since the

action of τ extends to the extended positive part, the requirement that $\tau(\Phi(\alpha|f|)) < \infty$, always makes sense.

REMARK 5.33. We refine Remark 4.15 in the context of Young functions. Let $a \in {}^\eta\mathcal{M}$ be given and let Φ be a general Young function. Then $\Phi(|a|)$ will in general not be a member of $\widetilde{\mathcal{M}}$ (unless of course $a \in \widetilde{\mathcal{M}}$ and $\mathrm{sp}(|a|) \subseteq [0, b_\Phi)$). However we are able to give meaning to $\Phi(|a|)$ as an element of $\widehat{\mathcal{M}}_+$ (and hence also give meaning to $\tau(\Phi(|a|))$). We pause to give some details of how this works: Suppose $\mathcal{M}$ acts on the Hilbert space H. If $b_\Phi = \infty$, we have that $\mathrm{sp}(|a|) \subseteq [0, b_\Phi) = [0, \infty)$, in which case we can then use the continuous functional calculus to see that $\Phi(|a|) \in \widetilde{\mathcal{M}}$. Now suppose that $b_\Phi < \infty$. There are two cases to consider here, namely $\Phi(b_\Phi) = \infty$, and $\Phi(b_\Phi) < \infty$. Suppose $\Phi(b_\Phi) = \infty$. If we attempt to use the spectral resolution $|a| = \int_0^\infty \lambda \, de_\lambda(|a|)$ to define $\Phi(|a|)$ by means of the prescription $\Phi(|a|) = \int_0^\infty \Phi(\lambda) \, de_\lambda(|a|)$, we find that $\Phi(|a|)$ exists as a densely-defined closed operator on $\chi_{[0,b_\Phi)}(|a|)(H)$ (which commutes with all the unitaries in the commutant of $\mathcal{M}$), but that $\int_0^\infty \Phi(\lambda) \, d\langle e_\lambda(|a|)\xi, \xi\rangle = \infty$ for all $\xi \in \chi_{[b_\Phi, \infty)}(H)$. By Theorem 1.133, such objects are all part of $\widehat{\mathcal{M}}_+$. The only difference in the case $\Phi(b_\Phi) < \infty$, is that here $\Phi(|a|)$ makes sense as a densely defined operator on $\chi_{[0,b_\Phi]}(|a|)(H)$ not $\chi_{[0,b_\Phi)}(|a|)(H)$. As noted in Remark 4.15, we have that $\tau(\Phi(|a|)) < \infty$ if and only if $\Phi(|a|)$ corresponds to an element of $L^1(\mathcal{M}, \tau)$. However more is true in this case. Recall that the right inverse Φ^{-1}, is continuous on $[0, \infty)$ with $\Phi^{-1}(\Phi(t)) \geq t$ for all $t \geq 0$. Thus if indeed $\Phi(|a|) \in L^1(\mathcal{M}, \tau)$, then by the functional calculus for positive operators, $\Phi^{-1}(\Phi(|a|))$ will be a τ-measurable element of $\widehat{\mathcal{M}}$ such that $\Phi^{-1}(\Phi(|a|)) \geq |a|$. This ensures that $|a|$, and hence also a, is then a τ-measurable element of $\widehat{\mathcal{M}}$.

To sum up, in terms of the action of τ on $\widehat{\mathcal{M}}_+$, we have that a given $a \in {}^\eta\mathcal{M}$ will belong to $L^\Phi(\mathcal{M}, \tau)$ if and only if $\tau_\mathcal{M}(\Phi(\alpha|a|)) < \infty$ for some $\alpha > 0$.

We now use the ideas described in the preceding remark to prove a deep fact regarding noncommutative Orlicz spaces. This fact is an extremely useful tool for lifting the classical theory to the noncommutative context.

THEOREM 5.34. Let Φ be a Young function and let $f \in \widetilde{\mathcal{M}}$ be given. Then $\tau(\Phi(|f|)) = \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds$. (Here $\Phi(|f|)$ may not be in $\widetilde{\mathcal{M}}$, but is given meaning as an object in the extended positive part of $\mathcal{M}$.) Moreover whenever $\tau(\Phi(|f|))$ is finite, we have that $\Phi(|f|) \in \widetilde{\mathcal{M}}$

PROOF. If indeed $\tau(\Phi(|f|))$ is finite, then by Remark 4.15, $\Phi(|f|) \in L^1(\mathcal{M}, \tau)$. The fact that in this case $\tau(\Phi(|f|)) = \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds$, then follows from Corollary 4.14. To conclude the proof, we need to show that $\tau(\Phi(|f|))$ will be finite, whenever $\int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds$ is finite. So with this in mind, suppose we are given that $\int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds < \infty$. We consider two cases.

Case 1, $(b_\Phi = \infty)$: In this case Φ is just a convex non-decreasing continuous function on $[0, \infty)$. It then follows from the continuous functional calculus that $\Phi(|f|)$ is again τ-measurable. But that means we can apply Corollary 4.14, to conclude that $\tau(\Phi(|f|)) = \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds < \infty$ as required.

Case 2, $(b_\Phi < \infty)$: Recall that Φ is infinite-valued on (b_Φ, ∞). So here the only way we could have $\int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds < \infty$, is if $\mathbf{m}_f(s) \leq b_\Phi$ for all $s > 0$. By the right-continuity of $s \mapsto \mathbf{m}_f(s)$, we will then have that $\|f\|_\infty = \mathbf{m}_f(0) \leq b_\Phi < \infty$. If in fact $\Phi(b_\Phi) < \infty$, it would then follow from the continuous functional calculus that $\Phi(|f|) \in \mathcal{M}$, in which case we could then apply Corollary 4.14, to see that $\tau(\Phi(|f|)) = \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds < \infty$. Hence assume that $\Phi(b_\Phi) = \infty$. For any $0 < \epsilon < 1$, we will then have that $\mathrm{sp}(\epsilon|f|) \subseteq [0, \epsilon\|f\|_\infty] \subseteq [0, b_\Phi)$. It will then follow from the continuous functional calculus that $\Phi(\epsilon|f|)$ even belongs to $\mathcal{M}$. Next recall that $\Phi(|f|)$ may be realised as a member of the extended positive part of $\mathcal{M}$. Suppose that $\mathcal{M}$ acts on the Hilbert space H. Recall from Remark 5.33 that in this case we could make sense of $\Phi(|f|)$ as a densely defined closed operator on $\chi_{[0, b_\Phi)}(|a|)(H)$, but that (formally) $\langle \Phi(|f|)^{1/2}\xi, \xi \rangle = \infty$ for all $\xi \in \chi_{[b_\Phi, \infty)}(|a|)(H)$. Now select a sequence $(\epsilon_n) \subseteq (0, 1)$ increasing to 1. Using the very specific structure of the function Φ in this case, it is then a somewhat non-trivial exercise to see that as members of the extended positive part $\widehat{\mathcal{M}}_+$, the operators $(\Phi(\epsilon_n|f|))$ increase to $\Phi(|f|)$. (Although infinite-valued on $[b_\Phi, \infty)$, Φ is here continuous on "all" of $[0, \infty)$ in the sense that it is continuous on $[0, b_\Phi)$ with $\Phi(t)$ increasing to ∞ as t increases to b_Φ.) But the extension of the trace to $\widehat{\mathcal{M}}_+$, respects such suprema (Theorem 3.21). Therefore $\tau(\Phi(|f|)) = \sup \tau(\Phi(\epsilon_n|f|))$. Since each $\Phi(\epsilon_n|f|)$ belongs to $\mathcal{M}$ and since $\mathbf{m}_{\epsilon_n f} \leq \mathbf{m}_f$, we may use Corollary 4.14 to see that $\tau(\Phi(\epsilon_n|f|)) = \int_0^\infty \Phi(\mathbf{m}_{\epsilon_n f}(s)) \, ds \leq \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds < \infty$. Hence $\tau(\Phi(|f|)) \leq \int_0^\infty \Phi(\mathbf{m}_f(s)) \, ds$ as required. $\qquad\square$

DEFINITION 5.35. Let $L^\Phi(\mathcal{M}, \tau)$ be as before. Define the Luxemburg-Nakano norm on $L^\Phi(\mathcal{M}, \tau)$ to be $\|f\|_\Phi = \inf\{\epsilon > 0 : \tau(\Phi(\epsilon^{-1}|f|)) \leq 1\}$.

The Orlicz norm is defined to be the quantity

$$\|f\|_\Phi^O = \inf\{\tau(|fg|) \colon g \in L^{\Phi^*}(\mathcal{M}, \tau), \tau(\Phi^*(|g|)) \leq 1\}.$$

The first task that now befalls us, is to prove that $L^\Phi(\mathcal{M}, \tau)$ is a linear space and that these quantities are in fact norms. After that we will compare these norms and investigate questions of completeness and duality. Our first result strengthens the link between the classical and noncommutative theory noted in Theorem 5.34 above.

COROLLARY 5.36. *Let Φ be a Young function and let $f \in \widetilde{\mathcal{M}}$ be given. Then $\mathbf{m}_f \in L^\Phi(0, \infty)$ if and only if $f \in L^\Phi(\mathcal{M}, \tau)$. Moreover if indeed $f \in L^\Phi(\mathcal{M}, \tau)$, then $\|f\|_\Phi = \|\mathbf{m}_f\|_\Phi$.*

THEOREM 5.37. *Let Φ be a Young function. Then $L^\Phi(\mathcal{M}, \tau)$ is a linear space, and $\| \cdot \|_\Phi$ a norm for $L^\Phi(\mathcal{M}, \tau)$.*

PROOF. Given $f \in L^\Phi(\mathcal{M}, \tau)$, it is an easy exercise to see that for any $\alpha \in \mathbb{C}$, αf is again in $L^\Phi(\mathcal{M}, \tau)$. Next let $f, g \in L^\Phi(\mathcal{M}, \tau)$ be given. By Corollary 5.36, we may select $\alpha_f, \alpha_g > 0$ so that $\int_0^\infty \Phi(\alpha_f \mathbf{m}_f(s))\, ds < \infty$ and $\int_0^\infty \Phi(\alpha_g \mathbf{m}_g(s))\, ds < \infty$. For $\alpha = \frac{1}{2}\min(\alpha_f, \alpha_g)$, it will then follow from Theorem 4.22 that

$$
\begin{aligned}
\int_0^\infty \Phi(\alpha \mathbf{m}_{f+g}(s))\, ds &= \int_0^\infty \Phi(\mathbf{m}_{\alpha(f+g)}(s))\, ds \\
&\leq \int_0^\infty \Phi(\mathbf{m}_{\alpha f}(s) + \mathbf{m}_{\alpha g}(s))\, ds \\
&\leq \int_0^\infty \Phi(\alpha(\mathbf{m}_f(s) + \mathbf{m}_g(s)))\, ds \\
&\leq \int_0^\infty \Phi\left(\tfrac{\alpha_f \mathbf{m}_f(s)}{2} + \tfrac{\alpha_g \mathbf{m}_g(s)}{2}\right) ds.
\end{aligned}
$$

We may now use the convexity of Φ to see that we then further have that

$$
\begin{aligned}
\int_0^\infty \Phi(\alpha \mathbf{m}_{f+g}(s))\, ds &\leq \int_0^\infty \Phi\left(\tfrac{\alpha_f \mathbf{m}_f(s)}{2} + \tfrac{\alpha_g \mathbf{m}_g(s)}{2}\right) ds \\
&\leq \int_0^\infty \tfrac{1}{2}[\Phi(\alpha_f \mathbf{m}_f(s)) + \Phi(\alpha_g \mathbf{m}_g(s)]\, ds \\
&< \infty.
\end{aligned}
$$

But then by Corollary 5.36, $f + g \in L^\Phi(\mathcal{M}, \tau)$.

We now show that $\| \cdot \|_\Phi$ is a norm. Let $f \in L^\Phi(\mathcal{M}, \tau)$ be given with $\|f\|_\Phi = 0$, or equivalently $\|\mathbf{m}_f\|_\Phi = 0$. For any $\epsilon > 0$ we will then by

definition have that $\int_0^\infty \Phi(\epsilon^{-1}\mathbf{m}_f(s))\,ds \le 1$. By convexity and the fact that $\Phi(0) = 0$, we have that $\Phi(rt) = \Phi((1-r)0 + rt) \le r\Phi(t)$ for any $0 < r \le 1$, $t \ge 0$. Equivalently $\Phi(\gamma t) \ge \gamma\Phi(t)$ for any $\gamma \ge 1$. Given $0 < \epsilon$, we will then for any $\gamma \ge 1$ have that

$$\gamma \int_0^\infty \Phi(\epsilon^{-1}\mathbf{m}_f(s))\,ds \le \int_0^\infty \Phi(\gamma\epsilon^{-1}\mathbf{m}_f(s))\,ds \le 1.$$

This can in turn only be true if $\int_0^\infty \Phi(\epsilon^{-1}\mathbf{m}_f(s))\,ds = 0$. That means that for any $\epsilon > 0$, $\Phi(\epsilon^{-1}\mathbf{m}_f(s))$ is 0 almost everywhere. Since $\mathbf{m}_f$ is non-increasing, and Φ non-zero on some connected portion of $(0, \infty)$, the only way this can be, is if $\mathbf{m}_f$ is 0 on $(0, \infty)$. The right-continuity of $\mathbf{m}_f$ then ensures that $\|f\|_\infty = \mathbf{m}_f(0) = 0$, in other words that $f = 0$. It is a simple exercise to see that for any $f \in L^\Phi(\mathcal{M}, \tau)$ and any $\gamma \in \mathbb{C}$, $\|\gamma f\|_\Phi = |\gamma|\,\|f\|_\Phi$. We proceed to prove the triangle inequality.

Let $f, g \in L^\Phi(\mathcal{M}, \tau)$ be given. For any $\epsilon > 0$, we will by the definition of the Luxemburg-Nakano norm then have that $\int_0^\infty \Phi((\|f\|_\Phi + \epsilon)^{-1}\mathbf{m}_f(s))\,ds \le 1$ and $\int_0^\infty \Phi((\|g\|_\Phi + \epsilon)^{-1}\mathbf{m}_g(s))\,ds \le 1$. (Here we silently used Corollary 5.36.) We may then use Theorem 4.22 and the convexity of Φ, to see that

$$\int_0^\infty \Phi((\|f\| + \|g\| + 2\epsilon)^{-1}\mathbf{m}_{f+g}(s))\,ds$$

$$\le \int_0^\infty \Phi((\|f\| + \|g\| + 2\epsilon)^{-1}(\mathbf{m}_f(s) + \mathbf{m}_g(s))\,ds$$

$$\le \int_0^\infty \Phi\big(\tfrac{\|f\|+\epsilon}{\|f\|_\Phi+\|g\|+2\epsilon}(\|f\| + \epsilon)^{-1}\mathbf{m}_f(s)$$

$$+ \tfrac{\|g\|+\epsilon}{\|f\|+\|g\|+2\epsilon}(\|g\| + \epsilon)^{-1}\mathbf{m}_g(s)\big)\,ds$$

$$\le \tfrac{\|f\|+\epsilon}{\|f\|_\Phi+\|g\|+2\epsilon}\int_0^\infty \Phi((\|f\| + \epsilon)^{-1}\mathbf{m}_f(s))\,ds$$

$$+ \tfrac{\|g\|+\epsilon}{\|f\|+\|g\|+2\epsilon}\int_0^\infty \Phi((\|g\| + \epsilon)^{-1}\mathbf{m}_g(s))\,ds$$

$$\le \tfrac{\|f\|+\epsilon}{\|f\|_\Phi+\|g\|+2\epsilon} + \tfrac{\|g\|+\epsilon}{\|f\|+\|g\|+2\epsilon}$$

$$= 1$$

(Here we dropped the subscripts of the norms for the sake of clarity.) The above clearly shows that $\|f + g\|_\Phi = \|\mathbf{m}_{f+g}\|_\Phi \le \|\mathbf{m}_f\|_\Phi + \|\mathbf{m}_g\|_\Phi + 2\epsilon =$

$\|f\|_\Phi + \|g\|_\Phi + 2\epsilon$. Since $\epsilon > 0$ was arbitrary, we have that $\|f + g\|_\Phi \leq \|f\|_\Phi + \|g\|_\Phi$ as required. $\qquad\square$

EXERCISE 5.38. Show that $L^p(\mathcal{M}, \tau)$ $(1 \leq p \leq \infty)$ are Orlicz spaces. Also show that the Orlicz space corresponding to the Young function

$$\Phi_\infty(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ \infty & \text{if } 1 < t \end{cases}$$

is $L^\infty(\mathcal{M}, \tau)$.

REMARK 5.39. In the theory of Orlicz spaces two Young functions Φ and Ψ are said to be equivalent if there exists a constant $K > 0$ so that $K^{-1}\Phi \leq \Psi \leq K\Phi$. It is an interesting exercise to show that the norms $\| \cdot \|_\Phi$ and $\| \cdot \|_\Psi$ are equivalent whenever Φ and Ψ are.

We now show that $\| \cdot \|_\Phi^O$ is a seminorm, and then use that fact to prove that $L^\Phi(\mathcal{M}, \tau)$ injects continuously into $\widetilde{\mathcal{M}}$ and that it is in fact complete. After that we will show that $\| \cdot \|_\Phi^O$ is in fact a norm which is equivalent to $\| \cdot \|_\Phi$. To show that $\| \cdot \|_\Phi^O$ is a seminorm, all we need to do is to show that for each $f \in L^\Phi(\mathcal{M}, \tau)$, $\|f\|_\Phi^0$ is finite. We need the following lemma to prove this fact. Apart from other considerations, this lemma shows that in the definition of the Orlicz norm, the requirement $\tau(\Phi^*(|g|)) \leq 1$, can be replaced with the requirement that $\|g\|_{\Phi^*} \leq 1$.

LEMMA 5.40. *Let $f \in L^\Phi(\mathcal{M}, \tau)$ be given. If $f \neq 0$, then $\tau(\Phi((\|f\|_\Phi)^{-1}|f|)) \leq 1$. If $\|f\|_\Phi \leq 1$, we will have that $\tau(\Phi(|f|)) \leq \|f\|_\Phi$, whilst if $\|f\|_\Phi > 1$, we will have that $\tau(\Phi(|f|)) \geq \|f\|_\Phi$. Therefore $\tau(\Phi(|f|)) \leq 1$ if and only if $\|f\|_\Phi \leq 1$.*

PROOF. To prove the first claim, select a sequence $(\epsilon_n) \subseteq (\|f\|_\Phi, \infty)$ decreasing to $\|f\|_\Phi = \|\mathbf{m}_f\|_\Phi$. Then by the monotone convergence theorem $\int_0^\infty \Phi(\epsilon_n^{-1}\mathbf{m}_f(s)) \, ds$, will increase to $\int_0^\infty \Phi(\|\mathbf{m}_f\|_\Phi^{-1}\mathbf{m}_f(s)) \, ds$. Since by the definition of $\|\mathbf{m}_f\|_\Phi$ we have that $\int_0^\infty \Phi(\epsilon_n^{-1}\mathbf{m}_f(s)) \, ds \leq 1$ for each n, it is clear that $\tau(\Phi(\|f\|^{-1}|f|)) = \int_0^\infty \Phi(\|\mathbf{m}_f\|_\Phi^{-1}\mathbf{m}_f(s)) \, ds \leq 1$.

For the second claim, suppose that $\|f\|_\Phi \leq 1$. If $\|f\|_\Phi = 0$, then $f = 0$, whence $\tau(\Phi(|f|)) = 0 \leq 1$. So assume that $0 < \|f\|_\Phi \leq 1$. It then follows from the first part of the proof that $\tau(\Phi(\|f\|_\Phi^{-1}|f|)) \leq 1$, and hence that $\Phi(\|f\|_\Phi^{-1}|f|) \in \widetilde{\mathcal{M}}$. If we combine the convexity of Φ with the fact that $\Phi(0) = 0$, it is then clear that $\Phi(rt) \leq r\Phi(t)$ for any $t \geq 0$ and any $0 \leq r \leq 1$. Equivalently $\Phi(\gamma t) \geq \gamma\Phi(t)$ for any $t \geq 0$ and any $\gamma \leq 1$. Since by assumption $0 < \|f\|_\Phi \leq 1$, we have that $\Phi(\|f\|_\Phi^{-1}t) \geq \|f\|_\Phi^{-1}\Phi(t)$ for

any $t \geq 0$, and hence that $\Phi(\|f\|_\Phi^{-1}|f|) \geq \|f\|_\Phi^{-1}\Phi(|f|)$. Taking into account that $\Phi(\|f\|_\Phi^{-1}|f|) \in \widetilde{\mathcal{M}}$, we must therefore also have that $\|f\|_\Phi^{-1}\Phi(|f|) \in \widetilde{\mathcal{M}}$, with $\|f\|_\Phi^{-1}\tau(\Phi(|f|)) \leq \tau(\Phi(\|f\|_\Phi^{-1}|f|)) \leq 1$ as required.

Now suppose that $\|f\|_\Phi > 1$. The claimed inequality will clearly follow if $\tau(\Phi(|f|)) = \infty$. Hence assume that $\tau(\Phi(|f|)) < \infty$. Recall that this ensures that $\Phi(|f|) \in \widetilde{\mathcal{M}}$. Since $\|f\|_\Phi > 1$, we may select $\epsilon > 0$ such that $(\|f\|_\Phi - \epsilon) > 1$. Since $(\|f\|_\Phi - \epsilon) < \|f\|_\Phi$, we must by the definition of the Luxemburg-Nakano norm have that $\tau(\Phi((\|f\|_\Phi - \epsilon)^{-1}|f|)) > 1$. We once again note that the convexity of Φ ensures that $\Phi(rt) \leq r\Phi(t)$ for any $t \geq 0$ and any $r \leq 1$. Given that $(\|f\|_\Phi - \epsilon) > 1$, we therefore have that $\Phi((\|f\|_\Phi - \epsilon)^{-1}t) \leq (\|f\|_\Phi - \epsilon)^{-1})\Phi(t)$ for any $t \geq 0$, and hence that $\Phi((\|f\|_\Phi - \epsilon)^{-1}|f|) \leq (\|f\|_\Phi - \epsilon)^{-1}\Phi(|f|)$. But then $\Phi((\|f\|_\Phi - \epsilon)^{-1}|f|)$ must belong to $\widetilde{\mathcal{M}}$ since $(\|f\|_\Phi - \epsilon)^{-1}\Phi(|f|)$ does. On applying the trace, it follows that $1 < \tau(\Phi((\|f\|_\Phi - \epsilon)^{-1}f)) \leq (\|f\|_\Phi - \epsilon)^{-1}\tau(\Phi(f))$. Since $\epsilon > 0$ was arbitrary, we have that $\tau(\Phi(|f|)) \geq \|f\|_\Phi$ as required.

The one direction of the final claim clearly follows from the first claim, and the other from the definition of the Luxemburg-Nakano norm.   □

PROPOSITION 5.41. *For any Young function Φ and any $f \in L^\Phi(\mathcal{M}, \tau)$, we have that $\|f\|_\Phi^O \leq 2\|f\|_\Phi$.*

PROOF. Let f be a nonzero element of $L^\Phi(\mathcal{M}, \tau)$. Recall that the pair (Φ, Φ^*) satisfies the Hausdorff-Young inequality $uv \leq \Phi(u) + \Phi^*(v)$ $(u, v \geq 0)$. Given $g \in L^{\Phi^*}(\mathcal{M}, \tau)$ and $\alpha \geq 0$, we may then combine this fact with Theorem 5.2, to see that

$$
\begin{aligned}
\tau(|\alpha fg|) &= \int_0^\infty (\mathbf{m}_{\alpha fg}(s))\, ds \\
&\leq \int_0^\infty (\alpha \mathbf{m}_f(s)\mathbf{m}_g(s))\, ds \\
&\leq \int_0^\infty \Phi(\alpha \mathbf{m}_f(s))\, ds + \int_0^\infty \Phi^*(\mathbf{m}_g(s))\, ds \\
&= \tau(\Phi(\alpha|f|)) + \tau(\Phi^*(|g|)).
\end{aligned}
$$

Recall that in the proof of the lemma we showed that $\tau(\Phi(\|f\|_\Phi^{-1}|f|))$ $= \int_0^\infty \Phi(\|\mathbf{m}_f\|_\Phi^{-1}\mathbf{m}_f(s))\, ds \leq 1$. So if in addition $\tau(\Phi^*(|g|)) \leq 1$, we will have that $\tau(\|\|f\|_\Phi^{-1}fg|) \leq 2$, and hence that $\|\|f\|_\Phi^{-1}f\|_\Phi^O \leq 2$.   □

The above proposition clearly shows that the Orlicz norm is finite on all of $L^\Phi(\mathcal{M}, \tau)$. Having noted this fact, we are now ready for the following Hölder inequality for Orlicz spaces.

COROLLARY 5.42. *For any $f \in L^\Phi(\mathcal{M}, \tau)$ and any $g \in L^{\Phi^*}(\mathcal{M}, \tau)$ we have that $fg, gf \in L^1(\mathcal{M}, \tau)$, with $\tau(|fg|) \leq \|f\|_\Phi^O \|g\|_{\Phi^*}$. In particular $\|f\|_\Phi^O = \sup\{\tau(|fg|) : \tau(\Phi^*(|g|)) \leq 1\} = \sup\{|\tau(fg)| : \tau(\Phi^*(|g|)) \leq 1\} = \sup\{|\tau(gf)| : \tau(\Phi^*(|g|)) \leq 1\} = \sup\{\tau(|gf|) : \tau(\Phi^*(|g|)) \leq 1\}.$*

PROOF. Let $f \in L^\Phi(\mathcal{M}, \tau)$ and $g \in L^{\Phi^*}(\mathcal{M}, \tau)$ be given. The fact that $\tau(|fg|) \leq \|f\|_\Phi^O \|g\|_{\Phi^*}$ follows fairly directly from Lemma 5.40 and the definition of the Orlicz norm. This clearly ensures that $fg \in L^1(\mathcal{M}, \tau)$. Since $\tau(|gf|) \leq \|f\|_\Phi \|g\|_{\Phi^*}^O$, we also have that $gf \in L^1(\mathcal{M}, \tau)$. It then follows from Theorem 5.22 that

$$\sup\{|\tau(fg)| : \tau(\Phi^*(|g|)) \leq 1\} = \sup\{|\tau(gf)| : \tau(\Phi^*(|g|)) \leq 1\}.$$

We prove that

$$\sup\{|\tau(fg)| : \tau(\Phi^*(|g|)) \leq 1\} = \sup\{\tau(|fg|) : \tau(\Phi^*(|g|)) \leq 1\}.$$

(The proofs of the remaining equalities are similar.) Since $|\tau(fg)| \leq \tau(|fg|)$ we clearly have that

$$\sup\{|\tau(fg)| : \tau(\Phi^*(|g|)) \leq 1\} \leq \sup\{\tau(|fg|) : \tau(\Phi^*(|g|)) \leq 1\}.$$

Let $g_0 \in L^{\Phi^*}(\mathcal{M}, \tau)$ be given with $\tau(\Phi^*(|g_0|)) \leq 1$, and let u be the partial isometry in the polar form $fg_0 = u|fg_0|$ of fg_0. Since $\mathbf{m}_{g_0 u^*} \leq \|u^*\| \mathbf{m}_{g_0} \leq \mathbf{m}_{g_0}$, we clearly have that $\int_0^\infty \Phi^*(\alpha \mathbf{m}_{g_0 u^*}(s))\, ds \leq \int_0^\infty \Phi^*(\alpha \mathbf{m}_{g_0}(s))\, ds$ for any $\alpha > 0$, and hence that $g_0 u^* \in L^{\Phi^*}(\mathcal{M}, \tau)$. By construction, we then have that $\tau(|fg_0|) = \tau(u^* fg_0) = \tau(fg_0 u^*) \leq \sup\{|\tau(fg)| : \tau(\Phi^*(|g|) \leq 1\}$. In view of the fact that $g_0 \in L^{\Phi^*}(\mathcal{M}, \tau)$ was arbitrary, we are done.   □

COROLLARY 5.43. *For any $f \in \widetilde{\mathcal{M}}$, all of the operators f, f^* and $|f|$ will belong to $L^\Phi(\mathcal{M}, \tau)$ whenever one of them does. In that case $\|f\|_\Phi = \|f^*\|_\Phi = \| |f| \|_\Phi$ and $\|f\|_\Phi^O = \|f^*\|_\Phi^O = \| |f| \|_\Phi^O.$*

PROOF. The first claim as well as the equality of the Luxemburg-Nakano norm is an immediate consequence of Proposition 4.12 considered alongside Corollary 5.36. On using what we have just verified, it now

follows from Corollary 5.42 that

$$\|f\|_\Phi^O = \sup\{\tau(|fg|)\colon \tau(\Phi^*(|g|) \le 1\}$$
$$= \sup\{\tau(|g^*f^*|)\colon \tau(\Phi^*(|g|) \le 1\}$$
$$= \sup\{\tau(|gf^*|)\colon \tau(\Phi^*(|g|) \le 1\} = \|f^*\|_\Phi^O.$$

Since for any $f, g \in \widetilde{\mathcal{M}}$ we have that $|fg| = ||f|g|$, it follows from the definition of the Orlicz norm that $\|f\|_\Phi^O = \||f|\|_\Phi^O$. □

As was the case with L^p-spaces, $\widetilde{\mathcal{M}}$ turns out to also be a natural superspace for Orlicz spaces.

PROPOSITION 5.44. *For any Young function Φ, the space $L^\Phi(\mathcal{M}, \tau)$ continuously injects into $\widetilde{\mathcal{M}}$.*

PROOF. Let (a_n) be a sequence in $L^\Phi(\mathcal{M}, \tau)$ converging to some $a \in L^\Phi(\mathcal{M}, \tau)$ in the norm $\| \cdot \|_\Phi$. Since $\mathbf{m}_{a-a_n}$ is non-increasing, we will for any $t > 0$ then have that

$$\begin{aligned}
\mathbf{m}_{a-a_n}(t) &\le \frac{1}{t}\int_0^t \mathbf{m}_{a-a_n}(s)\,ds \\
&= \frac{1}{t}\int_0^\infty \chi_{[0,t]}(s)\mathbf{m}_{a-a_n}(s)\,ds \\
&\le \frac{1}{t}\|\chi_{[0,t]}\|_{\Phi^*}^O \cdot \|\mathbf{m}_{a-a_n}\|_\Phi \\
&= \frac{1}{t}\|\chi_{[0,t]}\|_{\Phi^*}^O \cdot \|a - a_n\|_\Phi.
\end{aligned}$$

The claim now follows from Proposition 4.23. □

With the above result at our disposal, we proceed with the proof of the completeness of $L^\Phi(\mathcal{M}, \tau)$.

THEOREM 5.45. *Let Φ be a Young function. Then $(L^\Phi(\mathcal{M}, \tau), \| \cdot \|_\Phi)$ is complete.*

PROOF. Let (f_n) be a Cauchy sequence in $L^\Phi(\mathcal{M}, \tau)$. By Proposition 5.44 the sequence is Cauchy in the topology of convergence in measure. This topology is known to be complete, and hence there exists $f \in \widetilde{\mathcal{M}}$ so that $f_n \to f$ in measure. Let $\epsilon > 0$ be given. For any fixed m, $(f_n - f_m)$ will trivially converge in measure to $f - f_m$. Recall that $\mathbf{m}_{f-f_m}$ is finite-valued and monotone on $(0, \infty)$. Since by the Lebesgue-Young theorem such functions are known to be differentiable almost everywhere

(and hence continuous almost everywhere), we will by Lemma 4.25, then have that $\mathbf{m}_{f-f_m}(s) = \lim_{n\to\infty} \mathbf{m}_{f_n-f_m}(s)$ for almost every $s \geq 0$. Next select $N \in \mathbb{N}$ so that $\|f_n - f_m\|_\Phi = \|\mathbf{m}_{f_n-f_m}\|_\Phi < \epsilon$ for any $n, m \geq N$. We henceforth fix m as a natural number for which $m \geq N$. By the definition of the Luxemburg-Nakano norm (for $\mathbf{m}_{f_n-f_m}$), the fact that then $\|\mathbf{m}_{f_n-f_m}\|_\Phi < \epsilon$ for any $n \geq N$, means that

$$\int_0^\infty \Phi(\mathbf{m}_{\epsilon^{-1}(f_n-f_m)}(s))\, ds = \int_0^\infty \Phi(\epsilon^{-1}\mathbf{m}_{f_n-f_m}(s))\, ds \leq 1 \text{ for all } n \geq N.$$

In the case where $b_\Phi = \infty$, Φ is continuous and finite-valued on $[0,\infty)$, and hence in this case we will have that $\lim_{n\to\infty} \Phi((2\epsilon)^{-1}\mathbf{m}_{f_n-f_m}(s)) = \Phi((2\epsilon)^{-1}\mathbf{m}_{f-f_m}(s))$ for almost every s. Now suppose that $b_\Phi < \infty$. Then as in case 2 of the proof of Theorem 5.34, the fact that

$$\int_0^\infty \Phi(\mathbf{m}_{\epsilon^{-1}(f_n-f_m)}(s))\, ds \leq 1 < \infty \text{ for all } n \geq N$$

means that $\mathbf{m}_{\epsilon^{-1}(f_n-f_m)}(0) = \epsilon^{-1}\|f_n - f_m\|_\infty \leq b_\Phi$ for all $n \geq N$. Equivalently $(2\epsilon)^{-1}\mathbf{m}_{f_n-f_m}(s) \leq \frac{b_\Phi}{2}$ for all $s \geq 0$ and all $n \geq N$. Since Φ is continuous and finite-valued on $[0, \frac{b_\Phi}{2}]$, we will in this case also have that $\lim_{n\to\infty} \Phi((2\epsilon)^{-1}\mathbf{m}_{f_n-f_m}(s)) = \Phi((2\epsilon)^{-1}\mathbf{m}_{f-f_m}(s))$ for almost every s. We may therefore apply the standard Fatou lemma to see that

$$\int_0^\infty \Phi((2\epsilon)^{-1}\mathbf{m}_{f-f_m}(s))\, ds \leq \liminf_{n\to\infty} \int_0^\infty \Phi((2\epsilon)^{-1}\mathbf{m}_{f_n-f_m}(s))\, ds \leq 1$$

for all $n \geq N$. But by definition, this means that $\mathbf{m}_{f-f_m} \in L^\Phi(0,\infty)$ with $\|\mathbf{m}_{f-f_m}\|_\Phi \leq 2\epsilon$. Corollary 5.36 informs us that this is equivalent to the statement that $f - f_m$ (and hence f) belongs to $L^\Phi(\mathcal{M}, \tau)$, with $\|f - f_m\|_\Phi \leq 2\epsilon$. Since $\epsilon > 0$ and $m \geq N$ were arbitrary, it follows by definition that (f_m) converges to f in the $\|\cdot\|_\Phi$ norm. $\qquad\square$

5.2.1   The Orlicz norm and Köthe duality for Orlicz spaces

We start by introducing the concept of Köthe duality. We first briefly review the concept of a Banach function space of measurable functions on a measure space (X, Σ, ν). Readers who wish to have a fuller account may consult one of [BS88] or [KPS82]. Though there are subtly different ways in which one can approach the theory, at its most basic level, one starts by defining a so-called Banach function norm ρ on $M_0(X, \Sigma, \nu)$ (the almost

everywhere finite measurable functions) to be a mapping $\rho: M_0^+ \to [0, \infty]$ on the positive cone satisfying

[F1] $\rho(f) = 0$ if and only if $f = 0$ a.e.
[F2] $\rho(\lambda f) = \lambda \rho(f)$ for all $f \in M_0^+, \lambda > 0$.
[F3] $\rho(f + g) \leq \rho(f) + \rho(g)$ for all $f, g \in M_0^+$.
[F4] $f \leq g$ implies $\rho(f) \leq \rho(g)$ for all $f, g \in M_0^+$.

Such a ρ may be extended to all of M_0 by setting $\rho(f) = \rho(|f|)$, in which case we may then define $L^\rho(X, \Sigma, \nu) = \{f \in M_0(X, \Sigma, \nu): \rho(f) < \infty\}$. If indeed $L^\rho(X, \Sigma, \nu)$ turns out to be a Banach space when equipped with the norm $\|\cdot\|_\rho = \rho(\cdot)$, we refer to it as a Banach function space. If we add to the above list the so-called Fatou property, namely

[F5] for any sequence $(f_n) \subseteq M_0(X, \Sigma, \nu)$ we have that $0 \leq f_n \nearrow f$ implies $\rho(f_n) \nearrow \rho(f)$,

then $L^\rho(X, \Sigma, \nu)$ will automatically be complete. If further the situation $\mathbf{m}_f = \mathbf{m}_g$, $f \in L^\rho(X, \Sigma, \nu)$ and $g \in M_0$ ensures that $g \in L^\rho(X, \Sigma, \nu)$, we call $L^\rho(X, \Sigma, \nu)$ *rearrangement invariant*.

If we wish to ensure regular behaviour of the Banach function norm with respect to characteristic functions, we may additionally add the requirements that

[F6] for any measurable set E, $\nu(E) < \infty$ implies $\rho(\chi_E) < \infty$,
[F7] given any measurable set E with $\nu(E) < \infty$, there exists a constant $C_E > 0$ so that $\int_e f \, d\nu \leq C_E \rho(f)$ for any $f \in M_0(X, \Sigma, \nu)$.

For such a Banach function space the Köthe dual is defined to be the space $L^{\rho'}(X, \Sigma, \nu) = \{f \in M_0(X, \Sigma, \nu): fg \in L^1(X, \Sigma, \nu) \text{ for all } g \in L^\rho(X, \Sigma, \nu)\}$, with the canonical norm being given by

$$\|f\|_{\rho'} = \sup\{\int |fg| \, d\nu : g \in L^\rho(X, \Sigma, \nu), \|g\|_\rho \leq 1\}.$$

The additional regularity criteria ensure that ρ' is in fact a Banach function norm, and $L^{\rho'}(X, \Sigma, \nu)$ the corresponding Banach function space. Whenever referring to Banach function spaces in the ensuing text, we shall generally assume that each of [F1] – [F7] holds.

However our objective here is not to do a detailed study of Banach function spaces. Instead we will show that for any Young function Φ, $L_\Phi(\mathcal{M}, \tau)$ is the noncommutative Köthe dual of $L^{\Phi^*}(\mathcal{M}, \tau)$, in the sense that as linear spaces $L^\Phi(\mathcal{M}, \tau) = \{f \in \widetilde{\mathcal{M}}: fg \in L^1(\mathcal{M}, \tau) \text{ for all } g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$, with $\|f\|_\Phi^O = \sup\{\tau(|fg|): g \in L^{\Phi^*}(\mathcal{M}, \tau), \|g\|_\Phi \leq 1\}$. In

proving this, we will also show that the Luxemburg-Nakano and Orlicz norms are equivalent.

PROPOSITION 5.46. *Let Φ be a Young function.*

(a) *For any $g \in \widetilde{\mathcal{M}}$, the following are equivalent:*
   (i) *$gf \in L^1(\mathcal{M},\tau)$ for every $f \in L^\Phi(\mathcal{M},\tau)$;*
   (ii) *$fg \in L^1(\mathcal{M},\tau)$ for every $f \in L^\Phi(\mathcal{M},\tau)$;*
   (iii) *$\sup\{\tau(|gf|)\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\} < \infty$.*
(b) *Given some $g \in \widetilde{\mathcal{M}}$ satisfying the condition that $fg \in L^1(\mathcal{M},\tau)$ for every $f \in L^\Phi(\mathcal{M},\tau)$, we have that*

$$\sup\{\tau(|fg|)\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\}$$
$$= \sup\{|\tau(fg)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\}$$
$$= \sup\{|\tau(gf)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\}$$
$$= \sup\{\tau(|gf|)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\}$$

Moreover if g is as before and additionally $g \geq 0$, then we also have that

$$\sup\{|\tau(fg)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \leq 1\}$$
$$= \sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f \geq 0, \tau(\Phi(f)) \leq 1\}.$$

PROOF. We first prove the equivalence of (i) and (ii) in part (a). The proofs being similar, we only prove that (i)⇒(ii). Suppose (i) holds and let $g^* = u|g^*|$ be the polar decomposition of g^*. For any $f \in L^\Phi(\mathcal{M},\tau)$, $|f|$ will of course also belong to $L^\Phi(\mathcal{M},\tau)$. Moreover since $\mathbf{m}_{u|f|} \leq \mathbf{m}_{|f|}$, it is clear from Corollary 5.36, that in fact $u|f| \in L^\Phi(\mathcal{M},\tau)$. But then we must by hypothesis have that $g(u|f|) = (|g^*|u^*)(u|f|) = |g^*||f| \in L^1(\mathcal{M},\tau)$. On taking the adjoint, it follows that $|f||g^*| \in L^1(\mathcal{M},\tau)$. Let v be the partial isometry in the polar decomposition $f = v|f|$ of f. Since $L^1(\mathcal{M},\tau)$ is an $L^\infty(\mathcal{M},\tau)$-bimodule, it follows that $fg = v|f||g^*|u^* \in L^1(\mathcal{M},\tau)$ as required.

Having established the equivalence of (i) and (ii), the first half of part (b) now follows by same argument used in Corollary 5.42. We next prove the second part of (b). Let $f \in L^\Phi(\mathcal{M},\tau)$ be given with $\tau(\Phi(|f|)) \leq 1$. We remind the reader that this condition is equivalent to requiring $\|f\|_\Phi \leq 1$ (see Lemma 5.40).

It is clear that

$$\sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f = f^*, \tau(\Phi(f)) \le 1\}$$
$$\le \sup\{|\tau(fg)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \le 1\}.$$

We prove that equality holds. Let $f_0 \in L^\Phi(\mathcal{M},\tau)$ satisfies $\tau(\Phi(|f_0|)) \le 1$. For some $\alpha \in \mathbb{R}$, we have that $\tau(e^{i\alpha}f_0 g) = |\tau(f_0 g)|$. Since for $\operatorname{Im}(e^{i\alpha}f_0)$ we have that $\tau(\operatorname{Im}(e^{i\alpha}f_0)g) = \tau(g^{1/2}\operatorname{Im}(e^{i\alpha}f_0)g^{1/2}) \in \mathbb{R}$ (and similarly $\tau(\operatorname{Im}(e^{i\alpha}f_0)g) \in \mathbb{R}$), the equality $\tau(e^{i\alpha}f_0 g) = |\tau(f_0 g)|$ ensures that

$$\tau(\operatorname{Im}(e^{i\alpha}f_0)g) = \operatorname{Im}(\tau(e^{i\alpha}f_0 g)) = 0.$$

Moreover $\|\operatorname{Re}(e^{i\alpha}f_0)\|_\Phi \le \frac{1}{2}(\|e^{i\alpha}f_0\|_\Phi + \|(e^{i\alpha}f_0)^*\|_\Phi) = \|f_0\|_\Phi$. This inequality combined with Lemma 5.40 and the fact that $\tau(\Phi(|f_0|)) \le 1$, ensures that $\tau(\Phi(|\operatorname{Re}(e^{i\alpha}f_0)|)) \le 1$. Therefore $|\tau(f_0 g)| = \tau(\operatorname{Re}(e^{i\alpha}f_0)g) \le \sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f = f^*, \tau(\Phi(f)) \le 1\}$. It is now clear that

$$\sup\{|\tau(fg)|\colon f \in L^\Phi(\mathcal{M},\tau), \tau(\Phi(|f|)) \le 1\}$$
$$= \sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f = f^*, \tau(\Phi(|f|)) \le 1\}.$$

As in Remark 5.10 we now set $f_+ = f\chi_{[0,\infty)}(f)$ and $f_- = -f\chi_{(-\infty,0)}(f)$. Recall that for f_+ and f_- we have that $f_\pm \ge 0$, $f = f_+ - f_-$ and $|f| = f_+ + f_-$, with $\mathbf{m}_{f_\pm} \le \mathbf{m}_f$. It is now an exercise to use Corollary 5.36 to show that this last fact ensures that $f_\pm \in L^\Phi(\mathcal{M},\tau)$ with $\|f_\pm\|_\Phi = \|\mathbf{m}_{f_\pm}\|_\Phi \le \|\mathbf{m}_f\|_\Phi = \|f\|_\Phi \le 1$. It now follows from Theorem 5.22 that $\tau((f_+)g) = \tau(g^{1/2}(f_+)g^{1/2}) \ge 0$ and similarly that $\tau((f_-)g) \ge 0$. Suppose that $\tau((f_+)g) \ge \tau((f_-)g)$. Then $|\tau(fg)| = |\tau((f_+)g) - \tau((f_-)g)| \le \tau((f_+)g)$. This then shows that

$$\sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f = f^*, \tau(\Phi(|f|)) \le 1\}$$
$$= \sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M},\tau), f \ge 0, \tau(\Phi(f)) \le 1\},$$

which together with the previous centred equality, proves the second part of (b).

It remains to prove the equivalence of (i) and (iii). The implication (iii)$\Rightarrow$(i) is obvious. Hence we pass to showing that (i)$\Rightarrow$(iii). Suppose by way of contradiction that for some fixed $g \in \widetilde{\mathcal{M}}$, (i) holds, but that (iii) fails. Let $g = u|g|$ be the polar decomposition of g. We clearly have that $|g|f = u^*gf \in L^1(\mathcal{M},\tau)$ for any $f \in L^\Phi(\mathcal{M},\tau)$. Since in addition $|gf| = ||g|f|$ for any $f \in L^\Phi(\mathcal{M},\tau)$, it follows that we may assume that

$g \geq 0$. By the second part of (b) and the assumption regarding (iii), we then have that $\sup\{\tau(fg)\colon f \in L^\Phi(\mathcal{M}, \tau), f \geq 0, \tau(\Phi(f)) \leq 1\} = \infty$.

On taking note of Lemma 5.40, we may then select a sequence (f_n) of positive elements in the unit ball of $L^\Phi(\mathcal{M}, \tau)$, such that $\tau(gf_n) > n^3$ for each $n \in \mathbb{N}$. The formal sum $f_0 = \sum_{n=1}^{\infty} n^{-2} f_n$ converges absolutely in $L^\Phi(\mathcal{M}, \tau)$, and since this space is known to be a Banach space, f_0 must correspond to a well-defined element of $L^\Phi(\mathcal{M}, \tau)$. So we must have that $gf_0 \in L^1(\mathcal{M}, \tau)$, and hence that $\tau(gf_0) < \infty$. Since $f_0 = \sup_N \sum_{n=1}^{N} n^{-2} f_n$, we clearly have that $f_0 \geq n^{-2} f_n$ for any $n \in \mathbb{N}$. But by Theorem 5.22, this results in the situation that $\tau(gf_0) = \tau(g^{1/2} f_0 g^{1/2} f_0) \geq n^{-2}\tau(g^{1/2} f_n g^{1/2}) = n^{-2}\tau(gf_n) > n$ for any n. This is a clear contradiction. So (iii) must hold, if one of (ii) or (i) holds. $\qquad\square$

We need one more technical fact — important in its own right — before we are ready to prove the promised Köthe duality for the Orlicz spaces $L^\Phi(\mathcal{M}, \tau)$.

LEMMA 5.47. *Let Φ be a Young function. We may formally extend the norms $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^O$ to possibly infinite-valued quantities on $\widetilde{\mathcal{M}}$, by applying exactly the same prescriptions as those given in Definition 5.35. Denote these extensions by ρ_Φ and ρ_Φ^O respectively. Given $f_0, f_1 \in \{f \in \widetilde{\mathcal{M}}\colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$ with $0 \leq f_0 \leq f_1$, we have that $\rho_\Phi(f_0) \leq \rho_\Phi(f_1)$ and $\rho_\Phi^O(f_0) \leq \rho_\Phi^O(f_1)$. More generally if $(f_\alpha) \subseteq \{f \in \widetilde{\mathcal{M}}\colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$ is a net of positive elements increasing to $f_0 \in \{f \in \widetilde{\mathcal{M}}\colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$, then $(\rho_\Phi(f_\alpha))$ and $(\rho_\Phi^O(f_\alpha))$ respectively increase to $\rho_\Phi(f_0)$ and $\rho_\Phi^O(f_0)$.*

PROOF. We use the same notation ρ_Φ for the analogue of ρ_Φ on $M_0^+[0, \infty)$ — the cone of non-negative finite almost everywhere Borel-measurable functions on $[0, \infty)$. The same argument used to prove Corollary 5.36, then suffices to prove that $\rho_\Phi(f) = \rho_\Phi(\mathbf{m}_f)$. If therefore we consider part (iii) of Proposition 4.12 alongside this fact, it is clear that in the case of the Luxemburg-Nakano norm, the claim follows from the corresponding fact for classical Orlicz spaces. We therefore need only prove the claim regarding the quantity ρ_Φ^O. Let $f_0, f_1 \in \{f \in \widetilde{\mathcal{M}}\colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$ and $g_0 \in L^{\Psi^*}(\mathcal{M}, \tau)$ be given with $g_0 \geq 0$ and $0 \leq f_0 \leq f_1$. By Proposition 5.46, we may apply Theorem 5.22 to the products $f_0 g_0$ and $f_1 g_0$, to see that $\tau(f_0 g_0) = \tau(g_0^{1/2} f_0 g_0^{1/2}) \leq \tau(g_0^{1/2} f_1 g_0^{1/2}) = \tau(f_1 g_0)$. The fact that $\rho_\Phi^O(f_0) \leq \rho_\Phi^O(f_1)$ then follows from the final claim of part

(b) of Proposition 5.46. Now suppose we are given a net $(f_\alpha) \subseteq \{f \in \widetilde{\mathcal{M}} \colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$ of positive elements increasing to $f_0 \in \{f \in \widetilde{\mathcal{M}} \colon fg \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$. It is clear from what we just proved, that $\sup_\alpha \rho_\Phi^O(f_\alpha) \leq \rho_\Phi^O(f_0)$. It remains to prove the converse inequality.

Let $N \in (0, \rho_\Phi^O(f_0))$ be given. By part (b) of Proposition 5.46, $\rho_\Phi^O(f_0) = \sup\{\tau(f_0 g) \colon g \in L^{\Phi^*}(\mathcal{M}, \tau), g \geq 0, \tau(\Phi(g)) \leq 1\}$. We may therefore select $g_0 \in L^{\Phi^*}(\mathcal{M}, \tau)$ with $g_0 \geq 0$ and $\tau(\Phi(g_0)) \leq 1$, so that $\tau(f_0 g_0) > N$. Now notice that $g_0^{1/2} f_\alpha g_0^{1/2}$ increases to $g_0^{1/2} f_0 g_0^{1/2}$. By Proposition 4.17, we will then have that $\sup_\alpha \tau(f_\alpha g_0) = \sup_\alpha \tau(g_0^{1/2} f_\alpha g_0^{1/2}) = \tau(g_0^{1/2} f_0 g_0^{1/2}) = \tau(f_0 g_0)$. So there must exist an α such that $\tau(f_\alpha g_0) > N$, whence $\rho_\Phi^O(f_\alpha) > N$. Since $N \in (0, \rho_\Phi^O(f_0))$ was arbitrary, we have that $\sup_\alpha \rho_\Phi^O(f_\alpha) \geq \rho_\Phi^O(f_0)$ as required. $\square$

THEOREM 5.48. *Let Φ be a Young function. For any $f \in \widetilde{\mathcal{M}}$ we have that $\sup\{\tau(|fg|) \colon g \in L^{\Phi^*}(\mathcal{M}, \tau), \|g\|_\Phi^* \leq 1\} < \infty$ if and only if $f \in L^\Phi(\mathcal{M}, \tau)$, in which case $\|f\|_\Phi \leq \|f\|_\Phi^O \leq 2\|f\|_\Phi$. This in particular ensures that $L_\Phi(\mathcal{M}, \tau)$ is the Köthe dual of $L^{\Phi^*}(\mathcal{M}, \tau)$.*

PROOF. We saw in Proposition 5.41 that $\|f\|_\Phi^O \leq 2\|f\|_\Phi$ for $f \in L^\Phi(\mathcal{M}, \tau)$. Notice that by Lemma 5.40, we then have that $\sup\{\tau(|fg|) \colon g \in L^\Phi(\mathcal{M}, \tau), \|g\|_\Phi^* \leq 1\} = \|f\|_\Phi^O < \infty$. If therefore we are able to show that the condition $\sup\{\tau(|fg|) \colon g \in L^\Phi(\mathcal{M}, \tau), \|g\|_\Phi^* \leq 1\} < \infty$ ensures that $f \in L^\Phi(\mathcal{M}, \tau)$, and that in this case $\|f\|_\Phi \leq \|f\|_\Phi^O$, we will be done. We may clearly assume that $f \neq 0$.

Hence let $f \in \widetilde{\mathcal{M}}$ be given with $\sup\{\tau(|fg|) \colon g \in L^\Phi(\mathcal{M}, \tau), \|g\|_\Phi^* \leq 1\} < \infty$. Recall that this is equivalent to requiring that $f \in \{a \in \widetilde{\mathcal{M}} \colon ag \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$. Note that $|fg| = ||f|g|$ for any $g \in L^\Phi(\mathcal{M}, \tau)$. On considering this fact alongside Corollary 5.43, it is clear that we may assume that $f = |f|$. Having made this assumption, we will actually prove that $\rho_\Phi(f) \leq \rho_\Phi^O(f)$. Since by Proposition 5.46 ρ_Φ^O is finite on $\{a \in \widetilde{\mathcal{M}} \colon ag \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$, this will force $\rho_\Phi(f) < \infty$, which ensures that $f \in L^\Phi(\mathcal{M}, \tau)$. Since then $\rho_\Phi(f) = \|f\|_\Phi$ and $\rho_\Phi^O(f) = \|f\|_\Phi^O$, that will prove the theorem.

Recall that any positive measurable function may be written as the increasing limit of a sequence of positive simple functions. If we combine this fact with the Borel functional calculus, it is clear that f may written as an increasing limit of a sequence of operators (f_N) – all commuting

with f – of the form $\sum_{k=1}^{n} \alpha_k e_k$ where the α_k's are positive real numbers, and the e_k's mutually orthogonal projections. This ensures that in addition $f_N^2 \leq f^2$ for every N, and hence that $|f_N g|^2 \leq |fg|^2$ for every $g \in L^{\Phi^*}(\mathcal{M}, \tau)$. Taking square roots preserves the order, and hence for any $g \in L^{\Phi^*}(\mathcal{M}, \tau)$, $\tau(|f_N g|) \leq \tau(|fg|) < \infty$. So $(f_N) \subseteq \{a \in \widetilde{\mathcal{M}} : ag \in L^1(\mathcal{M}, \tau), g \in L^{\Phi^*}(\mathcal{M}, \tau)\}$. By Lemma 5.47, we may therefore pass to the case where $f = \sum_{k=1}^{n} \alpha_k e_k$, with the e_k's non-zero mutually orthogonal projections, and the α_k's positive. The projection $e_1 \in \mathcal{M}$ may in turn be written as the supremum of an increasing net (e_β) of subprojections of e_1 with finite trace. (See the proof of Corollary 5.21 for justification of this claim.) The operators $f_\beta = e_\beta f = \alpha_1 e_\beta + \sum_{k=2}^{n} \alpha_k e_k$ therefore increase to f. Since $\tau(|f_\beta|) = \tau(|e_\beta fg| \leq \tau(|fg|)$, we may argue as before to see that we may assume that $\tau(e_1) < \infty$. On inductively applying the same argument to each of the e_k's, it is now clear that we may in fact assume that each of the e_k's have finite trace.

For some $\gamma > 0$, we will have that $\tau(\Phi^*(\gamma e_1)) = \Phi^*(\gamma)\tau(e_1) \leq 1$. This ensures that $0 < \alpha_1 \gamma \tau(e_1) = \tau(|f(\gamma e_1)|) \leq \rho_\Phi^O(f)$. We may now rescale f to pass to the case where $\rho_\Phi^O(f) = 1$. It is then incumbent on us to show that $\rho_\Phi(f) \leq 1$. By the definition of ρ_Φ, this will follow once we prove that $\tau(\Phi(\gamma f)) \leq 1$ for any $\gamma \in (0, 1)$.

We first claim that $f \leq b_\Phi \sum_{k=1}^{n} e_k$. This is of course trivial if $b_\Phi = \infty$, so assume that $b_\Phi < \infty$. The claimed operator inequality will hold if for each k, $\alpha_k \leq b_\Phi$. So if $f \leq b_\Phi \mathbb{1}$ were not true, there must then exist some k_0 and some $\epsilon > 0$, such that $\alpha_{k_0} \geq b_\Phi + \epsilon$. Now consider the operator $h = (b_\Phi \tau(e_{k_0}))^{-1} e_{k_0}$. We may then use part (a) of Remark 5.30 to see that $\tau(\Phi^*(h)) \leq b_\Phi \tau((b_\Phi \tau(e_{k_0}))^{-1} e_{k_0}) \leq 1$. But this would force

$$\rho_\Phi^O(f) \geq \tau(|fh|) = (b_\Phi \tau(e_{k_0}))^{-1} \tau(\alpha_{k_0} e_{k_0}) = \frac{\alpha_{k_0}}{b_\Phi} \geq \frac{b_\Phi + \epsilon}{b_\Phi} > 1,$$

which contradicts our assumption that $\rho_\Phi^O(f) = 1$. Hence the claimed operator inequality holds. Since for any $0 < \gamma < 1$, we have $\Phi(\gamma b_\Phi) < \infty$, this in turn ensures that $\tau(\Phi(\gamma f)) \leq \Phi(\gamma b_\Phi) \tau(\sum_{k=1}^{n} e_k) < \infty$. (Recall that by Remark 4.15, this forces $\Phi(\gamma f) \in \widetilde{\mathcal{M}}$.)

Now let $\gamma \in (0, 1)$ be given. Recall that Φ is of the form $\Phi(t) = \int_0^t \phi(s)\, ds$ for some non-negative left-continuous non-decreasing function ϕ on $[0, \infty)$, which is infinite-valued on (b_Φ, ∞), but neither identically 0 nor infinite-valued on all of $(0, \infty)$. Since ϕ is bounded and increasing on

$[0, \gamma b_\Phi]$, the Borel functional calculus ensures that $g = \phi(\gamma f)$ is a well-defined element of $\mathcal{M}$, supported on $\sum_{k=1}^n e_k$. However more is true. Suppose we are given $0 \leq s, t < \infty$ with $s = \phi(t)$. It then follows from the equality criteria for the Hausdorff-Young inequality (see part (b) of Remark 5.30), that $\Phi^*(\phi(t)) = t\phi(t) - \Phi(t) \leq t\phi(t)$. Thus if ϕ is bounded on some interval $[0, r]$, then so is $\Phi^* \circ \phi$. Now recall that ϕ is non-decreasing and finite-valued on $[0, b_\Phi)$, and that by construction, $\gamma\|f\|_\infty < b_\Phi$. This ensures that ϕ, and therefore also $\Phi^* \circ \phi$, is bounded on $[0, \gamma\|f\|_\infty]$. It therefore follows that $\Phi^* \circ \phi(\gamma f) = \Phi^*(g) \in \mathcal{M}$.

Since the operators $\Phi(\gamma f)$ and $\Phi^*(g)$ are commuting operators affiliated to the von Neumann algebra generated by the spectral projections of f, we may apply the Borel functional calculus for f to the equality criteria for the Hausdorff-Young inequality (see part (b) of Remark 5.30), to see that $\gamma f g = \Phi(f) + \Phi^*(g)$. All operators in this expression belong to $\widetilde{\mathcal{M}}_+$, and hence we may apply the trace to see that

$$\tau(\gamma f g) = \tau(\Phi(\gamma f)) + \tau(\Phi^*(g)). \tag{5.1}$$

We have already seen that $\tau(\Phi(\gamma f)) < \infty$. Since by construction $\phi(\gamma f) \leq \phi(\gamma\|f\|_\infty)\sum_{k=1}^n e_k$, we also have $0 \leq \gamma f g \leq \phi(\gamma\|f\|_\infty)\gamma\|f\|_\infty\sum_{k=1}^n e_k$, and hence $\tau(\gamma f g) < \infty$. Thus by equation (5.1), we must have that $\tau(\Phi^*(g))$ is also finite. But then $g \in L^{\Phi^*}(\mathcal{M}, \tau)$. By the definition of ρ_Φ^O, we then have that $\tau(\gamma f g) \leq \rho_\Phi^O(\gamma f)\|g\|_{\Phi^*} \leq \|g\|_{\Phi^*}$. We now use Lemma 5.40 to see that

$$\tau(\gamma f g) \leq \|g\|_{\Phi^*} \leq \max(1, \tau(\Phi^*(g))) \leq 1 + \tau(\Phi^*(g)). \tag{5.2}$$

On considering equations (5.1) and (5.2) alongside each other, the fact that all terms are finite, ensures that $\tau(\Phi(\gamma f)) \leq 1$. This proves that $f \in L^\Phi(\mathcal{M}, \tau)$. However since $\gamma \in (0, 1)$ was arbitrary, we also have $\|f\|_\Phi \leq 1$, as required. $\qquad\square$

EXERCISE 5.49. Show that the space $L \log(L+1)(\mathcal{M}, \tau)$ is isomorphic to the Köthe dual of $L^{\cosh - 1}(\mathcal{M}, \tau)$. (Here $L \log(L+1)(\mathcal{M}, \tau)$ is the space produced by the Young function $t \mapsto t \log(t + 1)$.)

5.2.2   The Orlicz spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$

We close this very brief introduction to Orlicz spaces, with a description of the spaces $L^1 \cap L^\infty$ and $L^1 + L^\infty$. Both of these will be shown to be Orlicz spaces. We will in particular also see that these spaces in a very real sense represent the smallest and largest of all Orlicz spaces. For this

we will need the concept of a *fundamental function* of a rearrangement invariant Banach function space. For our purposes it is enough to at this point restrict attention to the measure space $((0, \infty), \mathcal{B}(0, \infty))$ equipped with Lebesgue measure.

DEFINITION 5.50. Given a rearrangement invariant Banach function space $L^\rho(0, \infty)$, define the associated fundamental function $\mathbf{f}_\rho : [0, \infty) \to [0, \infty)$ by the prescription that $\mathbf{f}_\rho(t) = \|\chi_E\|_\rho$, where E is a Borel set with measure t.

Any two Borel sets E_1 and E_2 with the same finite measure will have the same decreasing rearrangement. It is therefore the rearrangement invariance of the space $L^\rho(0, \infty)$ that ensures that $\mathbf{f}_\rho$ is well-defined. We proceed to list the basic properties of $\mathbf{f}_\rho$. We will merely sketch the proof. Readers who wish to have full details, may refer to section II.5 of [**BS88**].

PROPOSITION 5.51. *Let $L^\rho(0, \infty)$ be a rearrangement invariant Banach function space, and $L^{\rho'}(0, \infty)$ its Köthe dual. Both $\mathbf{f}_\rho$ and $\mathbf{f}_{\rho'}$ are so called* quasiconcave functions, *meaning that they are non-decreasing, continuous on $(0, \infty)$, zero-valued at precisely 0, and with both $t \mapsto \frac{\mathbf{f}_\rho(t)}{t}$ and $t \mapsto \frac{\mathbf{f}_{\rho'}(t)}{t}$ non-increasing on $(0, \infty)$. In addition $\mathbf{f}_\rho(t)\,\mathbf{f}_{\rho'}(t) = t$ for any $t \geq 0$.*

SKETCH OF PROOF. We shall not prove the final statement, but will merely indicate how that statement may be used to prove the rest of the claims. We firstly note that the non-degeneracy of the norm $\| \cdot \|_\rho$ ensures that $\mathbf{f}_\rho$ is zero-valued at precisely 0. If $0 \leq t_0 \leq t_1$, then $\chi_{[0,t_0]} \leq \chi_{[0,t_1]}$. It then follows from property [F4] in subsection 5.2.1, that $\mathbf{f}_\rho(t_0) = \|\chi_{[0,t_0]}\|_\rho \leq \|\chi_{[0,t_1]}\|_\rho = \mathbf{f}_\rho(t_1)$. The fact that $t \mapsto \frac{\mathbf{f}_\rho(t)}{t}$ is non-increasing, follows from the final claim combined with the fact that $\mathbf{f}_{\rho'}$ is non-decreasing. To see that $\mathbf{f}_\rho$ is continuous, observe that it is a non-decreasing function which cannot have any jump discontinuities on $(0, \infty)$, since a jump discontinuity at a point $t_0 > 0$, would mean that $t \mapsto \frac{\mathbf{f}_\rho(t)}{t}$ fails to be non-increasing at that point. $\square$

REMARK 5.52. We pause to note that the "converse" of Proposition 5.51 is also true in that every quasi-concave function $\mathbf{f}$ appears as the fundamental function of some rearrangement invariant Banach function space. (See [**BS88**, Proposition II.5.8].)

Let us now apply the above ideas to the classical Orlicz space $L^\Phi(0, \infty)$. Both the Luxemburg-Nakano and Orlicz norms turn out to be rearrangement invariant Banach function norms. (See section IV.8 of [**BS88**] for details.) For the Luxemburg-Nakano norm the rearrangement invariance follows from Corollary 5.36. The fundamental function corresponding to the Luxemburg-Nakano and Orlicz norms, will respectively be denoted by $\mathbf{f}^\Phi$ and $\mathbf{f}_\Phi$. We have the following very elegant formulae for these two fundamental functions:

PROPOSITION 5.53. *Let Φ be a Young function. The fundamental function corresponding to the Luxemburg-Nakano norm for the space $L^\Phi(0, \infty)$ is given by the formula $\mathbf{f}^\Phi(t) = \frac{1}{\Phi^{-1}(1/t)}$, and the one corresponding to the Orlicz norm of the same space by the formula $\mathbf{f}_\Phi(t) = t(\Phi^*)^{-1}(1/t)$. (Here Φ^{-1} and $(\Phi^*)^{-1}$ are as in part (d) of Remark 5.30.)*

PROOF. It is clear from the definition that $\mathbf{f}^\Phi(0) = 0$. Now let E be a Borel set of non-zero finite measure. First note that for any $0 < \alpha < b_\Phi$, the continuous functional calculus (applied to $L^\infty(0, \infty)$) ensures that $\Phi(\alpha\chi_E) = \Phi(\alpha)\chi_E$, and hence that

$$\int \Phi(\alpha\chi_E)\, d\mathrm{m}(t) = \Phi(\alpha) \int \chi_E\, d\mathrm{m}(t) = \Phi(\alpha)\mathrm{m}(E).$$

It is therefore clear that $\chi_E \in L^\Phi(0, \infty)$. To see the claim regarding $\mathbf{f}^\Phi$, we may use the above equality to see that

$$
\begin{aligned}
\|\chi_E\|_\Phi &= \inf\{\epsilon > 0 : \Phi\left(\frac{1}{\epsilon}\right) \int \chi_E\, d\mathrm{m}(t) \leq 1\} \\
&= \inf\left\{\epsilon > 0 : \Phi\left(\frac{1}{\epsilon}\right) \leq \left(\int \chi_E\, d\mathrm{m}(t)\right)^{-1}\right\} \\
&= \left[\sup\left\{\gamma > 0 : \Phi(\gamma) \leq \frac{1}{\int \chi_E\, d\mathrm{m}(t)}\right\}\right]^{-1} \\
&= \frac{1}{\Phi^{-1}(1/\int \chi_E\, d\mathrm{m}(t))}.
\end{aligned}
$$

We therefore have that $\mathbf{f}^\Phi(t) = \frac{1}{\Phi^{-1}(1/t)}$ for all $t > 0$ as required.

The claim regarding $\mathbf{f}_\Phi$ follows from the fact that the fundamental function of the Köthe dual of $L^{\Phi^*}(0, \infty)$ must satisfy the relation $\mathbf{f}_\Phi(t)\, \mathbf{f}^{\Phi^*}(t) = t$. $\qquad\square$

PROPOSITION 5.54. *Let Φ_1 and Φ_2 be Young functions. Algebraically the Orlicz space $L^{\Phi_1 \vee \Phi_2}(\mathcal{M}, \tau)$ agrees with $L^{\Phi_1}(\mathcal{M}, \tau) \cap L^{\Phi_2}(\mathcal{M}, \tau)$. (Here $\Phi_1 \vee \Phi_2$ is the Young function $(\Phi_1 \vee \Phi_2)(t) = \max(\Phi_1(t), \Phi_2(t))$.)*

PROOF. If $f \in L^{\Phi_1 \vee \Phi_2}(\mathcal{M}, \tau)$, then by definition there exists $\alpha > 0$ so that $\tau(\Phi_1 \vee \Phi_2(\alpha|a|)) < \infty$. Since $\Phi_1 \vee \Phi_2$ majorises both Φ_1 and Φ_2, it is clear that we then also have that $\tau(\Phi_1(\alpha|a|)) < \infty$ and $\tau(\Phi_2(\alpha|a|)) < \infty$.

Conversely if $f \in L^{\Phi_1}(\mathcal{M}, \tau) \cap L^{\Phi_2}(\mathcal{M}, \tau)$, there must exist $\alpha_1, \alpha_2 > 0$ so that $\tau(\Phi_1(\alpha_1|a|)) < \infty$ and $\tau(\Phi_2(\alpha_2|a|)) < \infty$. For $\alpha = \min(\alpha_1, \alpha_2)$ we will then clearly have that $\tau(\Phi_1(\alpha|a|) + \Phi_2(\alpha|a|)) < \infty$. Since the function $\Phi_1 + \Phi_2$ majorises $\Phi_1 \vee \Phi_2$, we will then also have that $\tau((\Phi_1 \vee \Phi_2)(\alpha|a|)) < \infty$, as required. $\square$

We now apply the above to show that the spaces $(L^1 \cap L^\infty)(\mathcal{M}, \tau)$ and $(L^1 + L^\infty)(\mathcal{M}, \tau)$ may be realised as Orlicz spaces

PROPOSITION 5.55. *$L^1(\mathcal{M}, \tau) \cap L^\infty(\mathcal{M}, \tau)$ is an Orlicz space corresponding to the Young function*

$$\Phi_{1 \cap \infty}(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ \infty & \text{otherwise} \end{cases} \, .$$

$L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$ is an Orlicz space corresponding to the conjugate Young function $\Phi_{1 + \infty} = \Phi_{1 \cap \infty}^*$, which is given by

$$\Phi_{1 + \infty}(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ t - 1 & \text{otherwise} \end{cases} \, .$$

Convention: *When considered as Orlicz spaces, we write $L^{1 \cap \infty}(\mathcal{M}, \tau)$ for $L^1(\mathcal{M}, \tau) \cap L^\infty(\mathcal{M}, \tau)$, and $L^{1 + \infty}(\mathcal{M}, \tau)$ for $L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$.*

PROOF. The Young function generating $L^1(\mathcal{M}, \tau)$ is clearly $\Phi(t) = t$. Hence, the fact that $L^1(\mathcal{M}, \tau) \cap L^\infty(\mathcal{M}, \tau)$ is an Orlicz space corresponding to the given Young function is a consequence of Exercise 5.38 and the preceding Proposition.

We pass to the claim regarding $L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$. The fact that $\Phi_{1 + \infty} = \Phi_{1 \cap \infty}^*$ is as stated, is left as an exercise. If $f \in L^1(\mathcal{M}, \tau)$, we will clearly have that $\tau(\Phi_{1 + \infty}(|f|)) \leq \tau(|f|) < \infty$. If on the other hand $f \in L^\infty(\mathcal{M}, \tau)$, then by the definition of $\Phi_{1 + \infty}$, we will have that $\Phi_{1 + \infty}(\|f\|_\infty^{-1}|f|) = 0$. This then in turn ensures that $L^{\Phi_{1 + \infty}}(\mathcal{M}, \tau)$ contains $L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$. Conversely suppose that $f \in L^{\Phi_{1 + \infty}}(\mathcal{M}, \tau)$, and let $\alpha > 0$ be given such that $\tau(\Phi_{1 + \infty}(\alpha|f|)) < \infty$. Next observe that on

applying the Borel functional calculus to the definition of $\Phi_{1+\infty}$, we have that $\Phi_{1+\infty}(\alpha|f|) = (\alpha|f| - \mathbb{1})\chi_{(1,\infty)}(\alpha|f|)$. So requiring $\tau(\Phi_{1+\infty}(\alpha|f|)) < \infty$, is the same as requiring $(\alpha|f| - \mathbb{1})\chi_{(1,\infty)}(\alpha|f|) \in L^1(\mathcal{M}, \tau)$. Since $(\alpha|f| - \mathbb{1})\chi_{[0,1]}(\alpha|f|) \in L^\infty(\mathcal{M}, \tau)$, we therefore clearly have that $\alpha|f| - \mathbb{1} \in L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$ and hence that $f \in L^1(\mathcal{M}, \tau) + L^\infty(\mathcal{M}, \tau)$. $\qquad\square$

REMARK 5.56. If we apply Proposition 5.53 to the above Proposition, it follows that the fundamental function of the space $L^{1 \cap \infty}(0, \infty)$ corresponding to the Luxemburg-Nakano norm, is given by $\mathbf{f}_{1 \cap \infty}(t) = \max(1, t)$. By the final claim in Proposition 5.51 fundamental function of the space $L^{1+\infty}(0, \infty)$ corresponding to the Orlicz norm, is given by $\mathbf{f}_{1+\infty}(t) = \min(1, t)$.

To compute the norms of $L^{1 \cap \infty}(\mathcal{M}, \tau)$ and $L^{1+\infty}(\mathcal{M}, \tau)$, we shall need the following result

THEOREM 5.57. *For any $f \in (L^1 + L^\infty)(\mathcal{M}, \tau)$ and any $t > 0$, we have that*

$$\inf\{\|f_1\|_1 + t\|f_\infty\|_\infty \colon f = f_1 + f_\infty, f_1 \in L^1(\mathcal{M}, \tau), f_\infty \in L^\infty(\mathcal{M}, \tau)\}$$
$$= \int_0^t \mathbf{m}_f(s)\, ds.$$

In particular for each $f \in (L^1 + L^\infty)(\mathcal{M}, \tau)$, $\int_0^t \mathbf{m}_f(s)\, ds$ is then finite for each $t > 0$

PROOF. Let $f \in (L^1 + L^\infty)(\mathcal{M}, \tau)$ be given and let $f = f_1 + f_\infty$ be an arbitrary decomposition of f with $f_1 \in L^1(\mathcal{M}, \tau)$ and $f_\infty \in L^\infty(\mathcal{M}, \tau)$. For $0 < \alpha < 1$ and $s > 0$, it follows from Proposition 4.12 that $\mathbf{m}_f(s) \leq \mathbf{m}_{f_1}(\alpha s) + \mathbf{m}_{f_\infty}((1-\alpha)s) \leq \mathbf{m}_{f_1}(\alpha s) + \|f_\infty\|_\infty$. We may then apply Proposition 4.13, to see that

$$\begin{aligned}
\int_0^t \mathbf{m}_f(s)\, ds &= \int_0^t \mathbf{m}_{f_1}(\alpha s)\, ds + t\|f_\infty\|_\infty \\
&\leq \int_0^\infty \mathbf{m}_{f_1}(\alpha s)\, ds + t\|f_\infty\|_\infty \\
&= \alpha^{-1}\int_0^\infty \mathbf{m}_{f_1}(r)\, dr + t\|f_\infty\|_\infty \\
&= \alpha^{-1}\|f_1\|_1 + t\|f_\infty\|_\infty.
\end{aligned}$$

On letting α increase to 1, it will follow that $\int_0^t \mathbf{m}_f(s)\, ds \leq \|f_1\|_1 + t\|f_\infty\|_\infty$. We may then take the infimum over all decompositions of f as a sum of

elements from $L^1(\mathcal{M}, \tau)$ and $L^\infty(\mathcal{M}, \tau)$, to see that

$$\int_0^t \mathbf{m}_f(s)\, ds \le \inf\{\|f_1\|_1 + t\|f_\infty\|_\infty : f = f_1 + f_\infty,$$

$$f_1 \in L^1(\mathcal{M}, \tau),\ f_\infty \in L^\infty(\mathcal{M}, \tau)\}.$$

This clearly also proves the final claim.

To prove the converse inequality, let $f = u|f|$ be the polar form of f, and let $|f| = \int_0^\infty \lambda\, de_\lambda$ be the spectral decomposition of $|f|$. For a fixed $t > 0$ we set $\alpha = \mathbf{m}_f(t)$, and then define f_1 and f_∞ by

$$f_1 = u\int_\alpha^\infty (\lambda - \alpha)\, de_\lambda \text{ and } f_\infty = f - f_1.$$

Since

$$g(\lambda) = \begin{cases} 0 & \text{if } 0 \le \lambda \le \alpha \\ \lambda - \alpha & \text{if } \lambda \ge \alpha \end{cases}$$

is a continuous increasing function with $|f_1| = \int_0^\infty (\lambda - \alpha)\, de_\lambda = g(|f|)$, it follows from Proposition 4.12 that

$$\mathbf{m}_{f_1}(s) = g(\mathbf{m}_f(s)) = \begin{cases} \mathbf{m}_f(s) - \alpha & \text{if } 0 < s < t \\ 0 & \text{if } s \ge t \end{cases}$$

For f_∞ we clearly have that

$$f_\infty = f - f_1 = u\int_0^\infty \lambda\, de_\lambda - u\int_\alpha^\infty \alpha\, de_\lambda = u\int_0^\alpha \lambda\, de_\lambda,$$

and hence that $f_\infty \in L^\infty(\mathcal{M}, \tau)$ with $\|f_\infty\|_\infty \le \alpha$. It therefore follows that

$$\|f_1\|_1 + t\|f_\infty\|_\infty \le \int_0^\infty \mathbf{m}_{f_1}(s)\, ds + t\alpha$$

$$= \int_0^t (\mathbf{m}_f(s) - \alpha)\, ds + t\alpha$$

$$= \int_0^t \mathbf{m}_f(s)\, ds.$$

Since $\int_0^t \mathbf{m}_f(s)\, ds$ must be finite, this also proves that $f_1 \in L^1(\mathcal{M}, \tau)$, which then proves the theorem. $\qquad\square$

It is a classical fact that if two Banach spaces X_0 and X_1 canonically embed into a Hausdorff topological vector space in such a way that one can give meaning to $X_0 \cap X_1$ and $X_0 + X_1$, that these spaces become Banach spaces when respectively equipped with the norms $\max(\|x\|_0, \|x\|_1)$ and

$\inf\{\|x_0\|_0 + \|x_1\|_1 : x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}$. In the following theorem we show how for the pair $(L^{1\cap\infty}(\mathcal{M},\tau), L^{1+\infty}(\mathcal{M},\tau))$, these natural norms on $L^{1\cap\infty}(\mathcal{M},\tau)$ and $L^{1+\infty}(\mathcal{M},\tau)$ may be realised as Luxemburg-Nakano and Orlicz norms. We will denote these natural norms by $\|\cdot\|_{1\cap\infty}$ and $\|\cdot\|_{1+\infty}$ respectively.

THEOREM 5.58. *For the spaces $L^{1\cap\infty}(\mathcal{M},\tau)$ and $L^{1+\infty}(\mathcal{M},\tau)$, the natural norms on these spaces respectively agree with the Luxemburg-Nakano and Orlicz norms. The spaces $L^{1\cap\infty}(\mathcal{M},\tau)$ and $L^{1+\infty}(\mathcal{M},\tau)$ are therefore Köthe duals of each other. The norm for $L^{1\cap\infty}(\mathcal{M},\tau)$ is given by $\|f\|_{1\cap\infty} = \max(\|f\|_1, \|f\|_\infty)$, and for $L^{1+\infty}(\mathcal{M},\tau)$ by $\|f\|_{1+\infty} = \int_0^1 \mathbf{m}_f(s)\, ds$.*

PROOF. We start with proving that the Luxemburg-Nakano norm on $L^1(\mathcal{M},\tau) \cap L^\infty(\mathcal{M},\tau)$ is as stated. Let $f \in L^1(\mathcal{M},\tau) \cap L^\infty(\mathcal{M},\tau)$ be given and consider $\epsilon = \max(\|f\|_1, \|f\|_\infty)$. We then clearly have that $\|\epsilon^{-1}|f|\,\|_\infty \leq 1$, which ensures that $\Phi_{1\cap\infty}(\epsilon^{-1}|f|) = \epsilon^{-1}|f|$. Hence

$$\tau(\Phi_{1\cap\infty}(\epsilon^{-1}|f|)) = \tau(\epsilon^{-1}|f|) = \epsilon^{-1}\|f\|_1 \leq 1.$$

This ensures that $\|f\|_{1\cap\infty} \leq \max(\|f\|_1, \|f\|_\infty)$.

Now suppose that $\epsilon < \max(\|f\|_1, \|f\|_\infty)$. Then one of $\epsilon < \|f\|_1$, or $\epsilon < \|f\|_\infty$ must hold. If $\epsilon < \|f\|_\infty$, then for $\gamma = \frac{\|f\|_\infty}{\epsilon}$, the spectral projection $\chi_{(1,\gamma]}(\epsilon^{-1}|f|)$ must be non-zero. Since $\epsilon^{-1}|f| \geq \epsilon^{-1}|f|\chi_{(1,\gamma]}(\epsilon^{-1}|f|)$ with $\Phi_{1\cap\infty}(\epsilon^{-1}|f|\chi_{(1,\gamma]}(\epsilon^{-1}|f|)) = \infty.\chi_{(1,\gamma]}(\epsilon^{-1}|f|))$, we must then have that $\tau(\Phi_{1\cap\infty}(\epsilon^{-1}|f|)) = \infty$. Now suppose that $\|f\|_\infty \leq \epsilon$, but that $\|f\|_1 > \epsilon$. Since in this case $\|\epsilon^{-1}|f|\,\|_\infty \leq 1$, we will have that $\Phi_{1\cap\infty}(\epsilon^{-1}|f|) = \epsilon^{-1}|f|$, which then ensures that $\tau(\Phi_{1\cap\infty}(\epsilon^{-1}|f|)) = \tau(\epsilon^{-1}|f|) = \epsilon^{-1}\|f\|_1 > 1$. We must therefore have that $\|f\|_{1\cap\infty} \geq \epsilon$, and hence that $\|f\|_{1\cap\infty} \geq \max(\|f\|_1, \|f\|_\infty)$ as required.

We now use the above and Remark 5.56 to prove the claims regarding $L^{1+\infty}(\mathcal{M},\tau)$. Let $f \in L^{1+\infty}(\mathcal{M},\tau)$ be given and suppose that $f = f_1 + f_\infty$, where $f_1 \in L^1(\mathcal{M},\tau)$ and $f_\infty \in L^\infty(\mathcal{M},\tau)$. Given any $g \in L^{1\cap\infty}(\mathcal{M},\tau)$ with $\|g\|_{1\cap\infty} \leq 1$, we may then select partial isometries u and v so that $|fg| \leq u|f_1 g|u^* + v|f_\infty g|v^*$ (Lemma 4.2). Therefore

$$\begin{aligned}\tau(|fg|) \ &\leq \tau(u|f_1 g|u^* + v|f_\infty g|v^*) \leq \tau(|f_1 g|) + \tau(|f_\infty g|) \\ &\leq \|f_1\|_1\|g\|_\infty + \|f_\infty\|_\infty\|g\|_1 \leq \|f_1\|_1 + \|f_\infty\|_\infty.\end{aligned}$$

On taking the infimum over all decompositions of the form $f = f_1 + f_\infty$, we see that $\tau(|fg|) \leq \|f\|_{1+\infty}$. Now take the supremum over all g's with $\|g\|_{1\cap\infty} \leq 1$, to see that $\|f\|_{\Phi_{1+\infty}}^O \leq \|f\|_{1+\infty}$.

For the converse notice that by Theorem 5.48, the space $L^{1\cap\infty}(0,\infty)$ equipped with the norm $\|\cdot\|_{\Phi_{1\cap\infty}}$ is the Köthe dual of $L^{1+\infty}(0,\infty)$ equipped with the Orlicz norm. It now follows from Theorem 5.57 that

$$\begin{aligned}
\|f\|_{1+\infty} &= \int_0^1 \mathbf{m}_f(s)\,ds \\
&\leq \|\chi_{[0,1]}\|_{1\cap\infty}\|\mathbf{m}_f\|_{\Phi_{1+\infty}}^O \\
&= \mathbf{f}_{1\cap\infty}(1)\|\mathbf{m}_f\|_{\Phi_{1+\infty}}^O \\
&= \|\mathbf{m}_f\|_{\Phi_{1+\infty}}^O \\
&= \|f\|_{\Phi_{1+\infty}}^O.
\end{aligned}$$

Recall that we already know that $\|f\|_{\Phi_{1+\infty}}^O \leq \|f\|_{1+\infty}$. Hence the norms are equal as claimed. $\qquad\square$

We close this chapter by justifying our earlier claim that $L^{1\cap\infty}(\mathcal{M},\tau)$ and $L^{1+\infty}(\mathcal{M},\tau)$ respectively represent the smallest and largest of all Orlicz spaces.

THEOREM 5.59. *Let Φ be a Young function. Then $L^{1\cap\infty}(\mathcal{M},\tau) \hookrightarrow L^\Phi(\mathcal{M},\tau) \hookrightarrow L^{1+\infty}(\mathcal{M},\tau)$ makes sense in the sense that $L^{1\cap\infty}(\mathcal{M},\tau)$ continuously injects into $L^\Phi(\mathcal{M},\tau)$, and $L^\Phi(\mathcal{M},\tau)$ continuously injects into $L^{1+\infty}(\mathcal{M},\tau)$.*

PROOF. By definition Φ is finite-valued on some interval $[0,\delta]$. So by convexity there must exist some $K > 0$ so that $\Phi(t) \leq Kt$ for all $t \in [0,\delta]$. Equivalently $\Phi(\delta t) \leq K\delta t$ for all $t \in [0,1]$. It then clearly follows that $\Phi(\delta t) \leq (K\delta)\Phi_{1\cap\infty}(t)$ for all $t \geq 0$. We may of course assume that $K\delta \geq 1$. The fact that $\Phi_{1\cap\infty}$ is both convex and 0 at 0, then ensures that $\Phi(\delta t) \leq (K\delta)\Phi_{1\cap\infty}(t) \leq \Phi_{1\cap\infty}(K\delta t)$ for all $t \geq 0$. Equivalently $\Phi(s) \leq \Phi_{1\cap\infty}(Ks)$ for all $s \geq 0$. So if for some $f \in L^{1\cap\infty}(\mathcal{M},\tau)$ we are given an $\epsilon > 0$ for which $\tau(\Phi_{1\cap\infty}(\epsilon^{-1}|f|)) \leq 1$, we must then have that $\tau(\Phi((K\epsilon)^{-1}|f|)) \leq \tau(\Phi_{1\cap\infty}(\epsilon^{-1}|f|)) \leq 1$. This ensures that $\|f\|_\Phi \leq K\|f\|_{1\cap\infty}$, proving the first claim.

For the remaining injection, we show that if the above situation pertains, then $L_{\Phi^*}(\mathcal{M},\tau)$ continuously injects into $L^{1+\infty}(\mathcal{M},\tau)$. So choose $g \in L_{\Phi^*}(\mathcal{M},\tau)$ and assume that f is a non-zero element of $L^{1\cap\infty}(\mathcal{M},\tau)$. Using what we have just proven, it then follows that

$$\frac{\tau(|gf|)}{\|f\|_{1\cap\infty}} \leq K\frac{\tau(|gf|)}{\|f\|_\Phi} \leq \|g\|_{\Phi^*}^O.$$

By the preceding theorem, taking the supremum over all non-zero elements f of $L^{1 \cap \infty}(\mathcal{M}, \tau)$, yields the fact that $\|g\|_{1+\infty} \leq K\|g\|_{\Phi^*}^O$. Since $g \in L_{\Phi^*}(\mathcal{M}, \tau)$ was arbitrary, we are done. $\square$

CHAPTER 6

Crossed products

In chapter 5 we were introduced to the very elegant theory of L^p and Orlicz spaces for semifinite algebras. What is very clear from that chapter is the central role that the algebra of τ-measurable operators played in that development. In trying to extend that theory to general algebras, a major difficulty we need to overcome is the fact that many von Neumann algebras do not admit a faithful normal semifinite trace. Hence for these von Neumann algebras no direct construction of an algebra of τ-measurable operators is possible. To overcome this challenge we appeal to the theory of crossed products. Using the theory of crossed products, any von Neumann algebra may in a canonical way be enlarged to an algebra which does admit a faithful normal semifinite trace. Via this enlarged algebra one may then gain access to the technology of τ-measurable operators. It is this specific aspect that is the focus of our interest in crossed products. We will therefore in no way attempt to give a comprehensive introduction to crossed products, but will content ourselves with familiarising the reader with those aspects essential to the theory of Haagerup L^p-spaces. Throughout this chapter, $\mathcal{M}$ will be a von Neumann algebra acting on a Hilbert space H, and equipped with a faithful normal semifinite weight φ. Readers who wish to get to the nuts and bolts of Haagerup L^p-spaces as quickly as possible may at a first reading merely familiarise themselves with the content of Theorems 6.50, 6.62, 6.65, 6.72, 6.74, and of Propositions 6.61, 6.67, and 6.70, and then move on to chapter 7.

A clear understanding of the fundamentals of Tomita-Takesaki modular theory, of Connes cocycles, and of conditional expectations and operator valued weights is absolutely crucial for the theory that will follow. For that reason we will in the first three sections pause to briefly lay a suitable foundation regarding these theories, before proceeding with the development of the theory of crossed products. Our presentation in section 6.1 of

the foundational material regarding modular automorphism groups borrows very heavily from the matching presentation in [**BR87a**]. In section 6.2 we present the essentials of Connes cocycles as introduced by Connes in his famous paper on the classification of type III factors [**Con73**], with the bulk of section 6.3 being based on the material introduced by Haagerup in [**Haa79b, Haa79c**]. Where no proofs are offered, interested readers will find proofs of these facts in the indicated references. Didactic expediency has lead us to for the most part focus on σ-finite von Neumann algebras in our presentation of the theory of modular automorphism groups in section 6.1. Readers eager to for this section see proofs that hold for general von Neumann algebras, may wish to consult for example [**Tak03a**] and [**Str81**]. Readers familiar with this theory may of course skip these sections, and proceed directly to section 6.4.

The material in section 6.4 is for the most part based on similar material in [**vD78**], with section 6.5 borrowing heavily from [**Haa78a**]. We do however note that the dual weight construction as presented in Theorem 6.55, extends the dual weight construction as presented by both Haagerup [**Haa78a, Haa78b**], and in the context of crossed products with modular groups, by Terp [**Ter81**]. Haagerup demonstrated the validity of the dual weight construction for possibly non-abelian groups, but only considered faithful and semifinite normal weights. In a more restricted context Terp managed to demonstrate the validity of Theorem 6.55 for semifinite but not necessarily faithful normal weights. Theorem 6.55 shows that only normality is required. The final section is an extension and modification of similar material in [**Ter81**].

6.1 Modular automorphism groups

The key ingredient to developing a theory of L^p spaces valid for possibly non-semifinite von Neumann algebras is unquestionably the theory of modular automorphism groups created by Minoru Tomita and Masamichi Takesaki. In view of this fact we pause to review the foundational theory regarding modular automorphism groups that we shall need in the subsequent development of the theory. Although this theory is essential background for Haagerup L^p-spaces, it is not yet part of the core of that theory. We shall therefore merely survey the theory rather than proving all claims from first principles. Our exposition is very strongly based on the discussion of this material in [**BR87a**] with some material from [**Tak03a, Haa75b**] being added to flesh out the exposition. Readers who

wish to see detailed proofs may consult these references as well as the very comprehensive review of modular theory presented in the classic work of Strătilă [**Str81**].

Unless otherwise stated, we will for the most part assume that we are working with a von Neumann algebra $\mathcal{M}$ equipped with a faithful normal state ω. The essence of the theory is easier to convey and formulation of results simpler in this case. However all results stated have counterparts which hold for von Neumann algebras equipped with a faithful normal semifinite weight, rather than a state. This assumption is therefore being made for purely didactic reasons.

6.1.1   Basic concepts

Recall that when a von Neumann algebra $\mathcal{M}$ equipped with a faithful normal state ω is identified with the GNS representation engendered by ω, the state ω then becomes a vector state corresponding to a cyclic and separating vector Ω. The vector Ω is then in fact cyclic and separating for both $\mathcal{M}$ and $\mathcal{M}'$ (See [**BR87a**, Propositions 2.5.3 and 2.5.6]). We may now use this vector to define antilinear operators S_0 and F_0 on dense subspaces of H, by means of the prescriptions

$$S_0(a\Omega) = a^*\Omega, \quad F_0(a'\Omega) = a'^*\Omega,$$

where $a \in \mathcal{M}$ and $a' \in \mathcal{M}'$.

REMARK 6.1. In the case where we have a normal semifinite weight ψ rather than a state, the Hilbert space H_ψ in the GNS-construction for the pair $(\mathcal{M}, \psi)$ is constructed from the quotient space $\mathfrak{n}_\psi/N_\psi$, where $N_\psi \subseteq \mathfrak{n}_\psi$ is the left-ideal $N_\psi = \{x \in \mathcal{M}: \psi(x^*x) = 0\}$. This quotient space becomes a pre-Hilbert space when equipped with the inner product $\langle x + N_\psi, y + N_\psi \rangle = \psi(y^*x)$ $(x, y \in \mathfrak{n}_\psi)$. The Hilbert space H_ψ is then just the completion of this pre-Hilbert space with respect to the inner-product topology, with the prescription $\eta_\psi : x \mapsto x + N_\psi$ $(x \in \mathfrak{n}_\psi)$ defining a dense embedding of $\mathfrak{n}_\psi$ into H_ψ. As in the state case there is a representation of $\mathcal{M}$ as a subalgebra of $B(H_\psi)$ realised by a *-homomorphism $\pi_\psi : \mathcal{M} \to B(H_\psi)$ satisfying $\langle \pi_\psi(a)\eta(b), \eta(c) \rangle = \psi(c^*ab)$ and $\pi_\psi(a)\eta(b) = \eta(ab)$ for all $a \in \mathcal{M}$ and $b, c \in \mathfrak{n}_\psi$. The triple $(\psi, H_\psi, \eta_\psi)$ is referred to as a semi-cyclic representation. Since in this case $\mathbb{1} \notin \mathfrak{n}_\psi$, it is clear that in this case the GNS construction corresponding to ψ cannot yield a cyclic and separating vector realising ψ as a state. In the case where we are dealing with a faithful normal semifinite weight φ, this construction is somewhat

simpler, as the faithfulness of φ then ensures that $N_\varphi = \{0\}$ and that π_φ is a *-isomorphism.

Despite the absence of a cyclic and separating vector, one may nevertheless still develop a modular theory that closely rivals that of the σ-finite setting. The primary ingredient one needs is a subspace of H_φ which admits an involutive structure that we can use to define an analogue of the operators S and F. The subspace $\eta(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ turns out to be just such a subspace. We may specifically equip this subspace with product and involution operations defined by the prescriptions

$$\begin{aligned} \eta_\varphi(x)\eta_\varphi(y) &= \eta_\varphi(xy) \\ \eta_\varphi(x)^\sharp &= \eta_\varphi(x^*) \end{aligned}$$

for all $x, y \in (\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$. Equipped with this structure $\eta_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ then becomes a so-called full left-Hilbert algebra [**Tak03a**, Theorem VII.2.6]. The completion of this full left-Hilbert algebra then yields all of H_φ [**Tak03a**, Theorems VII.2.5 & VII.2.6]. In direct analogy with the state case we may now densely define the operator S_0 on this subspace by means of the prescription $S_0 : \eta(a) \mapsto \eta(a^*)$. This operator extends to a closed anti-linear operator. The modular operator Δ is then $\Delta = |S|^2$ with the modular conjugation J the anti-linear isometry in the polar decomposition $S = J\Delta^{1/2}$ (consider the discussion preceding [**Tak03a**, Lemma VI.1.4] alongside [**Tak03a**, Lemma VI.1.5].).

Coming back to the case at hand, the operators S_0 and F_0 in fact both turn out to be closable, as can be seen from the following proposition.

PROPOSITION 6.2 ([**BR87a**, 2.5.9]). *The operators S_0 and F_0 defined above are both closable. Moreover $S_0^* = [F_0]$ and $F_0^* = [S_0]$ (square brackets denote the minimal closure). Also for any $\xi \in D([S_0])$ there exists a densely defined closed operator q affiliated to $\mathcal{M}$ such that $q\Omega = \xi$ and $q^*\Omega = [S_0]\xi$, with a similar claim holding for $[F_0]$.*

DEFINITION 6.3. We define the antilinear operators S and F to be $S = [S_0]$ and $F = [F_0]$. We let $\Delta_\omega = \Delta$ be the unique positive self-adjoint operator and $J_\omega = J$ the unique anti-unitary operator occurring in the polar decomposition $S = J\Delta^{1/2}$. We refer to Δ as the *modular operator* and J as the *modular conjugation* for the pair $(\mathcal{M}, \Omega)$.

We start our analysis by reviewing the basic properties and interrelation of the operators S, F, Δ, and J.

PROPOSITION 6.4 ([**BR87a**, 2.5.11]). *The following relations between S, F, Δ and J are valid:*

$$\Delta = FS \text{ and } \Delta^{-1} = SF,$$
$$S = J\Delta^{1/2} \text{ and } F = J\Delta^{-1/2},$$
$$J = J^* \text{ and } J^2 = \mathbb{1},$$
$$\Delta^{-1/2} = J\Delta^{1/2}J.$$

An easy consequence of the above, which is nevertheless worth noting, is the fact that the vector Ω is an eigenvector of Δ corresponding to the eigenvalue 1. This follows from the formula $\Delta = FS$ and the fact that by definition $S\Omega = \Omega = F\Omega$. One of the grand achievements of modular theory is the following theorem describing the action of the operators J and Δ on $\mathcal{M}$.

THEOREM 6.5 (Tomita-Takesaki theorem, cf. [**BR87a**, 2.5.14]). *Let $\mathcal{M}$ be a von Neumann algebra equipped with a cyclic and separating vector Ω, and let Δ and J respectively be the modular operator and modular conjugation corresponding to Ω. Then $J\mathcal{M}J = \mathcal{M}'$ with in addition $\Delta^{it}\mathcal{M}\Delta^{-it}$ for all $t \in \mathbb{R}$.*

The final fact noted in the above theorem now enables us to introduce the following definition. Because of its importance, we formulate this definition for the general case.

DEFINITION 6.6. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite weight φ. Let Δ_φ and J_φ be the modular operator and modular conjugation associated with the pair $(\pi_\varphi(\mathcal{M}), H_\varphi)$. The preceding theorem then ensures that the prescription $\sigma_t^\varphi(a) = \pi_\varphi^{-1}(\Delta_\varphi^{it}\pi_\varphi(a)\Delta_\varphi^{-it})$ where $a \in \mathcal{M}$ and $t \in \mathbb{R}$, yields a one-parameter group of $t \mapsto \sigma_t^\varphi$ of σ-weakly continuous $*$-automorphisms on $\mathcal{M}$, which we shall refer to as the *modular automorphism group associated to the pair* $(\mathcal{M}, \varphi)$.

REMARK 6.7.(1) The σ-weak continuity noted above follows from the fact that the unitary group $t \mapsto \Delta^{it}$ is strongly continuous by Stone's theorem. The automorphism group $t \mapsto \Delta_\varphi^{it}\pi_\varphi(\cdot)\Delta_\varphi^{-it}$ is therefore strong operator continuous.

(2) The faithful normal semifinite weight φ is invariant under the action of the modular group σ_t^φ. In the case where φ is a state this can easily be seen to follow from the fact noted earlier that $\Delta\Omega = \Omega$, and hence that $\Delta^{it}\Omega = \Omega$ for all $t \in \mathbb{R}$.

(3) Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal state ω corresponding to some cyclic and separating vector Ω. The defining property of traces is that operators "commute" under the trace. However even if φ is not a trace, then for all $a, b \in \mathcal{M}$ for which $\Delta^{1/2} a \Delta^{-1/2}$ and $\Delta^{1/2} b \Delta^{-1/2}$ uniquely extend to elements of $\mathcal{M}$, there is a sense in which the modular automorphism group may be used to swop the order of multiplication under φ. Specifically for such $a, b \in \mathcal{M}$, we have that

$$
\begin{aligned}
\langle (\Delta^{-1/2} b \Delta^{1/2})(\Delta^{1/2} a \Delta^{-1/2}) \Omega, \Omega \rangle &= \langle \Delta^{1/2} a \Omega, \Delta^{1/2} b^* \Omega \rangle \\
&= \langle JSa\Omega, JSb^*\Omega \rangle \\
&= \langle Ja^*\Omega, Jb\Omega \rangle \\
&= \overline{\langle a^*\Omega, b\Omega \rangle} \\
&= \langle b\Omega, a^*\Omega \rangle \\
&= \langle ab\Omega, \Omega \rangle.
\end{aligned}
$$

(Here we used the fact that the adjoint formula for antilinear operators $T : H \to H$ is of the form $\langle T\xi, \eta \rangle = \overline{\langle \xi, T^*\eta \rangle}$.) This may formally be written as the claim that $\omega(\sigma^\omega_{i/2}(b)\sigma^\omega_{-i/2}(a)) = \omega(ab)$. (See [**Str81**, Proposition 2.17] for details.)

(4) It is not difficult to see that the modular automorphism group of the pair $(\mathcal{M}, \tau)$, where τ is a faithful normal semifinite trace, is trivial. For the sake of lucidity of exposition, we proceed to substantiate this for the case where $\tau(a) = \langle a\Omega, \Omega \rangle$ for some cyclic and separating vector $\Omega \in H$. With τ being tracial, we will in this case for any $a \in \mathcal{M}$ have that $\|a\Omega\|^2 = \langle a^*a\Omega, \Omega \rangle = \tau(a^*a) = \tau(aa^*) = \langle aa^*\Omega, \Omega \rangle = \|a^*\Omega\|^2$. Thus here S is itself an antilinear isometry, ensuring that $\Delta = |S| = \mathbb{1}$.

The observations made above regarding the remnants of "trace-like" behaviour for weights, really come into their own on the portion of $\mathcal{M}$ on which (as is the case for a trace) the modular automorphism group is trivial. It is with respect to this portion that we see stronger evidence of trace-like behaviour. This is captured in the following definition and the theorem that follows it (which in view of its importance we once again formulate for weights).

DEFINITION 6.8. Let $\mathcal{M}$ be a von Neumann algebra equipped with a faithful normal semifinite weight φ. We define the *centralizer* of the pair $(\mathcal{M}, \varphi)$ to be the subalgebra $\mathcal{M}_\varphi = \{a \in \mathcal{M} \colon \sigma_t^\varphi(a) = a \text{ for all } t \in \mathbb{R}\}$.

THEOREM 6.9 ([**Tak03a**, VIII.2.6]). *Let φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$. A necessary and sufficient condition for $a \in \mathcal{M}$ to belong to the centralizer $\mathcal{M}_\varphi$ is that*

- $a\mathfrak{m}_\varphi \subseteq \mathfrak{m}_\varphi$ *and* $\mathfrak{m}_\varphi a \subseteq \mathfrak{m}_\varphi$,
- *and that* $\varphi(ax) = \varphi(xa)$ *for all* $x \in \mathfrak{m}_\varphi$.

6.1.2 The KMS-condition and analyticity

One of the crowning achievements of modular theory is Takesaki's theorem. This important theorem shows that modular automorphism group satisfies the so-called KMS-condition.

THEOREM 6.10 (cf. [**Tak03a**, Theorem VIII.1.2]). *Let φ be a faithful normal semifinite weight on a von Neumann algebra. Then the automorphism group σ_t^φ satisfies the KMS-condition, namely that it is the unique *-automorphism group $\sigma_t^\varphi = \sigma_t$ which satisfies the condition that*

(1) $\varphi \circ \sigma_t = \varphi$ *for all $t \in \mathbb{R}$,*
(2) *and that for any $x, y \in (\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ there exists a bounded continuous function $F_{x,y}$ on the closed strip $S = \{z \in \mathbb{C} \colon 0 \leq \operatorname{Im}(z) \leq 1\}$ which is analytic on the interior $S_o = \{z \in \mathbb{C} \colon 0 < \operatorname{Im}(z) < 1\}$ such that*

$$F_{x,y}(t) = \varphi(\sigma_t(x)y) \text{ and } F_{x,y}(t+i) = \varphi(y\sigma_t(x)) \text{ for all } t \in \mathbb{R}.$$

On a similar note one may now introduce the following notion.

DEFINITION 6.11. Let $t \mapsto \sigma_t$ be a one-parameter *-automorphism group on $\mathcal{M}$ for which $t \mapsto \sigma_t(a)$ is σ-weakly continuous for every $a \in \mathcal{M}$. An element $a \in \mathcal{M}$ is said to be σ_t-analytic if there exists a strip $S_\gamma = \{z \in \mathbb{C} \colon |\Im(z)| < \gamma\}$ in $\mathbb{C}$, and a function $F \colon S_\gamma \to \mathcal{M}$ such that

- $F(t) = \sigma_t(a)$ for each $t \in \mathbb{R}$,
- with $z \mapsto \rho(F(z))$ analytic for every $\rho \in \mathcal{M}_*$.

In such a case we write $\sigma_z(a)$ for $F(z)$. If F even extends to an entire-analytic function, we say that $a \in \mathcal{M}$ is entire-analytic.

The above definition raises the question of just how many analytic elements there are in $\mathcal{M}$. This has a very elegant answer.

PROPOSITION 6.12 ([**BR87a**, Proposition 2.5.22]). *For each $a \in \mathcal{M}$, the elements of the sequence (a_n) defined by*

$$a_n = \sqrt{\frac{n}{\pi}} \int \sigma_t(a) e^{-nt^2} \, dt$$

are entire-analytic with $\|a_n\| \leq \|a\|$ for each $n \in \mathbb{N}$, and with $a_n \to a$ in the σ-weak topology.

The set of entire-analytic elements of $\mathcal{M}$ actually forms a σ-weakly dense subalgebra. We specifically have the following:

THEOREM 6.13 ([**Tak03a**, Theorem VIII.2.3]). *Let $t \mapsto \sigma_t$ be a σ-weakly continuous one-parameter *-automorphism group on $\mathcal{M}$, and let $\mathcal{M}_\sigma^a$ be the set of all entire analytic elements. Then $\mathcal{M}_\sigma^a$ is a σ-weakly dense *-subalgebra of $\mathcal{M}$. Moreover for any $a, b \in \mathcal{M}_\sigma^a$ and $z, w \in \mathbb{C}$, we have that*

- $\sigma_z(ab) = \sigma_z(a)\sigma_z(b)$,
- $\sigma_{z+w}(a) = \sigma_z(a)\sigma_w(a)$,
- $\sigma_{\bar{z}}(a) = \sigma_z(a^*)^*$.

Any point to σ-weakly continuous one parameter *-automorphism group $t \mapsto \sigma_t$ on $\mathcal{M}$ will by general operator semigroup theory admit an infinitesimal generator $\delta : \mathcal{M} \supseteq \mathrm{dom}(\delta) \to \mathcal{M}$ for which $\mathrm{dom}(\delta) = \{a \in \mathcal{M} : \frac{d\sigma_t(a)}{dt}\big|_{t=0}$ exists$\}$. For each $a \in \mathrm{dom}(\delta)$, δ has the action of mapping a to $\frac{d\sigma_t(a)}{dt}\big|_{t=0}$. This operator turns out to be a σ-weakly closed operator with $\mathrm{dom}(\delta)$ a σ-weakly dense *-subalgebra of $\mathcal{M}$. The operator δ is a so-called *-derivation in that $\delta(ab) = a\delta(b) + \delta(a)b$ and $\delta(a^*) = \delta(a)^*$ for any $a, b \in \mathrm{dom}(\delta)$. It is a useful observation to note that for any $t \in \mathbb{R}$ we have that $\sigma_t(\mathrm{dom}(\delta)) = \mathrm{dom}(\delta)$ with $\delta(\sigma_t(a)) = \sigma_t(\delta(a))$ for any $\mathrm{dom}(\delta)$. (See chapter 3 of [**BR87a**] for details). One may alternatively define analyticity in terms of this infinitesimal generator, as described below.

DEFINITION 6.14. Let $t \mapsto \sigma_t$ be a σ-weakly continuous one-parameter *-automorphism group on $\mathcal{M}$ and let δ be the infinitesimal generator. We define an element $a \in \mathcal{M}$ to be an analytic element for δ if $a \in \mathrm{dom}(\delta^n)$ for each $n \in \mathbb{N}$ and if for some $t > 0$ we have that $\sum_{n=0}^\infty \frac{t}{n!}\|\delta^n(a)\| < \infty$. If in fact $\sum_{n=0}^\infty \frac{t}{n!}\|\delta^n(a)\| < \infty$ for all $t > 0$, we say that a is entire-analytic.

It is a beautiful and elegant fact that this definition of analyticity is entirely equivalent to the earlier one. If for example an element a is analytic in the above sense, then the function F defined by $F(t + z) =$

$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma_t(\delta^n(a))$ will be analytic for z in the radius of analyticity of $\sum_{n=0}^{\infty} \frac{z^n}{n!} \|\delta^n(a)\|$. It can now be verified that a then satisfies the criteria for σ_t-analyticity. If conversely a is σ_t analytic in the strip $S_\gamma = \{z \in \mathbb{C}: |\mathrm{Im}(z)| < \gamma\}$, then the well-known Cauchy inequalities yield

$$\|\delta^n(a)\| = \|\sigma_t(\delta^n(a))\| = \|\frac{d^n}{dt^n}\sigma_t(a)\| \leq \frac{n!M}{\gamma^n}$$

for some $M > 0$. But then $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \|\delta^n(a)\| \leq M \sum_{n=0}^{\infty} (\frac{|z|}{\gamma}) < \infty$ for all $|z| < \gamma$.

6.1.3 A Hilbert space approach

Modular theory may also be studied at Hilbert space level. For this part of the theory we shall not go into any measure of detail, but rather content ourselves with the very rudiments.

DEFINITION 6.15. Let $\mathcal{M}$ be a von Neumann algebra equipped with a cyclic and separating vector Ω. We define the *natural positive cone* $\mathscr{P}^\natural$ associated with the pair $(\mathcal{M}, \Omega)$ as the closure of the set $\{aj(a)\Omega: a \in \mathcal{M}\}$, where $j : \mathcal{M} \to \mathcal{M}'$ is the antilinear $*$-isomorphism given by $j(a) = JaJ$ (where $a \in \mathcal{M}$).

For the space H, $\mathscr{P}^\natural$ plays the same role that the cone of positive elements in $L^2(X, \Sigma, \mu)$ plays in this space. We present some basic technical facts regarding $\mathscr{P}^\natural$ before presenting a result substantiating this claim.

PROPOSITION 6.16 ([**BR87a**, 2.5.26]). *The closed subset $\mathscr{P}^\natural$ of H has the following properties:*
  (1) $\mathscr{P}^\natural = \overline{\Delta^{1/4}\mathcal{M}_+\Omega} = \overline{\Delta^{-1/4}\mathcal{M}'_+\Omega} = \overline{\Delta^{1/4}\overline{\mathcal{M}_+\Omega}} = \overline{\Delta^{-1/4}\overline{\mathcal{M}'_+\Omega}},$
      and hence $\mathscr{P}^\natural$ is convex.
  (2) $\Delta^{it}\mathscr{P}^\natural = \mathscr{P}^\natural$ *for all $t \in \mathbb{R}$.*
  (3) *For any positive-definite function f we have that $f(\log \Delta)\mathscr{P}^\natural \subseteq \mathscr{P}^\natural$.*
  (4) *For any $\xi \in \mathscr{P}^\natural$, we have that $J\xi = \xi$.*
  (5) *For any $a \in \mathcal{M}$ we have that $aj(a)\mathscr{P}^\natural \subseteq \mathscr{P}^\natural$.*

We end this discussion of $\mathscr{P}^\natural$ with the promised presentation of the geometric properties of $\mathscr{P}^\natural$.

PROPOSITION 6.17 ([**BR87a**, 2.5.28]).(1) $\mathscr{P}^\natural$ *is a self-dual cone in the sense that $\xi \in \mathscr{P}^\natural$ if and only if $\langle \xi, \eta \rangle \geq 0$ for all $\eta \in \mathscr{P}^\natural$.*

(2) $\mathscr{P}^{\natural}$ *is a pointed cone in the sense that* $\mathscr{P}^{\natural} \cap (-\mathscr{P}^{\natural}) = \{0\}$.

(3) *Any* $\xi \in H$ *for which we have that* $J\xi = \xi$, *admits a unique decomposition* $\xi = \xi_1 - \xi_2$, *where* $\xi_1, \xi_2 \in \mathscr{P}^{\natural}$ *with* $\xi_1 \perp \xi_2$.

(4) *The Hilbert space* H *is linearly spanned by* $\mathscr{P}^{\natural}$.

We close this section with a discussion regarding the uniqueness of the GNS-representation of the pair $(\mathcal{M}, \varphi)$ where φ is a faithful normal semifinite weight on the von Neumann algebra $\mathcal{M}$. Haagerup proved a very deep theorem essentially showing that any representation of $\mathcal{M}$ which admits objects that mimic the action of J_φ and $\mathscr{P}^{\natural}_\varphi$ is a faithful copy of the GNS-representation of the pair $(\mathcal{M}, \varphi)$. This claim may be made exact with the following definition:

DEFINITION 6.18. Given a von Neumann algebra $\mathcal{M}$ equipped with a faithful normal semifinite weight φ, a quadruple $(\pi_0(\mathcal{M}), H_0, J, \mathscr{P})$ where π_0 is a faithful representation of $\mathcal{M}$ on the Hilbert space H_0, $J : H_0 \to H_0$ anti-linear isometric involution, and $\mathscr{P}$ a self-dual cone of H_0, is said to be a *standard form* of $\mathcal{M}$ if the following conditions hold:

- $J\mathcal{M}J = \mathcal{M}'$ (the commutant of $\mathcal{M}$),
- $JzJ = z^*$ for all z in the centre of $\mathcal{M}$,
- $J\xi = \xi$ for all $\xi \in \mathscr{P}$,
- $a(JaJ)\mathscr{P} \subseteq \mathscr{P}$ for all $a \in \mathcal{M}$.

(Recall that when we say that $\mathscr{P}$ is a self-dual cone, we mean that $\xi \in \mathscr{P}$ if and only if $\langle \xi, \zeta \rangle \geq 0$ for all $\zeta \in \mathscr{P}$.)

The value of the above concept is derived from the following very deep and useful theorem:

THEOREM 6.19 ([**Haa75b**]). *The standard form of a von Neumann algebra* $\mathcal{M}$ *is unique in the sense that if*

$$(\pi_0(\mathcal{M}), H_0, J, \mathscr{P}) \text{ and } (\tilde{\pi}_0(\widetilde{\mathcal{M}}), \widetilde{H}_0, \widetilde{J}, \widetilde{\mathscr{P}})$$

are two standard forms, and $\alpha : \pi_0(\mathcal{M}) \to \tilde{\pi}_0(\widetilde{\mathcal{M}})$ *is a $*$-isomorphism, then there exists a unitary operator* $u : H_0 \to \widetilde{H}_0$ *such that*

- $\alpha(x) = uxu^*$ *for* $x \in \pi_0(\mathcal{M})$;
- $\widetilde{J} = uJu^*$;
- $\widetilde{\mathscr{P}} = u\mathscr{P}$.

6.2   Connes cocycle derivatives

For any serious study of modular automorphism groups, the theory of Connes cocycles is an essential companion, as it is par excellence the theory which provides us with the technology to compare the automorphism groups of two distinct faithful normal semifinite weights on a fixed von Neumann algebra. This technology will prove to be a vital ingredient in our development of the theory of crossed products. We therefore pause to briefly review the essentials of Connes cocycles as they relate to modular automorphism groups. In his paper on the classification of type III factors, Connes proved the following very important theorem:

THEOREM 6.20 ([**Tak03a**, VIII.3.3]). *Let $\mathcal{M}$ be a von Neumann algebra and ψ_1, ψ_2 faithful semifinite normal weights on $\mathcal{M}$. Then there exists a unique σ-strongly continuous one parameter family $\{u_t\}$ of unitaries in $\mathcal{M}$ with the following properties:*

- $u_{s+t} = u_s \sigma_s^{\psi_2}(u_t)$ *for all $s, t \in \mathbb{R}$;*
- $u_s \sigma_s^{\psi_2}(\mathfrak{n}_{\psi_1}^* \cap \mathfrak{n}_{\psi_2}) = \mathfrak{n}_{\psi_1}^* \cap \mathfrak{n}_{\psi_2}$ *for all $s \in \mathbb{R}$;*
- *For each $a \in \mathfrak{n}_{\psi_1} \cap \mathfrak{n}_{\psi_2}^*$ and $b \in \mathfrak{n}_{\psi_1}^* \cap \mathfrak{n}_{\psi_2}$, there exists a function F which is bounded and continuous on the closed strip $S = \{z \in \mathbb{C} \colon 0 \leq \Im(z) \leq 1\}$ and analytic on the open strip $S^o = \{z \in \mathbb{C} \colon 0 < \Im(z) < 1\}$, such that $F(t) = \psi_1(u_t \sigma_t^{\psi_1}(b)a)$ and $F(t+i) = \psi_2(au_t \sigma_t^{\psi_1}(b))$ for all $s \in \mathbb{R}$;*
- $\sigma_t^{\psi_1}(x) = u_t \sigma_t^{\psi_2}(x) u_t^*$ *for all $a \in \mathcal{M}, t \in \mathbb{R}$.*

In view of the uniqueness, we may make the following definition:

DEFINITION 6.21. The family $\{u_t\}$ described in the preceding theorem is called the cocycle derivative of ψ_1 with respect to ψ_2, and denoted by $(D\psi_1 : D\psi_2)_t = u_t$ $(t \in \mathbb{R})$.

One may now use the uniqueness clause in Theorem 6.20, to verify the following fact. This was first observed by Digernes [**Dig75**, Corollary 2.3].

COROLLARY 6.22. *If α is a $*$-automorphism of $\mathcal{M}$ and ψ_1, ψ_2 faithful semifinite normal weights, then $(D\psi_1 \circ \alpha : D\psi_2 \circ \alpha)_t = \alpha^{-1}((D\psi_1 : D\psi_2)_t)$ for all $t \in \mathbb{R}$.*

The following technical lemma is often useful.

LEMMA 6.23 ([**Con73**, Lemma 1.2.3(c)]). *Let ψ be a faithful semifinite normal weight on $\mathcal{M}$, and u a unitary in $\mathcal{M}$. For the weight ψ_u defined by*

$\psi_u(a) = \psi(uau^*)$ *for all* $a \in \mathcal{M}_+$, *we have that* $\sigma_t^\psi(u) = u(D(\psi_u) : D\psi)_t$
for all $t \in \mathbb{R}$.

The cocycle derivatives satisfy the following very elegant chain rule.

THEOREM 6.24 ([**Tak03a**, VIII.3.7]). *Let* ψ_1, ψ_2 *and* ψ_3 *be faithful
semifinite normal weights on* $\mathcal{M}$. *Then*

$$(D\psi_1 : D\psi_2)_t = (D\psi_1 : D\psi_3)_t (D\psi_3 : D\psi_2)_t$$

for all $t \in \mathbb{R}$.

There is a kind of converse to Theorem 6.20, in the form of the following theorem.

THEOREM 6.25 ([**Tak03a**, VIII.3.8]). *Let* ψ *be a faithful semifinite
normal weight on* $\mathcal{M}$, *and* $\{u_t\}$ *($t \in \mathbb{R}$) a* σ-*strongly continuous family
of unitaries in* $\mathcal{M}$ *satisfying the cocycle identity* $u_{s+t} = u_s \sigma_s^\psi(u_t)$ *for all
$s, t \in \mathbb{R}$. Then there exists a faithful semifinite normal weight* ψ_0 *on* $\mathcal{M}$
for which $u_t = (D\psi : D\psi_0)_t$ *for all* $t \in \mathbb{R}$.

The theory of cocycle derivatives also yields the following very deep
and useful characterisation of semifinite algebras. The beauty of this result
is that it not only characterises semifiniteness, but also gives a prescription
for constructing a faithful normal semifinite trace on the given algebra.
(The final claim is not formulated in [**Tak03a**], but can easily be seen to
hold from a consideration of the proof of that theorem.)

THEOREM 6.26 ([**Tak03a**, VIII.3.14]). *For a von Neumann algebra
$\mathcal{M}$ the following are equivalent:*

- $\mathcal{M}$ *is semifinite.*
- *For every faithful semifinite normal weight* ψ *on* $\mathcal{M}$, *the modular
  automorphism group* σ_t^ψ *($t \in \mathbb{R}$) is inner in the sense that there
  exists a strongly continuous unitary group* u_t *($t \in \mathbb{R}$) in* $\mathcal{M}$ *such
  that* $\sigma_t^\psi(a) = u_t a u_t^*$ *for all* $a \in \mathcal{M}$ *and all* $t \in \mathbb{R}$.
- *There exists a faithful semifinite normal weight* φ *on* $\mathcal{M}$ *for which
  the modular automorphism group* σ_t^φ *($t \in \mathbb{R}$) is inner in the above
  sense.*

*If the above conditions hold, then by Stone's theorem the unitary group
implementing* σ_t^φ *($t \in \mathbb{R}$) is of the form* $u_t = h^{it}$ *for some positive non-
singular operator affiliated to the centralizer* $\mathcal{M}_\varphi$, *with the prescription
$\tau = \lim_{\epsilon \searrow 0} \varphi((h_\epsilon^{-1})^{1/2} \cdot (h_\epsilon^{-1})^{1/2})$ where $h_\epsilon^{-1} = h^{-1}(1 + \epsilon h^{-1})^{-1}$, then
yielding a faithful semifinite normal trace on* $\mathcal{M}$.

When using the cocycle derivative to describe the domination of one weight by another, the following theorem is very useful.

THEOREM 6.27 ([**Tak03a**, VIII.3.17]). *For a pair ψ_1, ψ_2 of faithful semifinite normal weights on $\mathcal{M}$, the following conditions are equivalent:*

(1) *There exists $K > 0$ such that*

$$\psi_2(x) \leq K\psi_1(x), \quad x \in \mathcal{M}_+.$$

(2) *The cocycle derivative $(D\psi_1 : D\psi_2)_t \equiv u_t$ can be extended to an $\mathcal{M}$-valued σ-weakly continuous bounded function on the horizontal strip $\overline{D}_{\frac{1}{2}} = \{z \in \mathbb{C}: -\frac{1}{2} \leq \mathrm{Im}(z) \leq 0\}$ for which $\|u_{-\frac{i}{2}}\| \leq K^{1/2}$, and which is analytic on the interior of the strip.*

If these conditions hold, then

$$\psi_2(x) = \psi_1(u^*_{-\frac{i}{2}} x u_{-\frac{i}{2}}), \quad x \in \mathfrak{m}_{\psi_1}.$$

6.3 Conditional expectations and operator valued weights

DEFINITION 6.28. Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathcal{M}$, and $\mathcal{N}$ a von Neumann subalgebra for which the restriction $\varphi|_{\mathcal{N}}$ of φ to $\mathcal{N}$ is still semifinite. A linear map $\mathbb{E}$ from $\mathcal{M}$ onto $\mathcal{N}$ for which we have that

- $\|\mathbb{E}(a)\| \leq \|a\|$ for all $a \in \mathcal{M}$,
- $\mathbb{E}(a) = a$ for all $a \in \mathcal{N}$,
- and $\varphi \circ \mathbb{E} = \varphi$

is called the *conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ.*

REMARK 6.29. Let $\mathcal{C}$ be a unital C^*-algebra, and $\mathcal{B}$ a unital subalgebra. It is a well known result of Tomiyama [**Tom70**] that any unital contractive projection P from $\mathcal{C}$ onto $\mathcal{B}$ is completely contractive and satisfies $P(abc) = aP(b)c$ for all $a, c \in \mathcal{B}$ and all $b \in \mathcal{C}$. On applying this to a conditional expectation $\mathbb{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ, it trivially follows that any such conditional expectation is completely contractive and satisfies the condition that $\mathbb{E}(abc) = a\mathbb{E}(b)c$ for all $a, c \in \mathcal{N}$ and all $b \in \mathcal{M}$. Since by assumption $\mathbb{E}(\mathbb{1}) = \mathbb{1}$ with $\|\mathbb{E}\| = \|\mathbb{E}(\mathbb{1})\| = 1$, $\mathbb{E}$ must in fact be positivity preserving, and hence completely positive (not just completely contractive). In particular we also have that $\mathbb{E}(a^*)\mathbb{E}(a) \leq \mathbb{E}(a^*a)$ for all $a \in \mathcal{M}$.

The question of existence of such maps now arises. This is answered by the following theorem of Takesaki.

THEOREM 6.30 ([**Tak03a**, IX.4.2]). *Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathcal{M}$, and $\mathcal{N}$ a von Neumann subalgebra for which the restriction $\varphi|_{\mathcal{N}}$ of φ to $\mathcal{N}$ is still semifinite. We then have that $\sigma_t^{\varphi}(\mathcal{N}) = \mathcal{N}$ for each $t \in \mathbb{R}$ if and only if there exists a unique normal (σ-weakly continuous) conditional expectation $\mathbb{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ.*

REMARK 6.31. (i) Let $\mathbb{E}$ be a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$ with respect to φ. We then have that $\mathbb{E} \circ \sigma_t^{\varphi} = \sigma_t^{\varphi} \circ \mathbb{E}$. To see this note that for any $a \in \mathfrak{m}_{\varphi}$, we may conclude from the facts $\varphi \circ \mathbb{E} = \varphi$ and $\varphi \circ \sigma_t^{\varphi} = \varphi$, that $\varphi(|\mathbb{E}(\sigma_t^{\varphi}(a)) - \sigma_t^{\varphi}(\mathbb{E}(a))|^2) = 0$ for all t, and hence that $\mathbb{E} \circ \sigma_t^{\varphi} = \sigma_t^{\varphi} \circ \mathbb{E}$ on $\mathfrak{m}_{\varphi}$. The normality and σ-weak density of $\mathfrak{m}_{\varphi}$ in $\mathcal{M}$, now yields the claim.

(ii) In the case where $\mathcal{M}$ is equipped with a faithful semifinite normal trace τ, the criteria for the existence of a conditional expectation are much simpler. Recall that for a trace the modular automorphism group is trivial. So in this setting, the criteria for the existence of a normal conditional expectation $\mathbb{E}$ of $\mathcal{M}$ onto $\mathcal{N}$ with respect to τ is for the restriction of τ to $\mathcal{N}$ to still be semifinite.

Of course a conditional expectation of the above type may not always exist. In such cases a so-called operator-valued weight is often a good substitute.

DEFINITION 6.32. Let φ be a faithful semifinite normal weight on a von Neumann algebra $\mathcal{M}$, and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$. An *operator-valued weight* from $\mathcal{M}$ to $\mathcal{N}$ is a mapping $\mathscr{W} : \mathcal{M}_+ \to \widehat{\mathcal{N}}_+$ such that

(1) $\mathscr{W}(\gamma x) = \gamma \mathscr{W}(x) \quad \gamma \geq 0, x \in \mathcal{M}_+$;
(2) $\mathscr{W}(x + y) = \mathscr{W}(x) + \mathscr{W}(y) \quad x, y \in \mathcal{M}_+$;
(3) $\mathscr{W}(a^* x a) = a^* \mathscr{W}(x) a \quad x \in \mathcal{M}_+, a \in \mathcal{N}$.

We say that $\mathscr{W}$ is *normal* if $x_i \nearrow x \Rightarrow \mathscr{W}(x_i) \nearrow \mathscr{W}(x) \quad x_i, x \in \mathcal{M}_+$.

By analogy with ordinary weights, we set

$$\mathfrak{n}_{\mathscr{W}} = \{x \in \mathcal{M} : \|\mathscr{W}(x^* x)\| < \infty\}$$
$$\mathfrak{m}_{\mathscr{W}} = \mathfrak{n}_{\mathscr{W}}^* \mathfrak{n}_{\mathscr{W}} = \operatorname{span}\{x^* y : x, y \in \mathfrak{n}_{\mathscr{W}}\}$$

The weight $\mathscr{W}$ is called *faithful* if $\mathscr{W}(x^*x) = 0$ implies $x = 0$, and *semifinite* if $\mathfrak{n}_\mathscr{W}$ is σ-weakly dense in $\mathcal{M}$.

REMARK 6.33. In the case where $\mathscr{W}$ is normal, it allows for a natural extension to an affine normal map from $\widehat{\mathcal{M}}_+$ *onto* $\widehat{\mathcal{N}}_+$ [**Haa79b**, Porpo-sition 2.5]. Specifically given any $m \in \widehat{\mathcal{M}}_+$, one may pick $(x_i) \subseteq \mathcal{M}_+$ such that $x_i \nearrow m$, and then define $\mathscr{W}(m)$ to be $\sup_i \mathscr{W}(x_i)$ once the uniqueness of this supremum has been verified.

When working with operator-valued weights, the following technical facts often come in useful.

PROPOSITION 6.34 ([**Tak03a**, IX.4.13]). *Let $\mathscr{W}$ be as in the preceding definition. Then the following holds:*

- $\mathfrak{m}_\mathscr{W}$ *is spanned by its positive part* $\mathfrak{p}_\mathscr{W} = \{x \in \mathcal{M}_+ : \|\mathscr{W}(x)\| < \infty\}$.
- $\mathfrak{n}_\mathscr{W}$ *and* $\mathfrak{m}_\mathscr{W}$ *are two-sided modules over* $\mathcal{N}$.
- *The restriction of $\mathscr{W}$ to $\mathfrak{p}_\mathscr{W}$, extends to a linear map $\dot{\mathscr{W}} : \mathfrak{m}_\mathscr{W} \to \mathcal{N}$ which satisfies the "expectation-like" property that*

  $$\dot{\mathscr{W}}(axb) = a\dot{\mathscr{W}}(x)b \text{ for all } a, b \in \mathcal{N} \text{ and all } x \in \mathfrak{m}_\mathscr{W}.$$

  In particular if $\mathscr{W}(\mathbb{1}) = \mathbb{1}$, then $\dot{\mathscr{W}}$ is a contractive projection from $\mathcal{M}$ onto $\mathcal{N}$ (in which case Tomiyama's result applies - see Remark 6.29).

Let us close this survey by noting the behaviour of operator-valued weights with respect to tensor products.

THEOREM 6.35 ([**Haa79c**, Theorem 5.5]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be von Neumann algebras, and $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively be von Neumann subal-gebras of $\mathcal{M}_1$ and $\mathcal{M}_2$. For each $i \in \{1, 2\}$, let $\mathscr{W}_i$ be an operator-valued weight from $\mathcal{M}_i^+$ to $\widehat{\mathcal{N}}_i^+$. Then there is a unique operator valued weight $\mathscr{W}$ from $(\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_+$ to $(\widehat{\mathcal{N}_1 \overline{\otimes} \mathcal{N}_2})_+$ such that*

$$(\psi_1 \otimes \psi_2) \circ \mathscr{W} = (\psi_1 \circ \mathscr{W}_1) \otimes (\psi_2 \circ \mathscr{W}_2)$$

for any pair (ψ_1, ψ_2) of f.n.s. weights on the pair $(\mathcal{N}_1, \mathcal{N}_2)$.

6.4 Crossed products with general group actions

We start by introducing the concept of a group action.

DEFINITION 6.36. Let G be a locally compact group. We define an action of G on $\mathcal{M}$, to be a point-σ-weakly continuous mapping α from G into the group of $*$-automorphisms on $\mathcal{M}$, which respects the group action in the sense that $\alpha_s \circ \alpha_t = \alpha_{st}$ for all $s, t \in G$.

Throughout this section, G will denote a locally compact group admitting an action on the von Neumann algebra $\mathcal{M}$. The following proposition is a vital ingredient in the construction of the crossed product, and seems to be part of mathematical folklore by now. The interested reader may find a proof in the book of van Daele [**vD78**].

PROPOSITION 6.37. *Let $L^2(G)$ be the Hilbert space of square Haar-integrable functions on G. The Hilbert space tensor product $H \otimes L^2(G)$ is a copy of $L^2(G, H)$, the space of square Bochner-integrable functions from G to H. Hence the simple tensors $x \otimes f$ ($x \in H$, $f \in L^2(G)$) may be thought of as functions of the form $G \to H : s \mapsto f(s)x$.*

THEOREM 6.38. *For every $a \in \mathcal{M}$, the prescription*

$$\pi_\alpha(a)(\xi)(s) = \alpha_{s^{-1}}(a)(\xi(s)) \qquad \xi \in L^2(G, H)$$

is a well-defined bounded map on $L^2(G, H)$. Moreover the map

$$\pi_\alpha : \mathcal{M} \to B(L^2(G, H)) : a \mapsto \pi_\alpha(a) \qquad a \in \mathcal{M}$$

is a $*$-isomorphism from $\mathcal{M}$ into $B(L^2(G, H))$.
Convention: *When the group action α is understood, we shall simply write π for π_α.*

PROOF. For every element $\xi \in L^2(G, H)$ we have that

$$\|\pi(a)(\xi)\|^2 = \int_G \|\sigma_{s^{-1}}(a)(\xi(s))\|^2 \, ds \le \|a\|^2 \int_G \|\xi(s)\|^2 \, ds = \|a\|^2 \|\xi\|^2.$$

This shows that $\pi(a)$ is bounded map on $L^2(G, H)$. However this computation also shows that $\|\pi(a)\| \le \|a\|$. Hence the map $\pi : \mathcal{M} \to B(L^2(G, H)) :$ $a \mapsto \pi(a)$ is contractive. It is now an exercise to see that π is a $*$-homomorphism. To conclude the proof we show that it is injective and hence a $*$-isomorphism. So let $0 \ne a \in \mathcal{M}$ be given, and select $x, z \in H$ such that $\langle ax, z \rangle \ne 0$. By continuity we will then have that $0 \ne \langle \sigma_{s^{-1}}(a)x, z \rangle$ for all s in some compact neighbourhood K of the group unit e. That in turn ensures that $0 \ne \sigma_{s^{-1}}(a)x$ for all $s \in K$. We may in fact arrange matters so that for some $\epsilon > 0$ we have that $\|\sigma_{s^{-1}}(a)x\| \ge \epsilon$ on K. Now consider $\pi(a)(x \otimes \chi_K)$. Based on what we have noted thus far, direct

computation now shows that $\|\pi(a)(x \otimes \chi_K)\|^2 = \int_G \|\sigma_{s^{-1}}(a)(x)\|^2 \chi_K(s)\, ds = \int_K \|\sigma_{s^{-1}}(a)x\|^2\, ds \geq \epsilon \mu_H(K) > 0$. (Here μ_H denotes Haar measure where the subscript H serves to honour the inventor of this measure, Alfréd Haar.) Thus $a \mapsto \pi(a)$ is injective, and the claim therefore follows.

It remains to prove the normality of π_α. To this end let (a_γ) be a net in $\mathcal{M}_+$ increasing to $a \in \mathcal{M}$. The normality of the α_s's ensures that we will for any $s \in G$ and any $\xi \in L^2(G, H)$ then clearly have that $\sup_\gamma \alpha_{s^{-1}}(a_\gamma)\xi(s) = \alpha_{s^{-1}}(a)\xi(s)$. Using the fact that integration against μ_H is a normal weight (see for example [**Tak02**, Theorem III.1.18]), it now follows that we will for any $\xi \in L^2(G, H)$ have that

$$\sup_\gamma \langle \pi_\alpha(a_\gamma)\xi, \xi \rangle = \sup_\gamma \int_G \langle \alpha_{s^{-1}}(a_\gamma)\xi(s), \xi(s) \rangle\, ds$$

$$= \int_G \langle \alpha_{s^{-1}}(a)\xi(s), \xi(s) \rangle\, ds = \langle \pi_\alpha(a)\xi, \xi \rangle.$$

We therefore clearly have that $(\pi_\alpha(a_\gamma))$ increases to $\pi_\alpha(a)$, proving that $a \to \pi_\alpha(a)$ is normal. $\square$

DEFINITION 6.39. For every $g \in G$, define $\lambda_g \in B(L^2(G, H))$ to be the map $\lambda_g(\xi)(s) = \xi(g^{-1}s)$ where $\xi \in L^2(G, H)$.

PROPOSITION 6.40. *The prescription $g \mapsto \lambda_g$ is a strongly continuous unitary representation of G on $L^2(G, H)$, with $\lambda_{g^{-1}} = \lambda_g^*$. Moreover $\lambda_g \pi(a) \lambda_g^* = \pi(\sigma_g(a))$ for any $g \in G$ and any $a \in \mathcal{M}$.*

PROOF. It is a straightforward exercise to conclude from the left-translation invariance of Haar measure that each λ_g is a unitary operator with $\lambda_{g^{-1}} = \lambda_g^*$. We prove the claim regarding strong continuity. For any simple tensor $(x \otimes f)$ where $x \in H$ and $f \in L^2(G)$, $\pi(a)(x \otimes f)$, we have that $\lambda_g(x \otimes f)(s) = f(g^{-1}s)x$. On denoting left translation by elements of G on $L^2(G)$ by ℓ_g, we therefore have that $\|\lambda_g(x \otimes f) - (x \otimes f))\| = \|x\| \|f - \ell_g(f)\|_2$. From the basic theory of Haar measure, we know that $\|f - \ell_g(f)\|_2 \to 0$ as g tends to the group unit e. This proves strong continuity on the simple tensors. We may now use this fact to prove the claim for general elements of $L^2(G, H)$, by suitably approximating such elements with linear combinations of simple tensors.

Finally let $a \in \mathcal{M}$ and $\xi \in L^2(G, H)$ be given. On fixing $g \in G$, direct checking now shows that $\pi(a)(\lambda_g^*\xi)(s) = \sigma_{s^{-1}}(a)(\lambda_g^*\xi)(s) = \sigma_{s^{-1}}(a)\xi(g^{-1}s)$, and hence that

$$\lambda_g(\pi(a)\lambda_g^*\xi)(s) = \sigma_{gs^{-1}}(a)\xi(s) = \sigma_{s^{-1}}(\sigma_g(a))\xi(s) = \pi(\sigma_g(a))\xi(s).$$

$\square$

DEFINITION 6.41. We now define the crossed product of $\mathcal{M}$ with the group action of G, to be the von Neumann algebra on $L^2(G, H)$ generated by $\pi(\mathcal{M})$ and the translation operators λ_g where $g \in G$. We will denote this von Neumann algebra by $\mathcal{M} \rtimes_\alpha G$.

REMARK 6.42. Let G and $\mathcal{M}$ be as above. It is clear from Proposition 6.40 that $a \mapsto \lambda_g a \lambda_g^*$ defines an implemented action of the group G on $\mathcal{M} \rtimes_\alpha G$. This same proposition then also shows that in a very real sense this action may be thought of as an extension of the a priori given action of G on $\mathcal{M}$. On using the fact that $\lambda_g \pi(a) \lambda_g^* = \pi(\sigma_g(a))$ for any $g \in G$ and any $a \in \mathcal{M}$, it is an exercise to see that any algebraic combination of the λ_g's and elements from $\pi(\mathcal{M})$ may be written as a linear combination of terms of the form $\pi(a)\lambda_g$. Hence the crossed product corresponds to the σ-weak closure of $\mathrm{span}\{\pi(a)\lambda_g : a \in \mathcal{M}, g \in G\}$ in $B(L^2(G, H))$.

In view of the fact that $L^2(G, H) = H \otimes L^2(G)$, the question of how $\mathcal{M} \rtimes_\alpha G$ compares to $B(H) \overline{\otimes} B(L^2(G))$ now arises. This is answered by the following Proposition. For a proof see [**vD78**, Part I: Lemma 3.1].

PROPOSITION 6.43. *The crossed product $\mathcal{M} \rtimes_\alpha G$ is a subspace of the von Neumann algebra tensor product $\mathcal{M} \overline{\otimes} B(L^2(G))$. In particular each λ_g $(g \in G)$ is of the form $\mathbb{1} \otimes \ell_g$, where ℓ_g is the left-translation operator defined on $L^2(G)$ by $\ell_g(f)(s) = f(g^{-1}s)$. Moreover if the action α of G on $\mathcal{M}$ is implemented by some strongly continuous unitary group $\{u_g\} \subseteq B(H)$ (in the sense that $\alpha_g(a) = u_g a u_g^*$ for each $a \in \mathcal{M}$ and each $g \in G$), then $U^*(\mathcal{M} \otimes \mathbb{1})U = \pi(\mathcal{M})$ where U is the unitary defined by $U(\xi)(s) = u_s \xi(s)$ for each $\xi \in L^2(G)$*

The requirement in the last part of the above proposition that α be implemented is not too onerous. If the identity of the specific Hilbert space underlying $\mathcal{M}$ is not important, we may always arrange matters in such a way that this does hold. We may for example replace $\mathcal{M}$ by $\pi(\mathcal{M})$ to pass to a context where α is implemented (see Proposition 6.40).

With the previous proposition as background, we are now able to prove that *-isomorphic copies of a von Neumann algebra will yield *-isomorphic copies of the crossed product $\mathcal{M} \rtimes_\alpha G$. This then shows that up to *-isomorphic equivalence the crossed product is independent of the particular copy of a von Neumann algebra being used. We will revisit

this issue of uniqueness when we analyse the structure of crossed products with modular automorphism groups of a given canonical weight.

THEOREM 6.44. *Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras, and $\mathscr{I}$ a $*$-isomorphism from $\mathcal{M}$ onto $\mathcal{N}$. Let G be a locally compact group admitting actions α and β on $\mathcal{M}$ and $\mathcal{N}$ respectively. If for all $g \in G$ and $a \in \mathcal{M}$ we have that $\mathscr{I}(\alpha_g(a)) = \beta_g(a)$, then $\widetilde{\mathscr{I}} = \mathscr{I} \otimes \mathrm{Id}$ is a $*$-isomorphism $\widetilde{\mathscr{I}}$ from $\mathcal{M} \rtimes_\alpha G$ onto $\mathcal{N} \rtimes_\beta G$, for which we have that $\widetilde{\mathscr{I}}(\pi_\alpha(a)) = \pi_\beta(\mathscr{I}(a))$ for all $a \in \mathcal{M}$, and also that $\widetilde{\mathscr{I}}(\lambda_g) = \widetilde{\lambda}_g$ for all $g \in G$ where λ_g and $\widetilde{\lambda}_g$ respectively denote the left-shift operators corresponding to $\mathcal{M} \rtimes_\alpha G$ and $\mathcal{N} \rtimes_\beta G$.*

PROOF. We know, from the standard theory of von Neumann algebra tensor products, that $\widetilde{\mathscr{I}} = \mathscr{I} \otimes \mathrm{Id}$ is a *-isomorphism from $\mathcal{M} \overline{\otimes} B(L^2(G))$ to $\mathcal{N} \overline{\otimes} B(L^2(G))$. We need to show that $\widetilde{\mathscr{I}}$ maps the subspace $\mathcal{M} \rtimes_\alpha G$ onto the subspace $\mathcal{N} \rtimes_\beta G$ in the manner described in the hypothesis. Let λ_g denote the left-shift operators in $\mathcal{M} \rtimes_\alpha G$, and $\widetilde{\lambda}_g$ those in $\mathcal{N} \rtimes_\beta G$. Now recall that by Proposition 6.43 $\lambda_g = \mathbb{1}_\mathcal{M} \otimes \ell_g$ and $\widetilde{\lambda}_g = \mathbb{1}_\mathcal{N} \otimes \ell_g$. Thus we clearly have that $\widetilde{\mathscr{I}}(\lambda_g) = \widetilde{\lambda}_g$ for all $g \in G$. To complete the proof need we show that $\widetilde{\mathscr{I}}(\pi_\alpha(a)) = \pi_\beta(\mathscr{I}(a))$ for all $a \in \mathcal{M}$, since then $\widetilde{\mathscr{I}}$ will clearly map the von Neumann algebra generated by $\{\pi_\alpha(a), \lambda_g \colon a \in \mathcal{M}, \, g \in G\}$ (namely $\mathcal{M} \rtimes_\alpha G$) onto the von Neumann algebra generated by $\{\pi_\beta(b), \widetilde{\lambda}_g \colon b \in \mathcal{N}, \, g \in G\}$ (that is $\mathcal{N} \rtimes_\alpha G$).

For this part of the proof we need the fact that any element of $\mathcal{M} \overline{\otimes} B(L^2(G))$ may be represented as some sort of matrix. Select an orthonormal basis $\{f_i\}$ of $L^2(G)$. Any element $\widetilde{a} \in \mathcal{M} \overline{\otimes} B(L^2(G))$ may be written as the sum $\sum_{i,j \in I} (\mathbb{1} \otimes e_i) \widetilde{a} (\mathbb{1} \otimes e_j)$ where the e_i's are the projections $\langle \cdot, f_i \rangle f_i$, and convergence is in the σ-strong topology. In the specific case where $\widetilde{a} = \pi_\alpha(a)$ for some $a \in \mathcal{M}$ it is a not altogether trivial exercise to see that here $(\mathbb{1} \otimes e_i) \pi_\alpha(a) (\mathbb{1} \otimes e_j) = (\pi_\alpha(a)_{ij} \otimes u_{i,j})$ with $\pi_\alpha(a)_{ij} = \int_G \alpha_{s^{-1}}(x) \overline{f_i}(s) f_j(s) \, ds$ and $u_{i,j} = \langle \cdot, f_j \rangle f_i$. The $u_{i,j}$'s are a set of "matrix units" in that $u_{i,j}^* = u_{j,i}$ and $u_{i,j} u_{j,k} = u_{i,k}$ with $\sum_{i \in I} u_{i,i} = \sum_{i \in I} e_i = \mathbb{1}_{B(L^2(G))}$ (where the sum converges in the σ-strong* topology). So for such an a we must have that $\pi_\alpha(a) = \sum_{i,j} \pi_\alpha(a)_{ij} \otimes u_{i,j}$.

Given $a \in \mathcal{M}$, the normality of $\widetilde{\mathscr{I}}$ then ensures that $\widetilde{\mathscr{I}}(\pi_\alpha(a)) = \sum_{i,j} \widetilde{\mathscr{I}}(\pi_\alpha(a)_{ij} \otimes u_{i,j}) = \sum_{i,j} \mathscr{I}(\pi_\alpha(a)_{ij}) \otimes u_{i,j}$. Now observe that for

each $i, j \in I$ we have that

$$
\begin{aligned}
\mathscr{I}\big((\pi_\alpha(a))_{ij}\big) &= \int_G \mathscr{I}(\alpha_{s^{-1}}(a))\overline{f_i}(s) f_j(s)\, ds \\
&= \int_G \beta_{s^{-1}}(\mathscr{I}(a))\overline{f_i}(s) f_j(s)\, ds \\
&= \big(\pi_\beta(\mathscr{I}(a))\big)_{ij}.
\end{aligned}
$$

We therefore have that $\widetilde{\mathscr{I}}(\pi_\alpha(a)) = \pi_\beta(\mathscr{I}(a))$ as was required. $\square$

6.5   Crossed products with abelian locally compact groups

For the rest of this chapter we shall assume that G is abelian. When one passes to abelian locally compact groups, one gains access to the very rich theory of Pontryagin duality for locally compact groups. Our ultimate interest here is to first introduce the notion of a dual action, and then to use that to construct an operator valued weight from $\mathcal{M} \rtimes_\alpha G$ to $\mathcal{M}$. Although most of these constructs remain valid for more general groups when given suitable interpretations, the theory is more easily accessible in the setting abelian groups. As a start we shall in this context introduce the notion of a dual action of G on $\mathcal{M} \rtimes_\alpha G$, or more properly an action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$, and then show how this action may be used to describe $\pi(\mathcal{M})$ as a subspace of $\mathcal{M} \rtimes_\alpha G$. We remind the reader that the group $\widehat{G}$ is the group of all characters of G (continuous group-homomorphisms from G into $\mathbb{T}$), and that $\widehat{G}$ is itself again a locally compact abelian group. This allows for the following definition of the abstract Fourier transform.

DEFINITION 6.45. We may define the Fourier transform $\mathcal{F}$ on $L^1(G)$ by the prescription

$$
\mathcal{F}(f)(\gamma) = \int_G \overline{\gamma(g)} f(g)\, dg.
$$

We list some of the properties of this Fourier transform. For proofs of these facts refer to for example [**Tak03a**, Theorem VII.3.14].

- $\mathcal{F}$ is just the so-called Gelfand transform on $L^1(G)$, and maps $L^1(G)$ into $C_\infty(\widehat{G})$.
- **Abstract Plancherel formula:** $\mathcal{F}$ preserves the L^2-norm on $L^1(G) \cap L^2(G)$ and extends to unitary on $L^2(G)$. For every $f \in L^1(\widehat{G}) \cap L^2(\widehat{G})$, the map $\mathcal{F}^* = \mathcal{F}^{-1}$ agrees with the inverse

Fourier transform defined by $\mathcal{F}(f)(g) = \int_{\widehat{G}} \gamma(g) f(\gamma) \, d\gamma$, where $f \in L^1(\widehat{G})$.

- Let $VN(G)$ denote the von Neumann algebra generated by the left-shift operators ℓ_g on $L^2(G)$. Then $\mathcal{F}(VN(G))\mathcal{F}^{-1} \equiv L^\infty(\widehat{G})$. (More properly, $\mathcal{F}(VN(G))\mathcal{F}^{-1}$ agrees with the von Neumann algebra of multiplication operators on $L^2(\widehat{G})$ with symbols in $L^\infty(\widehat{G})$.)

Let $v_\gamma : L^2(G) \to L^2(G)$ ($\gamma \in \widehat{G}$) be the operator defined by $v_\gamma(f)(s) = \overline{\gamma(s)}f(s)$ for each $f \in L^2(G)$ and each $s \in G$. It is an exercise to see that these maps are actually unitaries. The maps $w_\gamma = \mathbb{1} \otimes v_\gamma$ on $H \otimes L^2(G) = L^2(G, H)$ are then also unitaries. It is these maps that we use to define an action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$. In their action on $L^2(G, H)$, they fulfil the prescription

$$w_\gamma(\xi)(s) = \overline{\gamma(s)}\xi(s) \qquad \xi \in L^2(G, H), s \in G, \gamma \in \widehat{G}.$$

(It is the unimodularity of the numbers $\overline{\gamma(s)}$, that ensure that each w_γ is a unitary.) For each simple tensor $x \otimes f$ ($x \in H$, $f \in L^2(G)$), we have that

$$\int_G \|(\mathbb{1} - w_\gamma)(x \otimes f)(s)\|^2 \, ds = \|x\|^2 \int_G |1 - \overline{\gamma(s)}|^2 |f(s)|^2 \, ds.$$

Standard estimates show that $\int_G |1 - \overline{\gamma(s)}|^2 |f(s)|^2 \, ds \to 0$ as γ tends to the group unit of $\widehat{G}$. Hence for each simple tensor $g \mapsto w_\gamma(x \otimes f)$ is continuous in $L^2(G, H)$-norm. By suitably approximating general elements of $L^2(G, H)$ with linear combinations of simple tensors, we may then show that $\gamma \mapsto w_\gamma$ is strongly continuous on all of $L^2(G, H)$. It is now easy to check that in addition $w_{\gamma_1} w_{\gamma_2} = w_{\gamma_1 \gamma_2}$. If now for each $b \in B(L^2(G, H))$ we define $\widehat{\alpha}_\gamma(b)$ to be $w_\gamma b w_\gamma^*$, the maps $\widehat{\alpha}_\gamma$ can easily be shown to be a group action of $\widehat{G}$ on $B(L^2(G, H))$. However more is true. Further checking reveals that

$$\widehat{\alpha}_\gamma(\pi(a)) = \pi(a) \text{ and } \widehat{\alpha}_\gamma(\lambda_g) = \overline{\gamma(g)}\lambda_g \text{ for each } a \in \mathcal{M} \text{ and } g \in G. \quad (6.1)$$

Since each of the maps $\widehat{\alpha}_\gamma$ map the generators of $\mathcal{M} \rtimes_\alpha G$ back into $\mathcal{M} \rtimes_\alpha G$, they must each preserve $\mathcal{M} \rtimes_\alpha G$. Thus the action $\widehat{\alpha}$ restricts to an action on $\mathcal{M} \rtimes_\alpha G$.

DEFINITION 6.46. The restriction of $\widehat{\alpha}$ to $\mathcal{M} \rtimes_\alpha G$ is defined to be the dual action of $\widehat{G}$ on $\mathcal{M} \rtimes_\alpha G$.

We have seen that the action $\widehat{\alpha}$ leaves $\pi(\mathcal{M})$ invariant. One of the primary results in this section asserts that the converse is also true. Proving this result is not difficult as such, but it does rely on a few rather non-trivial facts regarding Fourier analysis on locally compact groups. Readers not familiar with the finer points of that theory, may take the result at face value, and proceed with the rest of the analysis. The result first appeared in print in a paper of Haagerup [**Haa78a**, Lemma 3.6], but seems to be due to Landstad. Some preparation is needed before we are able to present the proof of the result.

THEOREM 6.47. *Let X be a locally compact space equipped with a Radon measure μ_R, and $\mathcal{A}$ the von Neumann algebra consisting of all multiplication operators on the Hilbert space $L^2(X, \mu_R)$, with symbols in $L^\infty(X, \mu_R)$. Then $\mathcal{A}$ is maximal abelian, that is $\mathcal{A} = \mathcal{A}'$. (This is proved in Takesaki [**Tak02**], Theorem III.1.2, for the case where μ_R is finite. That proof readily adapts to the present setting.)*

LEMMA 6.48. *Put $(X, d\mu_R) = (G, ds)$ in Theorem 6.47. Then $\mathcal{A} = \{v_\gamma \colon \gamma \in \widehat{G}\}''$.*

PROOF. Let ρ be a σ-weakly continuous functional on $\mathcal{A}$ such that $\rho(v_\gamma) = 0$ for all $\gamma \in \widehat{G}$. Let $h \in L^1(G)$ be the density $h = \frac{d\rho}{ds}$ of ρ with respect to ds. Then $\rho(v_\gamma) = \int \overline{\gamma(s)} h(s) \, ds = 0$, that is $\widehat{h}(\gamma) = 0$ for all $\gamma \in \widehat{G}$ where $\widehat{h}$ is the Fourier transform of h. Since the Fourier transform is injective on $L^1(G, ds)$, we have that $h = 0$, whence $\rho = 0$. The span of $\{v_\gamma \colon \gamma \in \widehat{G}\}$ is therefore σ-weakly dense in $\mathcal{A}$, with the result then following from the von Neumann double commutant theorem $\qquad\square$

LEMMA 6.49. $\{\ell_s, v_\gamma \colon s \in G, \gamma \in \widehat{G}\}'' = B(L^2(G))$.

PROOF. Let $x \in \{\ell_s, v_\gamma \colon s \in G, p \in \widehat{G}\}'$. By Theorem 6.47, x must be a multiplication operator by some function $f \in L^\infty(G, ds)$. If now $x\lambda_s = \lambda_s x$ for all $s \in G$, then, as

$$((x\lambda_s)(g))(t) = f(t)g(t - s),$$

and

$$((\lambda_s x)(g))(t) = x(g)(t - s) = f(t - s)g(t - s)$$

we have that $f(t)g(t-s) = f(t-s)g(t-s)$ for all $g \in L^2(G)$. It follows that f is constant almost everywhere, so that x is a multiple of the identity. $\quad\square$

THEOREM 6.50. *As a subspace of $\mathcal{M} \rtimes_\alpha G$, the algebra $\pi(\mathcal{M})$ corresponds to the fixed points of the dual action $\widehat{\alpha}$.*

PROOF. By the comment following Proposition 6.43, we may assume that the action α of G on $\mathcal{M}$ is implemented. Having made this assumption, let U be as in the hypothesis of Proposition 6.43.

We have already noted that $\pi(\mathcal{M}) \subseteq \mathcal{N} = \{\widetilde{a} \in \mathcal{M} \rtimes_\alpha G \colon \widehat{\alpha}_\gamma(\widetilde{a}) = \widetilde{a}, \gamma \in \widehat{G}\}$. Conversely let $\widetilde{a} \in \mathcal{N} \subseteq \mathcal{M} \rtimes_\alpha G$.

Let the v_γ's and w_γ's ($\gamma \in \widehat{G}$) be as in the discussion preceding Definition 6.46. The fact that $\widehat{\alpha}_\gamma(\widetilde{a}) = \widetilde{a}$ for each $\gamma \in \widehat{G}$, then corresponds to the claim that $\widetilde{a}$ commutes with each $w_\gamma = \mathbb{1} \otimes v_\gamma$. Direct checking now reveals $U^*(\mathbb{1} \otimes v_\gamma)U = (\mathbb{1} \otimes v_\gamma)$ for each $\gamma \in \widehat{G}$. We therefore have that $\widetilde{a} \in \{U^*(\mathbb{1} \otimes v_\gamma)U \colon \gamma \in \widehat{G}\}'$.

Since the group is abelian, the operators λ_g and their adjoints, all commute with each other. Note for example that $\lambda_s \lambda_t^* = \lambda_{st^{-1}} = \lambda_{t^{-1}s} = \lambda_t^* \lambda_s$. Now let the operators ℓ_g ($g \in G$) be as in Proposition 6.43. Each $\mathbb{1} \otimes \ell_g$ clearly commutes with each $u_g \otimes \mathbb{1}$, and hence we must have $U^*(\mathbb{1} \otimes \ell_g)U = (\mathbb{1} \otimes \ell_g)$ for each $g \in G$, or in different notation, that $U^* \lambda_g U = \lambda_g$. Since $\pi(\mathcal{M}) = U^*(\mathcal{M} \times \mathbb{1})U$, each element of $\pi(\mathcal{M})$ must therefore clearly commute with each $U^*(\mathbb{1} \otimes \ell_g)U$. With all the generators of $\mathcal{M} \rtimes_\alpha G$ now commuting with each $U^*(\mathbb{1} \otimes \ell_g)U$, it follows that all of $\mathcal{M} \rtimes_\alpha G$ must commute with each $U^*(\mathbb{1} \otimes \ell_g)U$.

If we combine the conclusions of the previous two paragraphs, we have that $U\widetilde{a}U^* \in \{\mathbb{1} \otimes \ell_g, \mathbb{1} \otimes v_\gamma \colon g \in G, \gamma \in \widehat{G}\}'$. However by Lemma 6.49, $\{\mathbb{1} \otimes \ell_g, \mathbb{1} \otimes v_\gamma \colon g \in G, \gamma \in \widehat{G}\}' = B(H) \otimes \mathbb{1}$. In other words $\widetilde{a}$ is of the form $U^*(b \otimes \mathbb{1})U$ for some $b \in B(H)$.

Finally observe that each $\lambda_g = \mathbb{1} \otimes \ell_g$ commutes with $\mathcal{M}' \otimes \mathbb{1}$. Recall that in the proof of Proposition 6.43, we saw that in addition $\pi(\mathcal{M}) \subseteq (\mathcal{M}' \otimes \mathbb{1})'$. Therefore $\mathcal{M} \rtimes_\alpha G \subseteq (\mathcal{M}' \otimes \mathbb{1})'$. Let $c \in \mathcal{M}'$ be given. As an element of $\mathcal{M} \rtimes_\alpha G$, $\widetilde{a} = U^*(b \otimes \mathbb{1})U$ must commute with $c \otimes \mathbb{1}$. So for each simple tensor $x \otimes f \in H \otimes L^2(G)$ and each $s \in G$, we must have $(cu_s^* b u_s x)f(s) = (c \otimes \mathbb{1})U^*(b \otimes \mathbb{1})U(x \otimes f)(s) = U^*(b \otimes \mathbb{1})U(c \otimes \mathbb{1})(x \otimes f)(s) = (u_s^* b u_s c x)f(s)$. If we take s to be the group unit e, we may conclude from this that b commutes with c, and hence that $b \in \mathcal{M}$. Therefore $\widetilde{a} = U^*(b \otimes \mathbb{1})U \in U^*(\mathcal{M} \otimes \mathbb{1})U = \pi(\mathcal{M})$ as required. $\qquad\square$

With Theorem 6.50 as foundation, we may now construct an operator-valued weight from $\mathcal{M} \rtimes_\alpha G$ to $\mathcal{M}$.

DEFINITION 6.51. We formally define the operator-valued weight $\mathscr{W}_G$ from $(\mathcal{M} \rtimes_\alpha G)_+$ onto the extended positive part of $\pi(M)$ by the prescription $\mathscr{W}_G(a) = \int_{\widehat{G}} \widehat{\alpha}_\gamma(a)\, d\gamma$ where $a \in (\mathcal{M} \rtimes_\alpha G)_+$.

Haagerup's result confirms that the above definition serves the purpose for which it was formulated.

PROPOSITION 6.52 ([**Haa78b**, Theorem 1.1]). *The prescription $a \mapsto \mathscr{W}_G(a)$ defined above yields a faithful normal semifinite operator valued weight from $(\mathcal{M} \rtimes_\alpha G)_+$ onto $\widehat{\pi(\mathcal{M})}$ for which we have that $\mathscr{W}_G \circ \widehat{\alpha}_\gamma = \mathscr{W}_G$ for each $\gamma \in \widehat{G}$.*

COROLLARY 6.53. *If the group G is discrete, the operator-valued weight $\mathscr{W}_G$ defined above is a positive scalar multiple of a faithful normal conditional expectation from $\mathcal{M} \rtimes_\alpha G$ onto $\pi(\mathcal{M})$. The action of this conditional expectation is uniquely determined by the formula*

$$\mathscr{W}_G(\lambda_g \pi(a)) = \left\{ \begin{array}{ll} \pi(a) & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right. \qquad g \in G, a \in \mathcal{M}.$$

PROOF. We have already noted that the group G is discrete if and only if the dual group $\widehat{G}$ is compact. Hence, Haar measure on $\widehat{G}$ will be finite. It is clear that in this case $\mathscr{W}_G(a) = \int_{\widehat{G}} \widehat{\alpha}_\gamma(a)\, d\gamma$ will be an element of $\mathcal{M}$ for each $a \in \mathcal{M} \rtimes_\alpha G$. In fact, on rescaling we may assume Haar measure on $\widehat{G}$ to be a probability measure, in which case $\mathscr{W}_G(\mathbb{1}) = \mathbb{1}$. The fact that the action of the conditional expectation on terms of the form $\lambda_g \pi(a)$ (where $g \in G, a \in \mathcal{M}$) uniquely determines the expectation, follows from Remark 6.42 and the noted normality of this expectation. Given such an element, we may apply Theorem 6.50 to see that $\mathscr{W}_G(\lambda_g \pi(a)) = \int_{\widehat{G}} \widehat{\alpha}_\gamma(\lambda_g \pi(a))\, d\gamma = \lambda_g \pi(a) \int_{\widehat{G}} \overline{\gamma(g)}\, d\gamma$. Assuming G to be additive, the claim now follows from the known fact that

$$\int_{\widehat{G}} \overline{\gamma(g)}\, d\gamma = \left\{ \begin{array}{ll} 1 & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right. .$$

(See Exercise VII.5.6 of [**Kat04**].)   □

We are now ready to introduce the notion of a dual weight, and study its properties.

DEFINITION 6.54. Given any normal weight ψ on $\mathcal{M}$, we define $\widetilde{\psi} = \widehat{\psi} \circ \widehat{\pi^{-1}} \circ \mathscr{W}_G$ to be the corresponding dual weight on $(\mathcal{M} \rtimes_\alpha G)$. Here $\widehat{\psi}$ is the extension of ψ to $\widehat{\mathcal{M}}$ and $\widehat{\pi^{-1}}$ the extension of π^{-1}, to $\widehat{\pi(\mathcal{M})}$.

We hasten to point out that Haagerup has demonstrated the existence of an operator valued weight from $(\mathcal{M} \rtimes_\alpha G)_+$ to $\widehat{\mathcal{M}}_+$, and of dual weights for general possibly non-abelian locally compact groups ([**Haa78b**], [**Haa78a**, Definition 3.1]).

We note that the fact proven below that $\psi \mapsto \widetilde{\psi}$ is a bijection on the full set of normal weights, does not seem to be recorded in the literature. (See the remark at the start of this chapter.)

THEOREM 6.55. *The mapping $\psi \mapsto \widetilde{\psi}$ is a bijection between the set of all normal weights on $\mathcal{M}$, and the set of normal weights on $\mathcal{M} \rtimes_\alpha G$ which are $\widehat{\alpha}$-invariant in the sense that $\widetilde{\psi} \circ \widehat{\alpha}_\gamma = \widetilde{\psi}$ for all $\gamma \in \widehat{G}$.*

For any two normal weights ψ_1 and ψ_2 on $\mathcal{M}$, and any $a \in \mathcal{M}$, we moreover have that

    (i) *$\psi_1 \leq \psi_2$ if and only if $\widetilde{\psi}_1 \leq \widetilde{\psi}_2$, and in addition $\psi_\alpha \nearrow \psi$ if and only if $\widetilde{\psi}_\alpha \nearrow \widetilde{\psi}$,*

    (ii) *$\widetilde{(\psi_1 + \psi_2)} = \widetilde{\psi_1} + \widetilde{\psi_2}$,*

    (iii) *$\pi(a^*).\widetilde{\psi}.\pi(a) = \widetilde{a^*.\psi.a}$,*

    (iv) *$e_0(\widetilde{\psi}) = \pi(e_0(\psi))$,*

    (v) *$e_\infty(\widetilde{\psi}) = \pi(e_\infty(\psi))$,*

It follows that ψ is faithful (respectively semifinite) if and only if $\widetilde{\psi}$ is. Hence the map $\psi \mapsto \widetilde{\psi}$ restricts to a bijection between the set of all normal semifinite weights on $\mathcal{M}$, and the set of normal semifinite $\widehat{\alpha}$-invariant weights on $\mathcal{M} \rtimes_\alpha G$.

The following proof is based on the argument presented in [**Ter81**, Lemma II.1].

PROOF. For the sake of simplicity, we will in the proof identify $\mathcal{M}$ with $\pi(\mathcal{M})$, and write $\mathscr{W}$ for $\mathscr{W}_G$ where convenient.

Firstly note that if ψ is normal, then so is its extension $\widehat{\psi}$ to the extended positive part of $\mathcal{M}$. The normality of $\mathscr{W}_G$ then ensures that $\widetilde{\psi} = \widehat{\psi} \circ \mathscr{W}_G$ is indeed normal. In addition the fact that $\mathscr{W}_G$ is $\widehat{\alpha}$-invariant, clearly ensures that the same is true of $\widetilde{\psi}$.

Since $\mathscr{W}_G$ maps onto $\widehat{\mathcal{M}}_+$, it is clear that $\psi_1 \leq \psi_2 \Leftrightarrow \widehat{\psi}_1 \leq \widehat{\psi}_2 \Leftrightarrow \widetilde{\psi}_1 \leq \widetilde{\psi}_2$ and similarly that $\psi_\alpha \nearrow \psi \Leftrightarrow \widehat{\psi}_\alpha \nearrow \widehat{\psi} \Leftrightarrow \widetilde{\psi}_\alpha \nearrow \widetilde{\psi}$. Hence (i) follows.

Given two normal semifinite weights ψ_1, ψ_2 on $\mathcal{M}$, it easily follows from the basic properties of the extension of a weight to the extended

positive part, that $\widehat{\psi_1 + \psi_2} = \widehat{\psi_1} + \widehat{\psi_2}$ (see [**Haa79b**, Proposition 1.10]). The validity of (ii) then follows by definition.

Let $a \in \mathcal{M}$ be given. Recall that for any m in the extended positive part $\widehat{M}_+$ of $\mathcal{M}$, a^*ma is defined to be that member of $\widehat{M}_+$ which maps a positive linear functional ρ on $\mathcal{M}$, onto $m(a^*\rho a)$, where $a^*\rho a$ is the positive linear functional given by $\rho(a \cdot a^*)$. Given a normal semifinite weight ψ on $\mathcal{M}$, we may then define $a^*\widehat{\psi}a$ to be that map on $\widehat{M}_+$ which maps each $m \in \widehat{M}_+$ onto $\widehat{\psi}(a^*ma)$. On once again considering the basic properties of the extension of a weight to the extended positive part, careful checking now reveals that $a^*\widehat{\psi}a = \widehat{a^*\psi a}$. If we combine this fact with the observation that $\mathscr{W}_G(a^*fa) = a^*\mathscr{W}_G(f)a$ for any $f \in (\mathcal{M} \rtimes_\alpha G)_+$, the validity of (iii) then follows by definition.

Let $p_0 \in \mathcal{M}$ and $q_0 \in \mathcal{M} \rtimes_\alpha G$ be those projections for which $\mathcal{M}p_0 = N_\psi = \{a \in \mathcal{M} \colon \psi(a^*a) = 0\}$ and $(\mathcal{M} \rtimes_\alpha G)q_0 = N_{\widetilde{\psi}} = \{f \in (\mathcal{M} \rtimes_\alpha G) \colon \widetilde{\psi}(f^*f) = 0\}$. We first show that in fact $q_0 \in \mathcal{M}$. Since $\widetilde{\psi}$ is $\widehat{\alpha}$-invariant, it easily follows that $f \in N_{\widetilde{\psi}}$ if and only if for any $\gamma \in \widehat{G}$, we have that $\widehat{\alpha}_\gamma(f) \in N_{\widetilde{\psi}}$. That means that $(\mathcal{M} \rtimes_\alpha G)q_0 = N_{\widetilde{\psi}}$ is $\widehat{\alpha}$-invariant, which can only be the case if q_0 itself is $\widehat{\alpha}$-invariant. But by Theorem 6.50, we then have that $q_0 \in \mathcal{M}$.

We next claim that $\mathfrak{n}_\mathscr{W} N_\psi \subseteq N_{\widetilde{\psi}}$. To see this observe that for any $a \in N_\psi$ and any $f \in \mathfrak{n}_\mathscr{W}$, we have that $\widetilde{\psi}(a^*f^*fa^*) = \widehat{\psi}(\mathscr{W}_G(a^*f^*fa)) = \widehat{\psi}(a^*\mathscr{W}_G(f^*f)a) \leq \|\mathscr{W}_G(f^*f)\|\widehat{\psi}(a^*a) = 0$. Since $\mathfrak{n}_\mathscr{W}$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$ and $N_\psi = \mathcal{M}p_0$, we therefore have that $\overline{\mathfrak{n}_\mathscr{W} N_\psi}^{w^*} = (\mathcal{M} \rtimes_\alpha G)p_0 \subseteq (\mathcal{M} \rtimes_\alpha G)q_0$. This can of course only be the case if $p_0 \leq q_0$.

Since $q_0 \in \mathcal{M}$, we need only show that $\psi(q_0) = 0$ (equivalently $q_0 \in N_\psi$), to see that equality holds. Recall that by definition $\psi(\dot{\mathscr{W}}_G(f)) = \widetilde{\psi}(f)$ for all $f \in \mathfrak{m}_\mathscr{W}$. Let $a, b \in \mathfrak{n}_\mathscr{W}$ be given. For any $k \in \{0, 1, 2, 3\}$, we will therefore have that

$$0 = \widetilde{\psi}(q_0(a + i^k b)^*(a + i^k b)q_0) = \psi(\dot{\mathscr{W}}_G(q_0(a + i^k b)^*(a + i^k b)q_0))$$
$$= \psi(q_0 \dot{\mathscr{W}}_G((a + i^k b)^*(a + i^k b))q_0).$$

Therefore $q_0 \dot{\mathscr{W}}_G((a+i^k b)^*(a+i^k b))q_0 \in N_\psi^* N_\psi$ for each $k \in \{0, 1, 2, 3\}$. We may now use the identity $q_0 b^*aq_0 = \frac{1}{4}\sum_{k=0}^3 q_0(a+i^k b)^*(a+i^k b)q_0$, to conclude that in fact $q_0 \dot{\mathscr{W}}_G(b^*a)q_0 \in N_\psi^* N_\psi$, and hence that $q_0 \dot{\mathscr{W}}_G(\mathfrak{m}_\mathscr{W})q_0 \subseteq N_\psi^* N_\psi$. Since $\dot{\mathscr{W}}_G(\mathfrak{m}_\mathscr{W})$ is σ-weakly dense in $\mathcal{M}$, it follows the fact that

$N_\psi = \mathcal{M}p_0$, that the above inclusion can only hold if $q_0\mathcal{M}q_0 \subseteq p_0\mathcal{M}p_0$. Therefore $q_0 \leq p_0$, and hence equality holds. this proves (iv).

With regard to (v), we will first show that $e_\infty(\psi) \leq e_\infty(\widetilde{\psi})$, and then prove that equality holds. To be able to do this we first need to show that if ψ is semifinite, then so is $\widetilde{\psi}$. So let ψ be a given normal semifinite weight on $\mathcal{M}$. The semifiniteness of $\mathscr{W}_G$ ensures that $\mathfrak{n}_\mathscr{W} = \{a \in \mathcal{M} \rtimes_\alpha G : \mathscr{W}_G(a^*a) \in \mathcal{M}\}$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$. By assumption $\mathfrak{n}_\psi = \{f \in \mathcal{M} : \psi(f^*f) < \infty\}$ is σ-weakly dense in $\mathcal{M}$. Hence we may select a net $(f_i) \subseteq \mathfrak{n}_\psi$ which is σ-weakly convergent to $\mathbb{1}$. For any $a \in \mathfrak{n}_\mathscr{W}$, the net (af_i) will then be σ-weakly convergent to a. In addition

$$\begin{aligned}
\widetilde{\psi}((af_i)^*(af_i)) &= \psi(\mathscr{W}_G(f_i^*a^*af_i)) \\
&= \psi(f_i^*\mathscr{W}_G(a^*a)f_i) \\
&\leq \|\mathscr{W}_G(a^*a)\|\psi(f_i^*f_i) < \infty.
\end{aligned}$$

In other words $(af_i) \subseteq \mathfrak{n}_{\widetilde{\psi}}$. But then $\mathfrak{n}_{\widetilde{\psi}}$ must be σ-weakly dense in $\mathfrak{n}_\mathscr{W}$, which in turn we know to be σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$. Hence $\mathfrak{n}_{\widetilde{\psi}}$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$. Thus if ψ is semifinite, then so is $\widetilde{\psi}$.

Returning to the general case, recall that for any normal weight ψ, the weight $e_\infty(\psi).\psi.e_\infty(\psi)$ is semifinite. It now follows from part (iii) and what we have just shown, that $e_\infty(\psi).\widetilde{\psi}.e_\infty(\psi)$ will then be semifinite. Writing e_∞ for $e_\infty(\psi)$, this means that $\mathfrak{n}_{e_\infty.\widetilde{\psi}.e_\infty}$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$. It is now an easy exercise to see that $\mathfrak{n}_{e_\infty.\widetilde{\psi}.e_\infty} e_\infty \subseteq \mathfrak{n}_{\widetilde{\psi}}$. On taking the σ-weak closure of both sides of this inclusion, we obtain that $(\mathcal{M} \rtimes_\alpha G)e_\infty \subseteq (\mathcal{M} \rtimes_\alpha G)e_\infty(\widetilde{\psi})$, which can only be the case if $e_\infty(\psi) \leq e_\infty(\widetilde{\psi})$.

We pass to showing that equality holds. Let $f \in \mathfrak{n}_\psi$ be given. For ease of notation write e_0 for $e_0(\psi)$. Observe that as a member of $\widehat{\mathcal{M}}_+$, $(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)$ has a spectral resolution of the form

$$(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0) = \int_0^\infty \lambda \, de_\lambda + \infty.p$$

where each e_λ is orthogonal to p. Since $(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)$ is "supported" on $\mathbb{1} - e_0$, we in particular also have that $p \leq (\mathbb{1} - e_0)$.

Since ψ is faithful on $(\mathbb{1} - e_0)\mathcal{M}(\mathbb{1} - e_0)$, and since

$$\psi((\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)) = \psi(\mathscr{W}_G(f^*f)) = \widetilde{\psi}(f^*f) < \infty,$$

we must have that $\psi(p) = 0$. Given that ψ is faithful on $(\mathbb{1} - e_0)\mathcal{M}(\mathbb{1} - e_0)$, this ensures that that $p = 0$. But then $(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)$ is a densely-defined operator affiliated to $\mathcal{M}$ with spectral resolution $(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0) = \int_0^\infty \lambda\, de_\lambda$. Setting $e_n = \chi_{[0,n]}((\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0))$ we will for each $n \in \mathbb{N}$ then have that $e_n(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)e_n = \int_0^n \lambda\, de_\lambda \in \mathcal{M}$ with in addition

$$
\begin{aligned}
\psi(e_n(\mathbb{1} - q_0)\mathscr{W}_G(f^*f)(\mathbb{1} - q_0)e_n) &\leq \psi((\mathbb{1} - q_0)\mathscr{W}_G(f^*f)(\mathbb{1} - q_0)) \\
&= \widehat{\psi}(\mathscr{W}_G(f^*f)) = \widetilde{\psi}(f^*f) < \infty,
\end{aligned}
$$

where we used the fact that $e_n(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)e_n = \int_0^n \lambda\, de_\lambda \leq \int_0^\infty \lambda\, de_\lambda = (\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)$. Hence $e_n(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)e_n \in \mathfrak{m}_\psi \subseteq e_\infty \mathcal{M} e_\infty$ for each $n \in \mathbb{N}$. If now we let $n \to \infty$, it will follow that $(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0) \,\eta\, e_\infty \mathcal{M} e_\infty$. Recall that $e_0 \leq e_\infty$, and that e_∞ and e_0 must therefore commute. For any $f \in \mathfrak{n}_\psi$ we will then have that

$$
\begin{aligned}
\widetilde{\psi}(f^*f) = \widehat{\psi}(\mathscr{W}_G(f^*f)) &= \widehat{\psi}(e_\infty(\mathbb{1} - e_0)\mathscr{W}_G(f^*f)(\mathbb{1} - e_0)e_\infty) \\
&= \widehat{\psi}(\mathscr{W}_G(e_\infty(\mathbb{1} - e_0)f^*f(\mathbb{1} - e_0)e_\infty)) \\
&= \widetilde{\psi}(e_\infty(\mathbb{1} - e_0)f^*f(\mathbb{1} - e_0)e_\infty) \\
&= \widetilde{\psi}((\mathbb{1} - e_0)e_\infty f^*f e_\infty(\mathbb{1} - e_0)) \\
&= \widetilde{\psi}(e_\infty f^*f e_\infty).
\end{aligned}
$$

Recall that $\mathfrak{m}_{\widetilde{\psi}} = \operatorname{span}\{g^*f \colon f, g \in \mathfrak{n}_{\widetilde{\psi}}\}$, and that $g^*f = \frac{1}{4}\sum_{k=0}^3 (f + i^k g)^*(f + i^k g)$ for any $f, g \in \mathfrak{n}_{\widetilde{\psi}}$. The equalities displayed above therefore shows that $\widetilde{\psi}$ and $e_\infty \widetilde{\psi} e_\infty$ agree on $\mathfrak{m}_{\widetilde{\psi}}$, and hence also on $\overline{\mathfrak{m}_{\widetilde{\psi}}}^{w^*} = e_\infty(\widetilde{\psi})(\mathcal{M} \rtimes_\alpha G)e_\infty(\widetilde{\psi})$. But since $e_\infty = e_\infty(\psi) \leq e_\infty(\widetilde{\psi})$, we must then have that

$$
\begin{aligned}
0 = (e_\infty.\widetilde{\psi}.e_\infty)(e_\infty(\widetilde{\psi}) - e_\infty) &= \widetilde{\psi}(e_\infty(\widetilde{\psi}) - e_\infty) \\
&= \widetilde{\psi}((\mathbb{1} - e_0)(e_\infty(\widetilde{\psi}) - e_\infty)(\mathbb{1} - e_0)).
\end{aligned}
$$

(Here we used the fact that $\widetilde{\psi}(x) = \widetilde{\psi}((\mathbb{1} - e_0)x(\mathbb{1} - e_0))$ for any $x \in (\mathcal{M} \rtimes_\alpha G)_+$.) Since $\widetilde{\psi}$ is faithful on $(\mathbb{1} - e_0)(\mathcal{M} \rtimes_\alpha G)(\mathbb{1} - e_0)$, it follows that $0 = (\mathbb{1} - e_0)(e_\infty(\widetilde{\psi}) - e_\infty)(\mathbb{1} - e_0)$. Given that $e_0 = e_0(\widetilde{\psi}) \leq e_\infty = e_\infty(\psi) \leq e_\infty(\widetilde{\psi})$, it now easily follows that $(\mathbb{1} - e_0)(e_\infty(\widetilde{\psi}) - e_\infty)(\mathbb{1} - e_0) = e_\infty(\widetilde{\psi}) - e_\infty$, and hence that $e_\infty(\widetilde{\psi}) = e_\infty = e_\infty(\psi)$. Thus the validity of (v) is established.

Since a normal weight ψ is semifinite if and only if $e_\infty(\psi) = \mathbb{1}$ and faithful if and only if $e_0(\psi) = 0$ (and similarly for $\widetilde{\psi}$), it is now clear that ψ is faithful (respectively semifinite) if and only if $\widetilde{\psi}$ is.

We show that the map $\psi \mapsto \widetilde{\psi}$ is injective. Hence suppose that for normal weights ψ_1 and ψ_2 on $\mathcal{M}$, we have that $\widetilde{\psi}_1 = \widetilde{\psi}_2$. These weights extend uniquely to weights $\widehat{\widetilde{\psi}}_1 = \widehat{\widetilde{\psi}}_2$ on $(\widehat{M \rtimes_\alpha G})_+$. This uniqueness of course ensures that these extensions must be of the form

$$\widehat{\widetilde{\psi}}_1 = \widehat{\psi}_1 \circ \widehat{\pi^{-1}} \circ \widehat{\mathscr{W}_G} \quad \widehat{\widetilde{\psi}}_2 = \widehat{\psi}_2 \circ \widehat{\pi^{-1}} \circ \widehat{\mathscr{W}_G}$$

where $\widehat{\mathscr{W}_G}$ is the unique extension of $\mathscr{W}_G$ to all of $(\widehat{M \rtimes_\alpha G})_+$. Recall that the normality of $\mathscr{W}_G$ ensures that $\widehat{\mathscr{W}_G}$ actually maps onto $\widehat{\mathcal{M}}_+$ (see Remark 6.33). So given $x \in \mathcal{M}_+$, there must exist some $m_x \in (\widehat{M \rtimes_\alpha G})_+$ such that $x = \widehat{\mathscr{W}_G}(m_x)$. By the previously displayed formulas we will then have that $\psi_1(x) = \widehat{\widetilde{\psi}}_1(m_x) = \widehat{\widetilde{\psi}}_2(m_x) = \psi_2(x)$. We therefore have that $\psi_1 = \psi_2$ as was required.

We need some additional technology before we are able to prove the claim regarding surjectivity of the map $\psi \mapsto \widetilde{\psi}$. We therefore defer the proof of surjectivity until after the requisite technology has been developed. $\qquad\square$

Some conceptual background is necessary to understand the thrust of the proof of the next theorem. For that reason we discuss the issue of generators of groups of isometries on von Neumann algebras. The fastidious reader may find details and proofs of the claims made below in [**CZ76**]. (In particular note [**CZ76**, Theorems 2.4 & 4.4].) Let $t \mapsto \sigma_t$ be a σ-weakly continuous group of isometries on some von Neumann algebra $\mathcal{R}$. For any $w \in \mathbb{C}$ with $\mathrm{Im}(w) > 0$, we may then define $D(\sigma_w)$ to be the set of all $a \in \mathcal{R}$ for which the map $t \mapsto \sigma_t(a)$ $(t \in \mathbb{R})$, may be extended to a σ-weakly continuous function f_w on the strip $\{z \in \mathbb{C} \colon 0 \leq \mathrm{Im}(z) \leq \mathrm{Im}(w)\}$ which is analytic on the interior of that strip. We then define σ_w on $D(\sigma_w)$ by setting $\sigma_w(a) = f_w(a)$ for any $a \in D(\sigma_w)$. In the case where $\mathrm{Im}(w) < 0$, the map σ_w is defined similarly using the strip $\{z \in \mathbb{C} \colon 0 \geq \mathrm{Im}(z) \geq \mathrm{Im}(w)\}$. The maps σ_w can then all be shown to be σ-weakly closed and σ-weakly densely defined linear operators on $\mathcal{R}$, for which we have that $\sigma_{w_1}\sigma_{w_2} \subseteq \sigma_{w_1+w_2}$ for all $w_1, w_2 \in \mathbb{C}$, with equality holding whenever $\mathrm{Im}(w_1) \cdot \mathrm{Im}(w_2) \geq 0$. In particular $\sigma_{-w} = \sigma_w^{-1}$. Among these maps, σ_{-i} plays a particularly crucial role, and is referred to as the analytic generator of the group. As we shall

shortly see the crucial aspect regarding this map is that two such groups coincide if their analytic generators coincide. We briefly present some basic technical facts regarding such groups before passing to the proof of the theorem of interest.

The first fact we need is a criterion describing membership of $D(\sigma^\varphi_{-i/2})$ where (σ^φ) is the modular automorphism group of φ.

LEMMA 6.56 ([**Haa79b**, Lemma 3.3]/[**Tak03a**, VIII.3.18(1)]). *Let φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$. For any $a \in \mathcal{M}$ and $k \geq 0$ the following are equivalent:*

- $\varphi(a \cdot a^*) \leq k^2 \varphi$;
- $a \in D(\sigma^\varphi_{-i/2})$ *and* $\|\sigma^\varphi_{-i/2}(a)\| \leq k$.

We also need criteria which ensure the agreement of two such groups.

PROPOSITION 6.57 ([**Tak03a**, VIII.3.24]/[**Haa79b**, Lemma 4.4]). *Let $\mathcal{M}$, $\mathcal{M}_0$ be von Neumann algebras, with $\mathcal{M}_0$ a von Neumann subalgebra of $\mathcal{M}$. Let σ_t and $\sigma^{(0)}_t$ (where $t \in \mathbb{R}$), be σ-weakly continuous one-parameter groups of isometries on $\mathcal{M}$ and $\mathcal{M}_0$ respectively. If $\sigma_{-i} \subseteq \sigma^{(0)}_{-i}$, then $\sigma^{(0)}_t(a) = \sigma_t(a)$ for all $a \in \mathcal{M}_0$ and all $t \in \mathbb{R}$.*

The above result may now be used to prove the following useful fact.

THEOREM 6.58 ([**Tak03a**, VIII.3.25]). *Let φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$. For any $a, b \in \mathcal{M}$, the following are equivalent:*

- (a, b) *belongs to the graph $\mathcal{G}(\sigma^\varphi_{-i})$, that is we have $a \in D(\sigma^\varphi_{-i})$ with $\sigma^\varphi_{-i}(a) = b$;*
- $a\mathfrak{n}^*_\varphi \subseteq \mathfrak{n}^*_\varphi$, $\mathfrak{n}_\varphi b \subseteq \mathfrak{n}_\varphi$, *and* $\varphi(ax) = \varphi(xb)$ *for all* $x \in \mathfrak{m}_\varphi$.

The proof of the following theorem is due to Haagerup. The proofs of the two bullets are respectively taken from [**Haa79b**] and [**Haa78a**].

THEOREM 6.59. *For any faithful normal semifinite weight ψ on $\mathcal{M}$, the action of $\sigma^{\widetilde{\psi}}_t$ on $\mathcal{M} \rtimes_\alpha G$ is uniquely determined by the prescriptions*

- $\sigma^{\widetilde{\psi}}_t(\pi(a)) = \pi(\sigma^\psi_t(a))$ *for all $a \in \mathcal{M}$ and all $t \in \mathbb{R}$;*
- $\sigma^{\widetilde{\psi}}_t(\lambda_g) = \lambda_g(D\widetilde{\psi \circ \alpha_g} : D\widetilde{\psi})_t$ *for all $g \in G$, with $\sigma^{\widetilde{\psi}}_t(\lambda_g) = \lambda_g$ if ψ is α_g invariant.*

PROOF. For the sake of simplicity we identify $\mathcal{M}$ and $\pi(\mathcal{M})$. We first prove the second bullet. Given any $f \in \mathcal{M} \rtimes_\alpha G$ and any $\xi \in L^2(G, H)$, it follows from the definition of the dual action $\widehat{\alpha}$, that $\widehat{\alpha}_\gamma(\lambda_g f \lambda_g^*)\xi(s) = \lambda_g \widehat{\alpha}_\gamma(f)\lambda_g^* \xi(s)$ for every $\gamma \in \widehat{G}$, and all $s, g \in G$ (see equation (6.1)).

For any $f \in (\mathcal{M} \rtimes_\alpha G)_+$, we may now apply Proposition 6.40 to see that

$$\mathscr{W}_G(\lambda_g f \lambda_g^*) = \int_{\widehat{G}} \widehat{\alpha}_\gamma(\lambda_g f \lambda_g^*)\, d\gamma = \lambda_g \int_{\widehat{G}} \widehat{\alpha}_\gamma(f)\, d\gamma \lambda_g^*$$
$$= \lambda_g \mathscr{W}_G(f)\lambda_g^* = \sigma_g(\mathscr{W}_G(f)).$$

But then $\widetilde{\psi}(\lambda_g f \lambda_g^*) = \widehat{\psi}(\mathscr{W}_G(\lambda_g f \lambda_g^*)) = \widehat{\psi}(\alpha_g(\mathscr{W}_G(f))) = \widetilde{\psi \circ \alpha_g}(f)$. The claim now follows from Lemma 6.23.

We pass to proving the first bullet. By Proposition 6.57, it will be enough to show that in their action on $\mathcal{M}$, we have that $\sigma_{-i}^\psi \subseteq \sigma_{-i}^{\widetilde{\psi}}$. In principle, we therefore need to show that if $(a, b) \in \mathcal{G}(\sigma_{-i}^\psi)$ (the graph of σ_{-i}^ψ), then $(a, b) \in \mathcal{G}(\sigma_{-i}^{\widetilde{\psi}})$. So let $(a, b) \in \mathcal{G}(\sigma_{-i}^\psi)$ be given. By the discussion preceding the theorem, we then have that $a \in D(\sigma_{-i}^\psi) \subseteq D(\sigma_{-i/2}^\psi)$, and $b \in D(\sigma_i^\psi) \subseteq D(\sigma_{i/2}^\psi)$. (The claim about b follows since σ_i^ψ is the inverse of σ_{-i}^ψ.) Since for any $t \in \mathbb{R}$ we have that $\sigma_t^\psi(b^*) = \sigma_t^\psi(b)^*$, careful checking shows that this ensures that then $b^* \in D(\sigma_{-i/2}^\psi)$. By Lemma 6.56, we then have that there exists some $k > 0$ such that

$$\psi(axa^*) \le k^2 \psi(x) \text{ and } \psi(b^*xb) \le k^2 \psi(x) \text{ for every } x \in \mathcal{M}_+.$$

Any element of the extended positive part of $\mathcal{M}$ may be written as the limit of an increasing net in $\mathcal{M}_+$. Since the extension $\widehat{\psi}$ of ψ to $\widehat{\mathcal{M}}_+$ is normal, it therefore follows that

$$\widehat{\psi}(ama^*) \le k^2 \widehat{\psi}(m) \text{ and } \widehat{\psi}(b^*mb) \le k^2 \widehat{\psi}(m) \text{ for every } m \in \widehat{\mathcal{M}}_+.$$

On using the fact that for any $x \in (\mathcal{M} \rtimes_\alpha G)_+$ we have that $\mathscr{W}_G(axa^*) = a\mathscr{W}_G(x)a^*$ and $\mathscr{W}_G(b^*xb) = b^*\mathscr{W}_G(x)b$, it is then clear that

$$\widetilde{\psi}(axa^*) \le k^2 \widetilde{\psi}(x) \text{ and } \widetilde{\psi}(b^*xb) \le k^2 \widetilde{\psi}(x) \text{ for every } x \in (\mathcal{M} \rtimes_\alpha G)_+.$$
$$(6.2)$$

(Note for example that

$$\widetilde{\psi}(axa^*) = \widehat{\psi}(\mathscr{W}_G(axa^*)) = \widehat{\psi}(a\mathscr{W}_G(x)a^*) \le k^2 \widehat{\psi}(\mathscr{W}_G(x)) = k^2 \widetilde{\psi}(x)$$

when $x \in (\mathcal{M} \rtimes_\alpha G)_+$.) It now trivially follows from the above inequalities, that $\mathfrak{n}_{\widetilde{\psi}} a^* \subseteq \mathfrak{n}_{\widetilde{\psi}}$, and $\mathfrak{n}_{\widetilde{\psi}} b \subseteq \mathfrak{n}_{\widetilde{\psi}}$. So if we can show that

$$\widetilde{\psi}(ax) = \widetilde{\psi}(xb) \text{ for all } x \in \mathfrak{m}_{\widetilde{\psi}}, \tag{6.3}$$

we would by Theorem 6.58, then have that $(a, b) \in \mathcal{G}(\sigma_{-i}^{\widetilde{\psi}})$, which would prove the theorem. So it remains to show that equation (6.3) holds. We first show that equation (6.3) holds for all x in the subspace $(\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})^*(\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}}) = \mathrm{span}\{g^*f \colon f, g \in (\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})\}$. The first step in doing this is showing that $\dot{\mathscr{W}}_G$ maps $(\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})^*(\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})$ into $\mathfrak{n}_{\psi}^* \mathfrak{n}_{\psi} = \mathfrak{m}_{\psi}$. To see this let $f, g \in (\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})$ be given. For any $k \in \{0, 1, 2, 3\}$ we then have that

$$\psi(\dot{\mathscr{W}}_G((f + i^k g)^*(f + i^k g))) = \widetilde{\psi}((f + i^k g)^*(f + i^k g)) < \infty,$$

and hence that $\dot{\mathscr{W}}_G((f + i^k g)^*(f + i^k g)) \in \mathfrak{m}_{\psi}$. Since $g^*f = \frac{1}{4}\sum_{k=0}^{3}(f + i^k g)^*(f + i^k g)$, the same is then true of $\dot{\mathscr{W}}_G(g^*g)$, which proves the claim. Using what we know about ψ, we therefore have that

$$\begin{aligned} \widetilde{\psi}(ax_0) &= \psi(\dot{\mathscr{W}}_G(ax_0)) = \psi(a\dot{\mathscr{W}}_G(x_0)) \\ &= \psi(\dot{\mathscr{W}}_G(x_0)b) = \psi(\dot{\mathscr{W}}_G(x_0 b)) \\ &= \widetilde{\psi}(x_0 b) \end{aligned} \tag{6.4}$$

for all $x_0 \in (\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})^*(\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})$. We will use this equality to prove equation (6.3). Observe that since $\mathfrak{m}_{\widetilde{\psi}} = \mathrm{span}\{g^*f \colon f, g \in \mathfrak{n}_{\widetilde{\psi}}\}$, and since $g^*f = \frac{1}{4}\sum_{k=0}^{3}(f + i^k g)^*(f + i^k g)$ for any $f, g \in \mathfrak{n}_{\widetilde{\psi}}$, it is enough to prove that equation (6.3) holds for terms of the form $x = f^*f$ where $f \in \mathfrak{n}_{\widetilde{\psi}}$.

Let such an f be given. Recall that as a member of $\widehat{\mathcal{M}}_+$, $\mathscr{W}_G(f^*f)$ has a spectral resolution of the form $\mathscr{W}_G(f^*f) = \int_0^\infty \lambda\, de_\lambda + \infty.p$ where each e_λ is orthogonal to p. Since $\widehat{\psi}(\mathscr{W}_G(f^*f)) = \widetilde{\psi}(f^*f) < \infty$, we must have that $\widehat{\psi}(p) = 0$ and hence that $p = 0$. But then $\mathscr{W}_G(f^*f)$ is a densely-defined operator affiliated to $\mathcal{M}$. For each $n \in \mathbb{N}$ we then have that

$$\mathscr{W}_G(e_n f^* f e_n) = e_n \mathscr{W}_G(f^*f)e_n = \int_0^n \lambda\, de_\lambda \in \mathcal{M},$$

with in addition

$$\begin{aligned} \widetilde{\psi}(e_n f^* f e_n) &= \widehat{\psi}(\mathscr{W}_G(e_n f^* f e_n)) = \widehat{\psi}(e_n \mathscr{W}_G(f^*f)e_n) \\ &\leq \widehat{\psi}(\mathscr{W}_G(f^*f)) = \widetilde{\psi}(f^*f) < \infty \end{aligned}$$

since $e_n \mathscr{W}_G(f^*f)e_n \leq \mathscr{W}_G(f^*f)$. Hence $fe_n \in (\mathfrak{n}_{\widetilde{\psi}} \cap \mathfrak{n}_{\mathscr{W}})$ for each $n \in \mathbb{N}$. Also notice that by the normality of ψ, $\widetilde{\psi}(e_n f^* f e_n) = \psi(e_n \mathscr{W}_G(f^*f)e_n) = \psi(\int_0^n \lambda\, de_\lambda)$ increases to $\widetilde{\psi}(f^*f) = \psi(\mathscr{W}_G(f^*f)) = \psi(\int_0^\infty \lambda\, de_\lambda)$ as $n \nearrow \infty$. Since $\psi(\int_0^\infty \lambda\, de_\lambda) < \infty$, this ensures that

$$
\begin{aligned}
\widetilde{\psi}((\mathbb{1} - e_n)f^* f(\mathbb{1} - e_n)) &= \psi(\mathscr{W}_G((\mathbb{1} - e_n)f^* f(\mathbb{1} - e_n))) \\
&= \psi((\mathbb{1} - e_n)\mathscr{W}_G(f^*f)(\mathbb{1} - e_n)) \\
&= \psi\Big(\int_n^\infty \lambda\, de_\lambda\Big) \to 0 \text{ as } n \nearrow \infty.
\end{aligned}
$$

We now use this fact to show that $\lim_{n\to\infty} \widetilde{\psi}(ae_n f^* f e_n) = \widetilde{\psi}(af^*f)$, and $\lim_{n\to\infty} \widetilde{\psi}(e_n f^* f e_n b) = \widetilde{\psi}(f^*fb)$.

For any $n \in \mathbb{N}$ we may write

$$
\widetilde{\psi}(af^*f - ae_n f^* f e_n) = \widetilde{\psi}(a(\mathbb{1} - e_n)f^*f) + \widetilde{\psi}(ae_n f^* f(\mathbb{1} - e_n)).
$$

By the Cauchy-Schwarz inequality for $\widetilde{\psi}$ on $\mathfrak{m}_{\widetilde{\psi}}$ and equation (6.2), we have that

$$
\begin{aligned}
|\widetilde{\psi}(a(\mathbb{1} - e_n)f^*f)| &\leq \widetilde{\psi}(a(\mathbb{1} - e_n)f^* f(\mathbb{1} - e_n)a^*)^{1/2}\widetilde{\psi}(f^*f)^{1/2} \\
&\leq k\widetilde{\psi}((\mathbb{1} - e_n)f^* f(\mathbb{1} - e_n))^{1/2}\widetilde{\psi}(f^*f)^{1/2}.
\end{aligned}
$$

This clearly ensures that $\widetilde{\psi}(a(\mathbb{1} - e_n)f^*f) \to 0$ as $n \to \infty$. An entirely similar proof shows that $\widetilde{\psi}(ae_n f^* f(\mathbb{1} - e_n))$ also tends to 0 as $n \to \infty$. This then ensures that $\lim_{n\to\infty} \widetilde{\psi}(ae_n f^* f e_n) = \widetilde{\psi}(af^*f)$, as required. To prove the second limit formula, we write $\widetilde{\psi}(f^*fb - e_n f^* f e_n b) = \widetilde{\psi}((\mathbb{1} - e_n)f^*fb) + \widetilde{\psi}(be_n f^* f(\mathbb{1} - e_n)b)$, and argue along similar lines.

By equation (6.4), we have that $\widetilde{\psi}(ae_n f^* f e_n) = \widetilde{\psi}(e_n f^* f e_n b)$ for every $n \in \mathbb{N}$. If we consider this fact alongside the limit formulae we have just proven, we have that $\widetilde{\psi}(af^*f) = \widetilde{\psi}(f^*fb)$ as required. $\qquad\square$

Given two faithful normal semifinite weights ψ_1 and ψ_2 on $\mathcal{M}$, our next result compares the Connes cocycle derivatives of the pair (ψ_1, ψ_2) to that of the pair $(\widetilde{\psi}, \widetilde{\psi}_2)$.

THEOREM 6.60. *For any two faithful normal semifinite weights ψ_1 and ψ_2 on $\mathcal{M}$, we have that $(D\widetilde{\psi}_1 : D\widetilde{\psi}_2)_t = \pi((D\psi_1 : D\psi_2)_t)$ for every $t \in \mathbb{R}$*

Haagerup provided two proofs of this fact — one in [**Haa78a**], and the other in [**Haa79b**]. We outline a modified version of the first proof.

PROOF. The proof relies on tricks which are fairly standard in the theory of Connes cocycle derivatives. However although standard, this theory is somewhat outside the scope of this manuscript. We therefore provide only an outline of the proof. The interested reader may find details of these tricks in §VIII.3 of [**Tak03a**], and also in [**Haa78a**].

Let M_2 be the 2×2 matrices over $\mathbb{C}$, and let $\{e_{i,j} : 1 \leq i, j \leq 2\}$ be the standard basis for M_2. Now consider $\mathcal{M} \overline{\otimes} M_2$. The action α of G on $\mathcal{M}$, then lifts to an action $\beta = \alpha \otimes \mathrm{id}$ of G on $\mathcal{M} \overline{\otimes} M_2$ (where id is the identity operator on M_2). One may then further show that

$$(\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G = (\mathcal{M} \rtimes_\alpha G) \overline{\otimes} M_2.$$

The next step is to define a weight ψ on $(\mathcal{M} \overline{\otimes} M_2)_+$ by setting

$$\psi \left(\left[\begin{array}{cc} x_{11} & x_{1,2} \\ x_{21} & x_{22} \end{array} \right] \right) = \psi_1(x_{11}) + \psi_2(x_{22}) \text{ for any } [x_{ij}] \in (\mathcal{M} \overline{\otimes} M_2)_+.$$

Next note that by Theorem 6.35, the canonical operator-valued weight on $((\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G)_+$ is just $\mathcal{W}_G \otimes \mathrm{id}$. One may then use this fact to show that the dual weight $\widetilde{\psi}$ on $((\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G)_+$ is of the form

$$\widetilde{\psi} \left(\left[\begin{array}{cc} y_{11} & y_{1,2} \\ y_{21} & y_{22} \end{array} \right] \right) = \widetilde{\psi}_1(y_{11}) + \widetilde{\psi}_2(y_{22}) \text{ for any } [y_{ij}] \in ((\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G)_+.$$

As we have done before many times, we will in the rest of the proof again identify $\mathcal{M}$ with $\pi(\mathcal{M})$ to simplify notation. Having done this, we next observe that the one parameter modular automorphism group induced by $\widetilde{\psi}$ on $(\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G$ is of the form

$$\sigma_t^{\widetilde{\psi}} \left(\left[\begin{array}{cc} y_{11} & y_{1,2} \\ y_{21} & y_{22} \end{array} \right] \right) = \left(\left[\begin{array}{cc} \sigma_t^{\widetilde{\psi}_1}(y_{11}) & \sigma_t^{\widetilde{\psi}_1, \widetilde{\psi}_2}(y_{1,2}) \\ \sigma_t^{\widetilde{\psi}_2, \widetilde{\psi}_1}(y_{21}) & \sigma_t^{\widetilde{\psi}_1}(y_{22}) \end{array} \right] \right)$$

for any $[y_{ij}] \in (\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G$, where $\sigma_t^{\widetilde{\psi}_1, \widetilde{\psi}_2}$ and $\sigma_t^{\widetilde{\psi}_2, \widetilde{\psi}_1}$ are groups of isometries on $(\mathcal{M} \overline{\otimes} M_2) \rtimes_\beta G$. (See the discussion preceding [**Tak03a**, Lemma VIII.3.5].) Similarly for σ_t^ψ, we have that

$$\sigma_t^\psi \left(\left[\begin{array}{cc} x_{11} & x_{1,2} \\ x_{21} & x_{22} \end{array} \right] \right) = \left(\left[\begin{array}{cc} \sigma_t^{\psi_1}(x_{11}) & \sigma_t^{\psi_1, \psi_2}(x_{1,2}) \\ \sigma_t^{\psi_2, \psi_1}(x_{21}) & \sigma_t^{\psi_1}(x_{22}) \end{array} \right] \right)$$

for any $[x_{ij}] \in \mathcal{M} \overline{\otimes} M_2$. In fact the cocycle derivatives $(D\widetilde{\psi}_1 : D\widetilde{\psi}_2)_t$ and $(D\psi_1 : D\psi_2)_t$ may respectively be *defined* by the prescriptions $\sigma_t^{\widetilde{\psi}_1, \widetilde{\psi}_2}(\mathbb{1}) =$

$(D\widetilde{\psi}_1 : D\widetilde{\psi}_2)t$ and $\sigma_t^{\widetilde{\psi}_1,\widetilde{\psi}_2}(\mathbb{1}) = (D\psi_1 : D\psi_2)_t$. (Once again see the discussion preceding [**Tak03a**, Lemma VIII.3.5].) Having noted these facts we may now use Theorem 6.59 to see that for any $t \in \mathbb{R}$

$$
\begin{aligned}
\begin{bmatrix} 0 & (D\widetilde{\psi}_1 : D\widetilde{\psi}_2)t \\ 0 & 0 \end{bmatrix}
&= \begin{bmatrix} 0 & \sigma_t^{\widetilde{\psi}_1,\widetilde{\psi}_2}(\mathbb{1}) \\ 0 & 0 \end{bmatrix} \\
&= \sigma_t^{\widetilde{\psi}}\left(\begin{bmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{bmatrix} \right) \\
&= \sigma_t^{\psi}\left(\begin{bmatrix} 0 & \mathbb{1} \\ 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & \sigma_t^{\psi_1,\psi_2}(\mathbb{1}) \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & (D\psi_1 : D\psi_2)t \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

The result now clearly follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now finally ready to complete the proof of Theorem 6.55.

PROOF OF THE SURJECTIVITY CLAIM IN THEOREM 6.55. The proof uses the technology of Connes cocycle derivatives. The reader unfamiliar with this theory should review the relevant material in section 6.2. Let ϑ be a normal semifinite $\widehat{\alpha}$-invariant weight on $\mathcal{M} \rtimes_\alpha G$.

We first consider the case where ϑ is faithful. Let ψ be any faithful normal semifinite weight on $\mathcal{M}$. By what we have already proven, $\widetilde{\psi}$ is then an $\widehat{\alpha}$-invariant faithful normal semifinite weight on $\mathcal{M} \rtimes_\alpha G$. It then directly follows from Corollary 6.22 that $\widehat{\alpha}_\gamma((D\vartheta : D\widetilde{\psi})_t) = (D\vartheta : D\widetilde{\psi})_t$ for any $t \in \mathbb{R}$ and any $\gamma \in \widehat{G}$. By Theorem 6.50, this ensures that each $(D\vartheta : D\widetilde{\psi})_t$ belongs to $\pi(\mathcal{M})$. Now let $u_t = \pi^{-1}((D\vartheta : D\widetilde{\psi})_t)$ for each $t \in \mathbb{R}$. We may now use Theorem 6.59 to see that

$$
\pi(u_{s+t}) = (D\vartheta : D\widetilde{\psi})_{s+t} = (D\vartheta : D\widetilde{\psi})_s \sigma_s^{\widetilde{\psi}}((D\vartheta : D\widetilde{\psi})_t)
$$

$$
= (D\vartheta : D\widetilde{\psi})_s \pi(\sigma_s^{\psi}(u_t)) = \pi(u_s \sigma_s^{\psi}(u_t)),
$$

or equivalently that $u_{s+t} = u_s \sigma_s^{\psi}(u_t)$ for all $s, t \in \mathbb{R}$. By Theorem 6.25 there must then exist a faithful normal semifinite weight ψ_0 on $\mathcal{M}$, such that $u_t = (D\psi_0 : D\psi)_t$ for all $t \in \mathbb{R}$. An application of Theorem 6.60 now

allows us to conclude that

$$(D\vartheta : D\widetilde{\psi})_t = \pi(u_t) = \pi((D\psi_0 : D\psi)_t) = (D\widetilde{\psi_0} : D\widetilde{\psi})_t.$$

One may now use Theorem 6.24 to conclude that $(D\vartheta : D\widetilde{\psi_0})_t = \mathbb{1}$ for all $t \in \mathbb{R}$. But then Theorem 6.27 ensures that we must have that $\widetilde{\psi_0} \leq \vartheta \leq \widetilde{\psi_0}$. In other words $\widetilde{\psi_0} = \vartheta$.

Now let ϑ be a normal $\widehat{\alpha}$-invariant weight. Recall that by definition we then have that $(\mathcal{M} \rtimes_\alpha G)e_0(\vartheta) = \{x \in \mathcal{M} \rtimes_\alpha G \colon \vartheta(x^*x) = 0\}$. Since ϑ is $\widehat{\alpha}$-invariant, we clearly have that $\widehat{\alpha}_\gamma(\{x \in \mathcal{M} \rtimes_\alpha G \colon \vartheta(x^*x) = 0\}) = \{x \in \mathcal{M} \rtimes_\alpha G \colon \vartheta(x^*x) = 0\}$ for each $\gamma \in \widehat{G}$, and hence that $\widehat{\alpha}_\gamma((\mathcal{M} \rtimes_\alpha G)e_0(\vartheta)) = (\mathcal{M} \rtimes_\alpha G)e_0(\vartheta)$ for each $\gamma \in \widehat{G}$. This can only be the case if $\widehat{\alpha}_\gamma(e_0(\vartheta)) = e_0(\vartheta)$ for each $\gamma \in \widehat{G}$, which ensures that in fact $e_0(\vartheta) \in \pi(\mathcal{M})$. The $\widehat{\alpha}$-invariance of ϑ similarly ensures that $\widehat{\alpha}_\gamma(\mathfrak{n}_\vartheta) = \mathfrak{n}_\vartheta$ for each $\gamma \in \widehat{G}$, which on taking the σ-weak closure, yields the fact that $\widehat{\alpha}_\gamma((\mathcal{M} \rtimes_\alpha G)e_\infty(\vartheta)) = (\mathcal{M} \rtimes_\alpha G)e_\infty(\vartheta)$ for each $\gamma \in \widehat{G}$. As before we may conclude from this that in fact $e_\infty(\vartheta) \in \mathcal{M}$. Having noted this fact, we will in the following simply write e_0 and e_∞, for $e_0(\vartheta)$ and $e_\infty(\vartheta)$.

Now suppose that ϑ is a normal semifinite weight (so $e_\infty = \mathbb{1}$) for which $e_0(\vartheta) \neq 0$. Let ψ_1 be any normal semifinite weight on $\mathcal{M}$ with $e_0(\psi_1) = \mathbb{1} - e_0$. It then follows from what we have already shown that $\widetilde{\psi_1}$ is a normal semifinite weight on $(\mathcal{M} \rtimes_\alpha G)_+$ with $e_0(\widetilde{\psi_1}) = \mathbb{1} - e_0$. It is now an exercise to see that $\nu = \vartheta + \widetilde{\psi_1}$ is then normal and faithful. Since $\vartheta = (\mathbb{1} - e_0).\vartheta.(\mathbb{1} - e_0)$ and $\widetilde{\psi_1} = e_0\widetilde{\psi_1}e_0$, it is clear that $\mathfrak{n}_\vartheta(\mathbb{1} - e_0) \subseteq \mathfrak{n}_\vartheta$ and $\mathfrak{n}_{\widetilde{\psi_1}} e_0 \subseteq \mathfrak{n}_{\widetilde{\psi_1}}$, and that $\mathfrak{n}_\vartheta(\mathbb{1} - e_0) + \mathfrak{n}_{\widetilde{\psi_1}} e_0 \subseteq \mathfrak{n}_\nu$. By the semi-finiteness of ϑ and $\widetilde{\psi_1}$, each of $\mathfrak{n}_\vartheta$ and $\mathfrak{n}_{\widetilde{\psi_1}}$ is σ-weakly dense in $\mathcal{M} \rtimes_\alpha G$, and hence so is $\mathfrak{n}_\vartheta(\mathbb{1} - e_0) + \mathfrak{n}_{\widetilde{\psi_1}} e_0$. The weight ν is therefore also semifinite. By construction ν is $\widehat{\alpha}$-invariant. Hence there must exist a faithful normal semifinite weight ς on $\mathcal{M}$ such that $\nu = \widetilde{\varsigma}$. By construction and part (iii) of the present theorem, we must then have that

$$\vartheta = (\mathbb{1} - e_0)\nu(\mathbb{1} - e_0) = (\mathbb{1} - e_0) \cdot \widetilde{\varsigma} \cdot (\mathbb{1} - e_0) = \widetilde{(\mathbb{1} - e_0) \cdot \varsigma \cdot (\mathbb{1} - e_0)}.$$

This shows that for the normal weight $\varrho = (\mathbb{1} - e_0) \cdot \varsigma \cdot (\mathbb{1} - e_0)$ on $\mathcal{M}$ we have $\vartheta = \widetilde{\varrho}$. (Since $\vartheta = \widetilde{\varrho}$ is semifinite, we in fact have that ϱ is semifinite.)

Finally suppose that ϑ is merely normal and $\widehat{\alpha}$-invariant. Since $e_\infty \cdot \vartheta \cdot e_\infty$ is semifinite and still $\widehat{\alpha}$-invariant (by the fact that $e_\infty \in \mathcal{M}$), there exists a normal semifinite weight ς on $\mathcal{M}$ such that $\widetilde{\varsigma} = e_\infty \cdot \vartheta \cdot e_\infty$. We will construct a normal weight ν from ς for which $\widetilde{\nu} = \vartheta$. We first note that

$\{x\colon (e_\infty \cdot \vartheta \cdot e_\infty)(x^*x) = 0\} = N_{e_\infty \cdot \vartheta \cdot e_\infty} = (\mathcal{M} \rtimes_\alpha G)(e_0 + (\mathbb{1} - e_\infty))$. To see this observe that $x \in N_{e_\infty \cdot \vartheta \cdot e_\infty}$ if and only if $xe_\infty \in N_\vartheta = (\mathcal{M} \rtimes_\alpha G)e_0$. Since we trivially have that $(\mathcal{M} \rtimes_\alpha G)(\mathbb{1} - e_\infty) \subseteq N_{e_\infty \cdot \vartheta \cdot e_\infty}$, the claim follows. By (iv) we therefore have that $e_0(\varsigma) = e_0(e_\infty \cdot \vartheta \cdot e_\infty) = e_0 + (\mathbb{1} - e_\infty)$.

It clearly follows that $\mathbb{1} - e_\infty \leq e_0(\varsigma)$, or equivalently that $\mathbb{1} - e_0(\varsigma) \leq e_\infty$. Since $(\mathbb{1} - e_0(\varsigma)) \cdot \varsigma \cdot (\mathbb{1} - e_0(\varsigma)) = \varsigma$, this ensures that $e_\infty \cdot \varsigma \cdot e_\infty = \varsigma$. Thus ς is in particular 0-valued on $(\mathbb{1} - e_\infty)\mathcal{M}_+(\mathbb{1} - e_\infty)$. We now define a new weight ϱ on $\mathcal{M}$ with the prescription that it must be infinite valued on all the non-zero elements of $(\mathbb{1} - e_\infty)\mathcal{M}_+(\mathbb{1} - e_\infty)$ and equal ς on $e_\infty \mathcal{M}_+ e_\infty$, with $\varrho(x) = \varsigma(e_\infty x e_\infty) + \varrho((\mathbb{1} - e_\infty)x(\mathbb{1} - e_\infty))$ for $x \in \mathcal{M}_+$. It is not difficult to verify that ϱ is normal. In addition (on using the fact that $e_\infty \cdot \varsigma \cdot e_\infty = \varsigma$), it is also clear that $e_\infty \cdot \varrho \cdot e_\infty = \varsigma$.

The fact we have just noted, clearly ensures that for any $x \in \mathcal{M}$, $\varsigma(x^*x) < \infty$ if and only if $\varrho(e_\infty x^* x e_\infty) < \infty$. So $\mathfrak{n}_\varsigma e_\infty \subseteq \mathfrak{n}_\varrho$. Also if $\varrho(x^*x) < \infty$, then we must have that $(\mathbb{1} - e_\infty)x^*x(\mathbb{1} - e_\infty) = 0$, in which case $x \in \mathcal{M}e_\infty$. Hence $\mathfrak{n}_\varrho \subseteq \mathcal{M}e_\infty$. Taken together, these facts ensure that the σ-weak closure of $\mathfrak{n}_\varrho$ is precisely $\mathcal{M}e_\infty$, and hence that $e_\infty(\varrho) = e_\infty$. Since $N_\varrho = \{x\colon \varrho(x^*x) = 0\} \subseteq \mathfrak{n}_\varrho$, it is clear from the above that $N_\varrho \subseteq \mathcal{M}e_\infty$. Using the facts that ϱ and ς agree on $e_\infty \mathcal{M}e_\infty$, we may then conclude that $N_\varrho = N_\varsigma e_\infty$. By part (iii) we have that $e_0(\varsigma) = e_0(\widetilde{\varsigma}) = e_0(e_\infty \cdot \vartheta \cdot e_\infty) = e_0 + (\mathbb{1} - e_\infty)$. Hence $N_\varrho = N_\varsigma e_\infty = \mathcal{M}(e_0 + (\mathbb{1} - e_\infty))e_\infty = \mathcal{M}e_0$; that is $e_0(\varrho) = e_0$. For the weight ϱ we therefore have by part (iv) and (v) that $e_0(\widetilde{\varrho}) = e_0 = e_0(\vartheta)$ and $e_\infty(\widetilde{\varrho}) = e_\infty = e_\infty(\vartheta)$. In addition

$$e_\infty \cdot \widetilde{\varrho} \cdot e_\infty = e_\infty \cdot \widetilde{\varrho \cdot e_\infty} = \widetilde{\varsigma} = e_\infty \cdot \vartheta \cdot e_\infty.$$

These facts are enough to ensure that $\widetilde{\varrho} = \vartheta$ as required. $\qquad\square$

6.6 Crossed products with modular automorphism groups

Let the von Neumann algebra be equipped with a faithful normal semifinite weight φ. Tomita-Takesaki theory then informs us that this weight induces a canonical one-parameter group of point to σ-weak continuous *-automorphisms σ_t^φ $(t \in \mathbb{R})$ on $\mathcal{M}$ — the so-called modular automorphism group — for which we have that $\varphi \circ \sigma_t^\varphi = \varphi$ for all $t \in \mathbb{R}$. This group is of course an action of $\mathbb{R}$ on $\mathcal{M}$, and hence we may construct the crossed product with respect to this action. In this case we will denote this crossed product by $\mathcal{M} \rtimes_\varphi \mathbb{R}$. This crossed product is absolutely central to everything that follows, and will be repeatedly used. It is now

commonly referred as the core of $\mathcal{M}$ in the literature. We will for the sake of brevity often simply write $\mathfrak{M}$ for $\mathcal{M} \rtimes_\varphi \mathbb{R}$. The dual group of $\mathbb{R}$ is of course again a copy of $\mathbb{R}$, with the characters in the "dual group" of the form $\gamma_t : \mathbb{R} \to \mathbb{T} : s \mapsto e^{ist}$ (here $\mathbb{T}$ is the circle group $\{z \in \mathbb{C} : |z| = 1\}$). So in this case the unitary group w_t ($t \in \mathbb{R}$) acting on $L^2(\mathbb{R}, H)$ that induces the dual action on $\mathfrak{M}$ is of the form

$$w_t(\xi)(s) = e^{-its}\xi(s) \qquad \xi \in L^2(\mathbb{R}, H), s, t \in \mathbb{R}.$$

Following convention, we will in this case write θ_t ($t \in \mathbb{R}$) for the dual action, with the action on the generators of $\mathfrak{M}$ being given by

$$\theta_t(\pi(a)) = a, \quad \theta_t(\lambda_s) = e^{-ist}\lambda_s, \qquad a \in \mathcal{M}, \quad s, t \in \mathbb{R}.$$

In this particular case, the crossed product has some special features not shared by crossed products with more general groups. Much of this follows from the fact that in this case the modular automorphism group of the dual weight $\widetilde{\varphi}$ on $\mathfrak{M}$ is implemented. (Note that by Theorem 6.55, $\widetilde{\varphi}$ is indeed a faithful normal semifinite weight on $\mathfrak{M}$.) On combining Proposition 6.40 and Theorem 6.59, it is clear that the modular automorphism group $\sigma_t^{\widetilde{\varphi}}$ is implemented by $\{\lambda_t\} \subseteq \mathfrak{M}$. Due to its significance, we state this as a proposition.

PROPOSITION 6.61. *The modular automorphism group $\sigma_t^{\widetilde{\varphi}}$ ($t \in \mathbb{R}$) on $\mathfrak{M}$ corresponding to the dual weight $\widetilde{\varphi}$ is implemented by the unitary group $\{\lambda_t\} \subseteq \mathfrak{M}$ in the sense that $\sigma_t^{\widetilde{\varphi}}(a) = \lambda_t a \lambda_t^*$ for all $t \in \mathbb{R}$ and all $a \in \mathfrak{M}$.*

The above fact has two very far-reaching consequences, which we summarise in the theorem below:

THEOREM 6.62. *Let $\mathcal{M}$ and $\mathfrak{M}$ be as above.*
  (1) *The centre of $\pi(\mathcal{M})$ is contained in the centre of $\mathfrak{M}$.*
  (2) *There exists a positive non-singular operator h affiliated with $\mathfrak{M}_{\widetilde{\varphi}}$ (the centralizer of $\widetilde{\varphi}$), such that $\lambda_t = h^{it}$ for all $t \in \mathbb{R}$. For this operator, the derived weight $\tau = \widetilde{\varphi}(h^{-1}\cdot)$ is a faithful normal semifinite trace which satisfies the identity $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$. When equipped with this trace, h is just the Radon-Nikodym derivative $\frac{d\widetilde{\varphi}}{d\tau}$.*

PROOF. To see that (1) holds, notice that on the centre of $\mathcal{M}$, φ behaves like a trace. Since traces are known to induce trivial modular automorphism groups, it is no surprise to find that the elements of the

centre are fixed points of the automorphism group σ_t^φ ($t \in \mathbb{R}$) (see Theorem 6.9). When passing to $\pi(\mathcal{M})$, we see from Proposition 6.40 that this means that for every element a of the centre, we have that $\lambda_t \pi(a) \lambda_{t*} = \pi(a)$ for all t, or equivalently that $\lambda_t \pi(a) = \pi(a)\lambda_t$ for all t. Since $\pi(a)$ commutes with each λ_t and also each element of $\pi(\mathcal{M})$, it must in fact commute with each element of the algebra generated by these objects, namely $\mathfrak{M}$. We pass to proving the second claim.

The fact that h is a positive non-singular operator affiliated with $\mathfrak{M}_{\widetilde\varphi}$, and τ a faithful normal semifinite trace, follows directly from Theorem 6.26. Since τ is a trace, its modular automorphism group is trivial. In addition by the choice of τ, $\widetilde\varphi$ is then just $\widetilde\varphi = \tau(h\cdot)$. So by definition, h is the Radon-Nikodym derivative $\frac{d\widetilde\varphi}{d\tau}$. It remains to check the claim that $\tau \circ \theta_s = e^{-s}\tau$ for all $s \in \mathbb{R}$. In this regard note that for a fixed s, we have that $\theta_s(\lambda_t) = \theta_s(h^{it}) = (\theta_s(h))^{it}$ for each $t \in \mathbb{R}$. However we also have that $\theta_s(\lambda_t) = e^{-ist}h^{it} = (e^{-s}h)^{it}$. Hence both $\theta_s(h)$ and $e^{-s}h$ are generators of the unitary group $\{\theta_s(\lambda_t)\}$ ($t \in \mathbb{R}$). By the uniqueness of such generators, we must have that $\theta_s(h) = e^{-s}h$. Since s was arbitrary, this holds for all s. But then for any s, $\tau \circ \theta_s = \widetilde\varphi(h^{-1}\theta_s(\cdot)) = e^{-s}\widetilde\varphi(\theta_s(h^{-1}\cdot))$. Since $\widetilde\varphi$ is invariant with respect to the dual action, we have that $\tau \circ \theta_s = e^{-s}\widetilde\varphi(\theta_s(h^{-1}\cdot)) = e^{-s}\widetilde\varphi(h^{-1}\cdot) = e^{-s}\tau$ as required. $\square$

COROLLARY 6.63. *The dual action $\{\theta_t \colon t \in \mathbb{R}\}$ extends to a continuous action on $\widetilde{\mathfrak{M}}$.*

PROOF. The equality $\tau \circ \theta_s = e^{-s}\tau$ ensures that in their action on the projection lattice of $\mathfrak{M}$, we have that $\tau \circ \theta_s$ and τ are mutually absolutely continuous with respect to each other in an ϵ-δ sense. Hence the claim follows from Proposition 2.73. $\square$

We close this discussion of the trace on $\mathfrak{M}$ with the following observation which will prove to be an important tool in investigating the uniqueness of the crossed product.

PROPOSITION 6.64. *Let τ be the canonical trace on $\mathfrak{M}$ as described above. Then $\lambda_t = (D\widetilde\varphi : D\tau)_t$ for all $t \in \mathbb{R}$.*

PROOF. Recall that the modular automorphism group of $\widetilde\varphi$ is implemented by the λ_t's, and that τ is a trace, so that its modular automorphism group is trivial. We leave it as an exercise to verify that the λ_t's fulfil all the requirements stipulated for the $(D\widetilde\varphi : D\tau)_t$'s in Theorem

6.20. So by the uniqueness criterion in that theorem, we must have that $\lambda_t = (D\widetilde{\varphi} : D\tau)_t$ as required. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We pass to proving that in a very concrete sense, the algebra $\mathfrak{M} = \mathcal{M} \rtimes_\varphi \mathbb{R}$ is, almost surprisingly, independent of the particular faithful normal semifinite weight used to construct it!

THEOREM 6.65. *Let φ_1 and φ_2 be two f.n.s. weights on $\mathcal{M}$. Both crossed products $\mathcal{M} \rtimes_{\varphi_1} \mathbb{R} = \mathfrak{M}_1$ and $\mathcal{M} \rtimes_{\varphi_2} \mathbb{R} = \mathfrak{M}_2$, are realised on the same Hilbert space $L^2(\mathbb{R}, H)$ and share the same shift operators λ_t $(t \in \mathbb{R})$. The action θ_s of the dual group therefore shares the same implementation for each of the crossed products. In addition there is a *-isomorphism $\mathscr{I}$ from $\mathcal{M} \rtimes_{\varphi_1} \mathbb{R} = \mathfrak{M}_1$ onto $\mathcal{M} \rtimes_{\varphi_2} \mathbb{R} = \mathfrak{M}_2$ implemented by a unitary element u of $B(L^2(\mathbb{R}, H))$ for which we have that $\tau_1 = \tau_2 \circ \mathscr{I}$. The *-isomorphism $\mathscr{I}$ extends to a *-isomorphism which homeomorphically maps $\widetilde{\mathfrak{M}}_1$ onto $\widetilde{\mathfrak{M}}_2$, and which leaves the dual action invariant in the sense that $\mathscr{I} \circ \theta_t = \theta_t \circ \mathscr{I}$ for every $t \in \mathbb{R}$.*

PROOF. A consideration of Definitions 6.41 and 6.46 reveals that the first claim is by construction. We now define the unitary u on $L^2(\mathbb{R}, H)$ by

$$(u\xi)(t) = (D\varphi_2 : D\varphi_1)_{-t}\xi(t) \qquad \text{for all } t \in \mathbb{R} \text{ and all } \xi \in L^2(\mathbb{R}, H).$$

For each $a \in \mathcal{M}$ and each $\xi \in L^2(\mathbb{R}, H)$, it now follows from Theorem 6.20 that

$$u\pi_1(a)u^*\xi(t) = (D\varphi_2 : D\varphi_1)_{-t}\sigma^{\varphi_1}_{-t}(a)(D\varphi_2 : D\varphi_1)^*_{-t}\xi(t)$$
$$= \sigma^{\varphi_2}_{-t}(a)\xi(t) = \pi_2(a)\xi(t).$$

We proceed to compute $u\lambda_t u^*$ for all $t \in \mathbb{R}$. For this we need the fact that the chain rule for cocycle derivatives (Theorem 6.24), ensures that $\mathbb{1} = (D\varphi_1 : D\varphi_1)_t = (D\varphi_1 : D\varphi_2)_t(D\varphi_2 : D\varphi_1)_t$ for each t, and hence that $(D\varphi_1 : D\varphi_2)^*_t = (D\varphi_2 : D\varphi_1)_t$ Given $s \in \mathbb{R}$, we have that

$$
\begin{aligned}
(u\lambda_s u^*)\xi(t) &= (D\varphi_2 : D\varphi_1)_{-t}(\lambda_s u^*)\xi(t) \\
&= (D\varphi_2 : D\varphi_1)_{-t}(u^*\xi)(t - s) \\
&= (D\varphi_2 : D\varphi_1)_{-t}(D\varphi_2 : D\varphi_1)^*_{s-t}\xi(t - s)
\end{aligned}
$$

for all $t \in \mathbb{R}$ and all $\xi \in L^2(\mathbb{R}, H)$. Since by Theorem 6.20

$$(D\varphi_2 : D\varphi_1)_{-t}\sigma^{\varphi_1}_{-t}((D\varphi_2 : D\varphi_1)_s) = (D\varphi_2 : D\varphi_1)_{s-t} \text{ for all } s, t \in \mathbb{R},$$

the above may be rewritten as

$$(u\lambda_s u^*)\xi(t)$$
$$= (D\varphi_2 : D\varphi_1)_{-t}[(D\varphi_2 : D\varphi_1)_{-t}\sigma^{\varphi_1}_{-t}((D\varphi_2 : D\varphi_1)_s)]^*\xi(t-s)$$
$$= (D\varphi_2 : D\varphi_1)_{-t}\sigma^{\varphi_1}_{-t}((D\varphi_2 : D\varphi_1)^*_s)(D\varphi_2 : D\varphi_1)^*_{-t}\xi(t-s)$$
$$= (D\varphi_2 : D\varphi_1)_{-t}\sigma^{\varphi_1}_{-t}((D\varphi_1 : D\varphi_2)_s)(D\varphi_2 : D\varphi_1)^*_{-t}\xi(t-s)$$
$$= \sigma^{\varphi_2}_{-t}((D\varphi_1 : D\varphi_2)_s)\xi(t-s)$$
$$= \sigma^{\varphi_2}_{-t}((D\varphi_1 : D\varphi_2)_s)\lambda_s\xi(t)$$
$$= \pi_2((D\varphi_1 : D\varphi_2)_s)\lambda_s\xi(t).$$

(In the fourth equality we silently applied Theorem 6.20 once again.) We therefore have that $u\lambda_s u^* = \pi_2((D\varphi_1 : D\varphi_2)_s)\lambda_s$ for each s. This is clearly an element of $\mathcal{M} \rtimes_{\varphi_2} \mathbb{R}$, and hence the prescription $a \mapsto uau^*$ maps $\mathfrak{M}_1$ — the algebra generated by $\pi_1(\mathcal{M})$ and the λ_t's — into $\mathfrak{M}_2$. By now swopping the roles of φ_1 and φ_2, we can similarly show that the prescription $a \mapsto u^*au$ maps $\mathfrak{M}_2$ into $\mathfrak{M}_1$. Hence we must have that $u\mathfrak{M}_1u^* = \mathfrak{M}_2$. The prescription $\mathscr{I}(a) = uau^*$ therefore clearly defines a *-isomorphism from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$.

It is now a simple matter to check that on $\mathfrak{M}_1$, $\theta_s \circ \mathscr{I} = \mathscr{I} \circ \theta_s$ for each $s \in \mathbb{R}$. Specifically Theorem 6.50 ensures that for any $a \in \pi_1(\mathcal{M})$ it holds that $\theta_s \circ \mathscr{I}(a) = \mathscr{I}(a) = \mathscr{I} \circ \theta_s(a)$. In addition for any $t \in \mathbb{R}$, we have that $\theta_s \circ \mathscr{I}(\lambda_t) = \theta_s(\pi_2((D\varphi_1 : D\varphi_2)_t)\lambda_t) = e^{-ist}\pi_2((D\varphi_1 : D\varphi_2)_t)\lambda_t = \mathscr{I} \circ \theta_s(\lambda_t)$. Hence the claim follows. If we can show that $\tau_2 \circ \mathscr{I} = \tau_1$, then the fact that the *-isomorphism $\mathscr{I}$ extends to a bi-continuous *-isomorphism from $\widetilde{\mathfrak{M}}_1$ onto $\widetilde{\mathfrak{M}}_2$ will follow from Proposition 2.73. By continuity the extension will still satisfy $\theta_s \circ \mathscr{I} = \mathscr{I} \circ \theta_s$ for each $s \in \mathbb{R}$.

We proceed to show that indeed $\tau_2 \circ \mathscr{I} = \tau_1$. The first technical fact we need is the observation that for each $a \in \mathfrak{M}_1^+$

$$\mathscr{W}(\mathscr{I}(a)) = \int_{\mathbb{R}} \theta_s(\mathscr{I}(a))\,ds = \int_{\mathbb{R}} \mathscr{I}(\theta_s(a))\,ds = u\mathscr{W}(a)u^*.$$

A subtlety we need to contend with here is that with two weights, we now have two ways in which to define the corresponding dual weight. Let $\mathscr{W}$ be the operator valued weight from $\mathfrak{M}_1$ to $\pi_1(\mathcal{M})_+$. This weight is defined by $\mathscr{W}(a) = \int_{\mathbb{R}} \theta_s(a)\,ds$. Observe that exactly the same prescription is used to define the operator valued weight from $\mathfrak{M}_2$ to $\pi_2(\mathcal{M})_+$. We will therefore use the same notation for both versions. The standard way of defining $\widetilde{\varphi}_1$ is in terms of $\mathfrak{M}_1$ by means of the prescription $\widehat{\varphi}_1 \circ \pi_1^{-1} \circ \mathscr{W}$. However

we can also define a dual version of φ_1 in terms of $\mathfrak{M}_2$. We shall write $\widetilde{\varphi}_1^{(2)}$ for this alternative version. In this case the prescription for $\widetilde{\varphi}_1^{(2)}$ is $\widehat{\varphi}_1 \circ \pi_2^{-1} \circ \mathscr{W}$. Using the technical facts we verified above, we can now see that these two versions are related by the formula

$$
\begin{aligned}
\widetilde{\varphi}_1(a) &= \widehat{\varphi}_1 \circ \pi_1^{-1} \circ \mathscr{W}(a) \\
&= \widehat{\varphi}_1 \circ \pi_2^{-1}(\pi_2(\pi_1^{-1}(\mathscr{W}(a)))) \\
&= \widehat{\varphi}_1 \circ \pi_2^{-1}(u\pi_1(\pi_1^{-1}(\mathscr{W}(a)))u^*) \\
&= \widehat{\varphi}_1 \circ \pi_2^{-1}(u\mathscr{W}(a)u^*) \\
&= \widehat{\varphi}_1 \circ \pi_2^{-1}(\mathscr{W}(\mathscr{I}(a))) \\
&= \widetilde{\varphi}_1^{(2)}(\mathscr{I}(a))
\end{aligned}
$$

for each $a \in \mathfrak{M}_1^+$. We saw earlier that $\mathscr{I}(\lambda_t) = \pi_2((D\varphi_1 : D\varphi_2)_t)\lambda_t$. By reversing the roles of φ_1 and φ_2 in the proof of that fact, we may show that $\mathscr{I}^{-1}(\lambda_t) = u^*\lambda_t u = \pi_1((D\varphi_2 : D\varphi_1)_t)\lambda_t$ for each t. We may now use these two facts alongside Corollary 6.22 to see that for each t

$$
(D\widetilde{\varphi}_1 : D(\tau_2 \circ \mathscr{I}))_t = (D(\widetilde{\varphi}_1^{(2)} \circ \mathscr{I}) : D(\tau_2 \circ \mathscr{I}))_t = \mathscr{I}^{-1}((D\widetilde{\varphi}_1^{(2)} : D\tau_2)_t).
$$

On bringing Theorems 6.24 and 6.60, and Proposition 6.64 into play, it then follows that

$$
\begin{aligned}
(D\widetilde{\varphi}_1 : D(\tau_2 \circ \mathscr{I}))_t &= \mathscr{I}^{-1}((D\widetilde{\varphi}_1^{(2)} : D\tau_2)_t) \\
&= \mathscr{I}^{-1}((D\widetilde{\varphi}_1^{(2)} : D\widetilde{\varphi}_2)_t).\mathscr{I}^{-1}((D\widetilde{\varphi}_2 : D\tau_2)_t) \\
&= \mathscr{I}^{-1}(\pi_2((D\varphi_1 : D\varphi_2)_t)).\mathscr{I}^{-1}(\lambda_t) \\
&= \pi_1((D\varphi_1 : D\varphi_2)_t).\pi_1((D\varphi_2 : D\varphi_1)_t)\lambda_t \\
&= \lambda_t \\
&= (D\widetilde{\varphi}_1 : D\tau_1)_t.
\end{aligned}
$$

This ensures that $(D(\tau_2 \circ \mathscr{I}) : D\tau_1)_t = (D(\tau_2 \circ \mathscr{I}) : D\widetilde{\varphi}_1)_t(D\widetilde{\varphi}_1 : D\tau_1)_t = (D\widetilde{\varphi}_1 : D\tau_1)_t^*(D\widetilde{\varphi}_1 : D\tau_1)_t = \mathbb{1}$ for all t. Thus by Theorem 6.27, we have that $\tau_2 \circ \mathscr{I} = \tau_1$ as required. $\qquad\square$

The presence of a trace on $\mathfrak{M}$, now enables us to reinterpret the dual weight map $\psi \mapsto \widetilde{\psi}$. Specifically with Theorem 3.24 as background, we are now able to prove the promised reformulation of Theorem 6.55.

DEFINITION 6.66. For each normal weight ψ on $\mathcal{M}$, we define m_ψ to be the unique element of $\widehat{\mathfrak{M}}_+$ corresponding to the dual weight $\widetilde{\psi}$ by means of the bijection described in Theorem 3.24.

PROPOSITION 6.67. *The mapping $\psi \mapsto m_\psi$ defined above is a bijection from the set of all normal weights on $\mathcal{M}$ onto the set of all elements m of $\widehat{\mathfrak{M}}_+$ satisfying $\theta_s(m) = e^{-s}m$. For normal weights ψ, ψ_1 and ψ_2 on $\mathcal{M}$, and any $a \in \mathcal{M}$, we moreover have that*

  (1) *$m_{\psi_1} \leq m_{\psi_2}$ if and only if $\psi_1 \leq \psi_2$, and in addition $m_{\psi_\alpha} \nearrow m_\psi$ if and only if $\psi_\alpha \nearrow \psi$;*
  (2) *$m_{\psi_1+\psi_2} = m_{\psi_1} + m_{\psi_2}$;*
  (3) *$a^* \cdot m_\psi \cdot a = m_{a^* \cdot \psi \cdot a}$;*
  (4) *with $\int_0^\infty \lambda\, de_\lambda + \infty \cdot p$ denoting the spectral resolution of m_ψ, we have that $\mathbb{1} - p = \pi(e_\infty(\psi))$ and $e_0 = \pi(e_0(\psi))$.*

PROOF. As we have done before, we will in this proof too identify $\pi(\mathcal{M})$ with $\mathcal{M}$. Properties (1)-(4) are fairly immediate consequences of Theorems 6.55 and 3.24. It therefore remains to prove the first claim. It is clear from Theorems 6.55 and 3.24, that we may prove that claim by proving that any normal weight ψ on $\mathfrak{M}$ is θ_s-invariant if and only if for each s we have that $\theta_s(m_\psi) = e^{-s}m_\psi$. By Theorem 3.24 a typical normal weight on $\mathfrak{M}$ is of the form ψ_m for some $m \in \widehat{\mathfrak{M}}_+$. For such a weight we will for any $s \in \mathbb{R}$ and any $a \in \mathfrak{M}$, have that

$$\psi_{e^s\theta_s(m)}(a^*a) = e^s(\tau \circ \theta_s)(\theta_{-s}(a)m\theta_{-s}(a^*))$$
$$= \tau(\theta_{-s}(a)m\theta_{-s}(a^*))$$
$$= (\psi_m \circ \theta_{-s})(a^*a).$$

This clearly ensures that

$$e^{-s}m = \theta_s(m) \Leftrightarrow \psi_{e^s\theta_s(m)} = \psi_m \Leftrightarrow \psi_m = \psi_m \circ \theta_{-s}.$$

$\square$

REMARK 6.68. It follows from part (3) of the preceding proposition that the mapping $\psi \mapsto m_\psi$ restricts to a bijection from the normal semifinite weights on $\mathcal{M}$, to the positive operators h affiliated with $\mathfrak{M}$ for which we have that $\theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}$. (This is the case where $\mathbb{1} - p = \pi(e_\infty(\psi)) = 0$.) *To distinguish these two settings, we shall when working with the restriction of this bijection to the set of normal semifinite weights on $\mathcal{M}$, denote the bijection by $\psi \mapsto h_\psi$. This bijection then*

restricts further to a bijection from the faithful normal semifinite weights on $\mathcal{M}$ to the positive non-singular operators affiliated with $\mathfrak{M}$ for which we have that $\theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}$. (This is the case where in addition $e_0 = \pi(e_0(\psi)) = 0$.)

We proceed to the final topic of this section, which is to show that the dual action $\{\theta_s\}$ may be used to identify isometric copies of both $\mathcal{M}$ and $\mathcal{M}_*$ inside $\widetilde{\mathfrak{M}}$. Our first result in this regard is a refinement of Theorem 6.50.

PROPOSITION 6.69. *For any $x \in \widetilde{\mathfrak{M}}$, we have that $\theta_s(x) = x$ for each $s \in \mathbb{R}$ if and only if $x \in \pi(\mathcal{M})$*

PROOF. The result will clearly follow from Theorem 6.50 if we are able to prove that if for some $x \in \widetilde{\mathfrak{M}}$, we have that $\theta_s(x) = x$ for each $s \in \mathbb{R}$, then x must belong to $\mathfrak{M}$. So let $x \in \widetilde{\mathfrak{M}}$ be given such that $\theta_s(x) = x$ for each $s \in \mathbb{R}$. Since x is τ-measurable, there must exist some $\lambda > 0$ such that $\tau(\chi_{(\lambda,\infty)}(|x|)) < \infty$. Given some $s \neq 0$, we may then apply Theorem 6.62 to see that

$$\begin{aligned}
\tau(\chi_{(\lambda,\infty)}(|x|)) &= \tau(\chi_{(\lambda,\infty)}(|\theta_s(x)|)) \\
&= \tau(\theta_s(\chi_{(\lambda,\infty)}(|x|))) \\
&= e^{-s}\tau(\chi_{(\lambda,\infty)}(|x|)).
\end{aligned}$$

The only way this can be is if $\tau(\chi_{(\lambda,\infty)}(|x|)) = 0$ and hence $\chi_{(\lambda,\infty)}(|x|) = 0$. Thus $|x|$, and hence x is bounded as required. $\square$

We will now refine the flow of ideas in Remark 6.68, and work toward establishing technology which will allow us to obtain a bijection on the positive normal functionals on $\mathcal{M}$, by further restricting the map in Proposition 6.67. That will then equip us with the necessary tools for the completion of the task of finding an isometric copy of $\mathcal{M}_*$ inside $\widetilde{\mathfrak{M}}$. The key result in this quest is an important result of Haagerup.

PROPOSITION 6.70. *Let ψ be a normal semifinite weight on $\mathcal{M}$. For any $\gamma > 0$ we will then have that $\tau_{\mathfrak{M}}(\chi_{(\gamma,\infty)}(h_\psi)) = \gamma^{-1}\psi(\mathbb{1})$.*

PROOF. Since for any $\gamma > 0$, $\tau_{\mathfrak{M}}(\chi_{(\gamma,\infty)}(h_\psi)) = \tau_{\mathfrak{M}}(\chi_{(1,\infty)}(\gamma^{-1}h_\psi))$ with $\gamma^{-1}\psi$ corresponding to $\gamma^{-1}h_\psi$ by means of the bijection defined earlier, it suffices to prove the Proposition for the case $\gamma = 1$. Let g_ψ be the positive operator affiliated to $\mathfrak{M}$ and commuting with h_ψ, for which

we have that $g_\psi h_\psi = \mathsf{s}(h_\psi)$. It is clear from Theorem 6.50 and Proposition 6.67(4), that $\mathsf{s}(h_\psi)$ is invariant under the action of the θ_t's. So since $\theta_t(h_\psi) = e^{-t} h_\psi$ for each t, we must have that $\theta_t(g_\psi) = e^t g_\psi$ for each t.

Let $h_\psi = \int_0^\infty \lambda \, de_\lambda$ be the spectral resolution of h_ψ. Then g_ψ is of course just $\int_0^\infty \lambda^{-1} \, de_\lambda$. Let $\xi \in L^2(\mathbb{R}, H)$ be a unit vector, and write $\rho_{\xi,\xi}$ for the functional $a \mapsto \langle a\xi, \xi \rangle$. We then have that

$$
\begin{aligned}
\mathscr{W}_{\mathbb{R}}(g_\psi \chi_{(1,\infty)}(h_\psi))(\rho_{\xi,\xi}) &= \int_0^\infty \theta_s(g_\psi \chi_{(1,\infty)}(h_\psi))(\rho_{\xi,\xi}) \, ds \\
&= \int_0^\infty (\theta_s(g_\psi) \chi_{(1,\infty)}(\theta_s(h_\psi))(\rho_{\xi,\xi}) \, ds \\
&= \int_0^\infty (e^s g_\psi \chi_{(1,\infty)}(e^{-s} h_\psi)(\rho_{\xi,\xi}) \, ds \\
&= \int_0^\infty \int_0^\infty e^s \lambda^{-1} \chi_{(1,\infty)}(e^{-s}\lambda) \, d\langle e_\lambda \xi, \xi \rangle \, ds \\
&= \int_0^\infty \lambda^{-1} \left(\int_{(-\infty, \log(\lambda))} e^s \, ds \right) d\langle e_\lambda \xi, \xi \rangle \\
&= \int_0^\infty \lambda^{-1} \lambda \, d\langle e_\lambda \xi, \xi \rangle \\
&= \| \mathsf{s}(h_\psi)(\xi) \|^2.
\end{aligned}
$$

Since ξ was an arbitrary unit vector, we therefore have that

$$
\mathscr{W}_{\mathbb{R}}(g_\psi \chi_{(1,\infty)}(h_\psi)) = \mathsf{s}(h_\psi) = e_0(\psi).
$$

Recalling that $\widetilde{\psi} = \tau(h_\psi \cdot)$, it therefore follows that

$$
\begin{aligned}
\tau(\chi_{(1,\infty)}(h_\psi)) &= \tau(h_\psi^{1/2}(g_\psi \chi_{(1,\infty)}(h_\psi)) h_\psi^{1/2}) \\
&= \widetilde{\psi}(g_\psi \chi_{(1,\infty)}(h_\psi)) \\
&= \psi(\mathscr{W}_{\mathbb{R}}(g_\psi \chi_{(1,\infty)}(h_\psi))) \\
&= \psi(e_0(\psi)) \\
&= \psi(\mathbb{1})
\end{aligned}
$$

as claimed. $\qquad\square$

The corollary stated below follows on noticing that both conditions correspond to the finiteness (for some γ) of the quantity discussed in the above proposition.

COROLLARY 6.71. *Let ψ be a normal semifinite weight on $\mathcal{M}$. Then ψ belongs to $\mathcal{M}_*$ if and only if $h_\psi \in \widetilde{\mathfrak{M}}$.*

THEOREM 6.72. *The mapping $\mathcal{M}_*^+ \to \widetilde{\mathfrak{M}}_+ : \omega \mapsto h_\omega$ extends to a linear bijection from $\mathcal{M}_*$ onto $\{h \in \widetilde{\mathfrak{M}} : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\}$. For all $\omega \in \mathcal{M}_*$ and all $a, b \in \mathcal{M}$, this bijection satisfies the following properties:*

(1) $h_{x \cdot \omega \cdot y} = \pi(x)h_\omega\pi(y)$;
(2) $h_{\omega^*} = h_\omega^*$, *where ω^* denotes the normal functional defined by* $\omega^*(a) = \overline{\omega(a^*)}$;
(3) *If $\omega = u \cdot |\omega|$ is the polar decomposition of ω, then $h_\omega = u|h_\omega|$ is the polar decomposition of h_ω.*

PROOF. As is our wont, we will in this proof too identify $\mathcal{M}$ with $\pi(\mathcal{M})$. Let $\omega_1, \omega_2 \in \mathcal{M}_*^+$ be given, and let $h_{\omega_1} = h_1$ and $h_{\omega_2} = h_2$ be the image of these functionals under the bijection described in Proposition 6.67. By Corollary 6.71 both these operators are τ-measurable. This in turn ensures that their strong sum is again a positive self-adjoint operator. So by Proposition 3.26, that strong sum $h_1 \bar{+} h_2$ must be $h_1 \hat{+} h_2$. Hence we have that $h_{\omega_1+\omega_2} = h_{\omega_1} + h_{\omega_2}$, where the right-hand side represents the sum in $\widetilde{\mathfrak{M}}$. Similar considerations show that for any $\omega \in \mathcal{M}_*$ and any $a \in \mathcal{M}$, we will have that $h_{a \cdot \omega \cdot a^*} = a \bar{\cdot} h_\omega \bar{\cdot} a^* = ah_\omega a^*$ where the right-hand product is in $\widetilde{\mathfrak{M}}$. If in this final formula we take a to be $\gamma^{1/2}\mathbb{1}$, it is now clear that the prescription $\omega \mapsto h_\omega$ is an affine map from the positive cone $\mathcal{M}_*^+$, onto the positive cone $\{h \in \widetilde{\mathfrak{M}}_+ : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\}$. Hence this map will extend to a linear map from $\mathcal{M}_*$ onto span$\{h \in \widetilde{\mathfrak{M}}_+ : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\} = \{h \in \widetilde{\mathfrak{M}} : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\}$. (It is an exercise to see that $\{h \in \widetilde{\mathfrak{M}} : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\}$ is spanned by its positive elements.) To see that this map is well-defined, let $\omega \in \mathcal{M}_*$ be a hermitian functional with $\omega = \omega_1 - \omega_2 = \rho_1 - \rho_2$, where $\omega_1, \omega_2, \rho_1$ and ρ_2 are positive normal functionals. Then of course $\omega_1 + \rho_2 = \rho_1 + \omega_2$ by Proposition 6.67, in which case we then have that $h_{\omega_1} + h_{\rho_2} = h_{\rho_1} + h_{\omega_2}$. Thus $h_{\omega_1} - h_{\omega_2} = h_{\rho_1} - h_{\rho_2}$, which ensures that h_ω is well-defined. A similar argument shows that if we are given $\omega \in \mathcal{M}_*$ such that $\omega = \omega_1 + i\omega_2 = \rho_1 + i\rho_2$ where $\omega_1, \omega_2, \rho_1$ and ρ_2 are now hermitian normal functionals, then $h_{\omega_1} + ih_{\omega_2} = h_{\rho_1} + ih_{\rho_2}$.

Given $\omega \in \mathcal{M}_*$, we may then write ω as the linear combination $\omega = \omega_1 - \omega_2 + i\omega_3 - i\omega_4$ of four positive normal functionals ω_i $(i = 1, 2, 3, 4)$. It is not difficult to see that then $\omega^* = \omega_1 - \omega_2 - i\omega_3 + i\omega_4$. So writing

h_i for h_{ω_i}, ω will by linearity map onto $h_1 - h_2 + ih_3 - ih_4$, and ω^* onto $h_1 - h_2 - ih_3 + ih_4 = (h_1 - h_2 + ih_3 - ih_4)^*$, thereby verifying (2). We know that for any $x \in \mathcal{M}$ we have that $xh_ix^* = h_{x \cdot \omega_i \cdot x^*}$. So also by linearity the same must then be true of $h_\omega = h_1 - h_2 + ih_3 - ih_4$. Given $a, b \in \mathcal{M}$, we may then use this fact and the polarisation identities $ah_\omega b = \frac{1}{4}\sum_{k=0}^{3}(b+i^k a^*)^* h(b+i^k a^*)$ and $a \cdot \omega \cdot b = \frac{1}{4}\sum_{k=0}^{3}(b+i^k a^*)^* \cdot \omega \cdot (b+i^k a^*)$ to see that (1) holds.

Now let $\omega = u \cdot |\omega|$ be the polar decomposition of ω. It then follows from (1) that $h_\omega = h_{u \cdot |\omega|} = uh_{|\omega|}$. Notice that the initial projection for the partial isometry u is $e_0(\omega)$. By Proposition 6.67 this is precisely $\mathfrak{s}(h_{|\omega|})$. It therefore follows that $h_\omega^* h_\omega = h_{|\omega|}^2$, and hence that $|h_\omega| = h_{|\omega|}$. Thus $h_\omega = h_{u \cdot |\omega|} = uh_{|\omega|}$ is indeed the polar decomposition of h_ω.

It remains to prove the injectivity of this map. To see this, notice that if $h_\omega = 0$, then $0 = |h_\omega| = h_{|\omega|}$. But then $|\omega| = 0$ and hence $\omega = 0$ by the injectivity of the affine map from $\mathcal{M}_*^+$ to $\{h \in \widetilde{\mathfrak{M}}_+ : \theta_s(h) = e^{-s}h$ for all $s \in \mathbb{R}\}$. $\square$

REMARK 6.73. Proposition 6.69 and Theorem 6.72 show how both $\mathcal{M}$ and $\mathcal{M}_*$ may be realised as concrete spaces of operators within the same algebra of τ-measurable operators, and provide strong circumstantial evidence that this algebra may be a natural home for a theory of L^p-spaces for type III von Neumann algebras. To gain some intuition of how such a type III theory may look, we will first see how the well understood L^p and Orlicz spaces of semifinite algebras, may be described using the crossed product technology developed in this chapter.

6.6.1   Crossed products of semifinite algebras

For the sake of facilitating the objective outlined in the above remark, we close this chapter by investigating the structure of $\mathcal{M} \rtimes_\tau \mathbb{R}$ in the case where $\mathcal{M}$ is semifinite, and $\tau_\mathcal{M}$ a faithful normal semifinite trace on $\mathcal{M}$. We shall write $\tau_\mathfrak{M}$ for the canonical trace on $\mathcal{M} \rtimes_\tau \mathbb{R}$.

For a trace the modular automorphism group is of the form $\sigma_t^\tau = \mathrm{Id}$ for every t. So U as defined in Proposition 6.43 is just $\mathbb{1}$. Thus by Proposition 6.43, $\pi(\mathcal{M})$ is just $\mathcal{M} \otimes \mathbb{1}$. The von Neumann algebra generated by the left shift operators $\lambda_t = \mathbb{1} \otimes \ell_t$ is of course just $\mathbb{1} \otimes VN(\mathbb{R})$ where $VN(\mathbb{R})$ is the group von Neumann algebra generated by the operators ℓ_t on $L^2(\mathbb{R})$. So $\mathcal{M} \rtimes_\tau \mathbb{R}$ is just $\mathcal{M}\overline{\otimes}VN(\mathbb{R})$. The unitary $\mathbb{1} \otimes \mathcal{F}$ clearly commutes with $\mathcal{M} \otimes \mathbb{1}$. Using the fact that $\mathcal{F}VN(\mathbb{R})\mathcal{F}^{-1} \equiv L^\infty(\mathbb{R})$ (see the discussion at

the start of section 6.5), it therefore follows that $(\mathbb{1} \otimes \mathcal{F})(\mathcal{M} \rtimes_\tau \mathbb{R})(\mathbb{1} \otimes \mathcal{F}^{-1}) = (\mathbb{1} \otimes \mathcal{F})(\mathcal{M} \overline{\otimes} VN(\mathbb{R})(\mathbb{1} \otimes \mathcal{F}^{-1}) \equiv \mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$. (Recall that properly $\mathcal{F} VN(\mathbb{R}) \mathcal{F}^{-1}$ is the algebra of multiplication operators with symbols in $L^\infty(\mathbb{R})$.) It is a refreshing exercise to show that we will for any $f \in L^2(\mathbb{R})$ have that $v_t \mathcal{F}(f) = \mathcal{F}(\ell_t f)$, and hence that $\ell_t f = (\mathcal{F}^{-1} v_t \mathcal{F})(f)$. We claim that this same unitary transformation, transforms the dual action to the action given by $\widetilde{a} \mapsto \lambda_t \widetilde{a}$ for all $\widetilde{a} \in \mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$ $(t \in \mathbb{R})$. We pause to justify this fact.

Given some $f \in L^\infty(\mathbb{R})$, let M_f be the associated multiplication operator on $L^2(\mathbb{R})$. It can then be shown that in their action on $L^2(\mathbb{R})$, $\ell_t M_f \ell_t^*$ agrees with $M_{\ell_t f}$. For simplicity of notation we write $U_\mathcal{F}$ for $\mathbb{1} \otimes \mathcal{F}$, and $\alpha_\mathcal{F}$ for the map $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R}) \to U_\mathcal{F}^* \mathcal{M} \overline{\otimes} L^\infty(\mathbb{R}) U_\mathcal{F} = \mathcal{M} \rtimes_\tau \mathbb{R}$. Together the above observations ensure that when passing from $\mathcal{M} \rtimes_\tau \mathbb{R} = \mathcal{M} \overline{\otimes} VN(\mathbb{R})$ to $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$ by means of the unitary transformation described above, the transformed dual action $\widetilde{\theta}_t = U_\mathcal{F} \theta_t(U_\mathcal{F}^*(\cdot)U_\mathcal{F})U_\mathcal{F}^*$ will have the form $\widetilde{\theta}_t(a \otimes f) = (a \otimes \ell_t f) = \lambda_t(a \otimes f)$ on the simple tensors of $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$. Since the simple tensors are σ-weakly dense in $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$, this formula will by continuity hold on all of $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$.

It remains to compute the precise forms of $\widetilde{\tau}_\mathcal{M} \circ \alpha_\mathcal{F}$ and $\tau_\mathfrak{M} \circ \alpha_\mathcal{F}$.

It follows from Proposition 6.64 that $(D\widetilde{\tau}_\mathcal{M} : D\tau_\mathfrak{M})_t = \lambda_t = \mathbb{1} \otimes \ell_t$ for each $t \in \mathbb{R}$. In the context of $\mathcal{M} \rtimes_\tau \mathbb{R}$, the density $h = \frac{d\widetilde{\tau}_\mathcal{M}}{d\tau_\mathfrak{M}}$ is the unique positive non-singular operator for which $h^{it} = \lambda_t$ for each $t \in \mathbb{R}$. (See Theorem 6.62.)

Recall that $\mathcal{F} \ell_t \mathcal{F}^{-1} = v_t$, or equivalently that $\alpha_\mathcal{F}^{-1}(\mathbb{1} \otimes \ell_t) = \mathbb{1} \otimes v_t$ for each $t \in \mathbb{R}$. By Corollary 6.22, this then ensures that $(D(\widetilde{\tau}_\mathcal{M} \circ \alpha_\mathcal{F}) : D(\tau_\mathfrak{M} \circ \alpha_\mathcal{F}))_t = \alpha_\mathcal{F}^{-1}(\mathbb{1} \otimes \ell_t) = \mathbb{1} \otimes v_t$. It is a not too trivial exercise to now use this equality to conclude that similarly the density $h_\mathcal{F} = \frac{d(\widetilde{\tau}_\mathcal{M} \circ \alpha_\mathcal{F})}{d(\tau_\mathfrak{M} \circ \alpha_\mathcal{F})}$ is the unique positive non-singular operator for which $h_\mathcal{F}^{it} = \mathbb{1} \otimes v_t$. (This involves a technical modification of the proof of Theorem 6.62.) Now observe that in this particular case, the operators v_t are of the form $v_t(g)(s) = e^{-its} g(s)$ for each $g \in L^2(G)$ and each $s, t \in \mathbb{R}$. We require a clear understanding of the manner in which the operators v_t act on the multiplication operators M_f where $f \in L^\infty(\mathbb{R})$. For each $f \in L^\infty(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, and each $s, t \in \mathbb{R}$, we have that $v_t(M_f g)(s) = v_t(fg)(s) = e^{-its} f(s) g(s) = (e^{-s})^{it} f(s) g(s)$. The positive non-singular operator $h_\mathcal{F}$ for which $h_\mathcal{F}^{it} = \mathbb{1} \otimes v_t$, is therefore

nothing but

$$h_{\mathcal{F}} = \mathbb{1} \otimes \frac{1}{\exp}. \tag{6.5}$$

Using the "transformed" version of the dual action described above, the prescription for the dual weight translates on simple tensors $a \otimes f$ of $(\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R}))_{+}$ to the prescription $\mathscr{W}_{\mathcal{F}}(a \otimes f) = \int_{\mathbb{R}} \widetilde{\theta}_{t}(a \otimes f) \, dt = \int_{\mathbb{R}}(a \otimes \ell_{t}f) \, d\mathbb{m}_{L}(t) = a \otimes (\int_{\mathbb{R}}(\ell_{t}f \, d\mathbb{m}(t)).$ (Here $\mathbb{m}$ denotes Lebesgue measure.) Recall that the translation invariance of Lebesgue measure ensures that for each fixed $s \in \mathbb{R}$ we have $\int_{\mathbb{R}} f(s-t) \, d\mathbb{m}(t) = \int_{\mathbb{R}} f(-t) \, d\mathbb{m}(t).$ So $\mathscr{W}_{\mathcal{F}}(a \otimes f)$ is just $a \otimes (\int_{\mathbb{R}} f(-t) \, d\mathbb{m}(t))$ (or more properly $a \otimes (\int_{\mathbb{R}} f(-t) \, d\mathbb{m}(t))\mathbb{1}$). (In the case where $\int_{\mathbb{R}} f(-t) \, d\mathbb{m}(t) = \infty$ we can give meaning to this object in the extended positive part of $\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R})$.) Therefore

$$(\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}})(a \otimes f) = \tau_{\mathcal{M}}(\mathscr{W}_{\mathcal{F}}(a \otimes f)) = \tau_{\mathcal{M}}(a) \cdot \int_{\mathbb{R}} f(-t) \, d\mathbb{m}(t).$$

However we know that $\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}} = \tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}}(h_{\mathcal{F}}(\cdot))$ where $h_{\mathcal{F}}$ is as in equation (6.5) above. So on simple tensors $(a \otimes f) \in (\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R}))_{+}$, $\tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}}$ must then have the form

$$\begin{aligned}(\tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}})(a \otimes f) &= (\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}})(h_{\mathcal{F}}^{-1}(a \otimes f)) \\ &= \tau_{\mathcal{M}}(a) \cdot \int_{\mathbb{R}} e^{t} f(-t) \, d\mathbb{m}(t).\end{aligned}$$

Since for any Borel set $E \subseteq \mathbb{R}$ we have that $\mathbb{m}(E) = \mathbb{m}(-E)$, we will for any positive Borel function g on $\mathbb{R}$ have $\int_{\mathbb{R}} g(-t) \, d\mathbb{m}(t) = \int_{\mathbb{R}} g(t) \, d\mathbb{m}(t).$ Hence we have that

$$(\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}})(a \otimes f) = \tau_{\mathcal{M}}(a) \cdot \int_{\mathbb{R}} f(t) \, d\mathbb{m}(t),$$

and

$$(\tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}})(a \otimes f) = \tau_{\mathcal{M}}(a) \cdot \int_{\mathbb{R}} f(t) \, e^{-t} d\mathbb{m}(t).$$

With $\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}}$ and $\tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}}$ represented in this form, the sign change effected in the variable of the second coordinate, ensures that in this context the density $\frac{d(\widetilde{\tau_{\mathcal{M}}} \circ \alpha_{\mathcal{F}})}{d(\tau_{\mathfrak{M}} \circ \alpha_{\mathcal{F}})}$ is (by slight abuse of notation) just the simple tensor $\mathbb{1} \otimes e^{t}$ $(t \in \mathbb{R})$. We collate the observations made above in the following result:

THEOREM 6.74. *Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau_{\mathcal{M}}$. Write $\tau_{\mathfrak{M}}$ for the canonical trace on $\mathcal{M} \rtimes_{\tau} \mathbb{R}$. Up to Fourier transform we may then identify $\mathcal{M} \rtimes_{\tau}$*

$\mathbb{R}$ *with the von Neumann algebra tensor product* $\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R})$. *Under this identification, the dual action takes the form* $\widetilde{\theta}_t(a\otimes f)=(a\otimes\ell_t f)$ *on the simple tensors of* $\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R})$. *In their action on* $\mathcal{M}\rtimes_\tau\mathbb{R}$, *the dual trace* $\widetilde{\tau}_{\mathcal{M}}$ *and* $\tau_{\mathfrak{M}}$ *will under this identification be of the form*

$$\widetilde{\tau}_{\mathcal{M}}\equiv\tau_{\mathcal{M}}\otimes\int_{\mathbb{R}}(\cdot)\,d\mathfrak{m}_L(t)\ \ and\ \ \tau_{\mathfrak{M}}\equiv\tau_{\mathcal{M}}\otimes\int_{\mathbb{R}}(\cdot)\,e^{-t}d\mathfrak{m}(t)$$

where $\mathfrak{m}$ *denotes Lebesgue measure. In this representation the derivative* $\frac{d\widetilde{\tau}_{\mathcal{M}}}{d\tau_{\mathfrak{M}}}$ *may (by slight abuse of notation) be identified with* $\mathbb{1}\otimes e^t$, *which clearly commutes with each element of* $\mathcal{M}\overline{\otimes}L^{\infty}(\mathbb{R})$ *(with respect to the strong product).*

L^p and Orlicz spaces for general von Neumann algebras

7.1 The semifinite setting revisited

In this section we will indicate how the theory of L^p and Orlicz spaces for semifinite algebras, may be realised using crossed product techniques. So throughout this section we will assume that $\mathcal{M}$ is a semifinite von Neumann algebra equipped with a faithful normal semifinite trace $\tau_{\mathcal{M}}$. Theorem 6.74 will play a crucial role in our analysis. We will in fact freely identify $\mathcal{M} \rtimes_\tau \mathbb{R}$ with $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$ throughout this section. Readers are therefore well-advised to keep one eye on Theorem 6.74 as they read this section. The trace on $\mathcal{M} \overline{\otimes} L^\infty(\mathbb{R})$ will be denoted by $\tau_{\mathfrak{M}}$.

The following adaptation of Proposition 6.70 is the fountainhead of the theory developed in this section.

LEMMA 7.1. *Let f, g be commuting positive operators affiliated to an arbitrary von Neumann algebra $\mathcal{M}$. Also let Ψ be an Orlicz function and $\mathbf{f}^\Psi$ be the fundamental function of $L^\Psi(\mathbb{R})$ equipped with the Luxemburg norm. Then $\chi_{(1,\infty)}(\Psi(g)f) = \chi_{(1,\infty)}(\mathbf{f}^\Psi(f)g)$*

PROOF. Let $\alpha, \beta > 0$ be given. We proceed to show that $\alpha\Psi(\beta) \leq 1 \Leftrightarrow \beta \leq \Psi^{-1}(\frac{1}{\alpha})$. Since $\Psi^{-1}(\Psi(t)) \geq t$ (see part(d) of Remark 5.30), it is clear that

$$\alpha\Psi(\beta) \leq 1 \Leftrightarrow \Psi(\beta) \leq \frac{1}{\alpha} \Rightarrow \beta \leq \Psi^{-1}(1/\alpha),$$

and hence it only remains to show that $\beta \leq \Psi^{-1}(\frac{1}{\alpha}) \Rightarrow \alpha\Psi(\beta) \leq 1$. Clearly

$$\beta \leq \Psi^{-1}(1/\alpha) \Rightarrow \Psi(\beta) \leq \Psi(\Psi^{-1}(1/\alpha)).$$

Since $\Psi(\Psi^{-1}(1/\alpha)) \leq 1/\alpha$ (by part(d) of Remark 5.30), we have that $\beta \leq \Psi^{-1}(\frac{1}{\alpha}) \Rightarrow \alpha\Psi(\beta) \leq 1$ as required. It therefore follows that

$$\alpha\Psi(\beta) \leq 1 \Leftrightarrow \beta\frac{1}{\Psi^{-1}(1/\alpha)} = \beta\mathbf{f}^\Psi(\alpha) \leq 1,$$

or equivalently

$$\alpha\Psi(\beta) > 1 \Leftrightarrow \beta\mathbf{f}_\Psi(\alpha) > 1.$$

Since f and g are commuting positive operators affiliated to $\mathcal{M}$, the von Neumann algebra generated by their spectral projections is abelian, and hence may be represented as some $L^\infty(\Omega, \Sigma, \nu)$-space. By the Borel functional calculus, both f and g then appear as almost everywhere finite Borel functions on the measure space (Ω, Σ, ν). However given two positive almost everywhere finite Borel functions f and g on $\mathbb{R}$, the above equivalence ensures that the sets $\{t \in \mathbb{R}\colon f(t)\Psi(g(t)) > 1\}$ and $\{t \in \mathbb{R}\colon \mathbf{f}_\Psi(f(t))g(t) > 1\}$ differ by a set of measure 0, and hence that the characteristic functions corresponding to these sets, agree almost everywhere. So as elements of $L^\infty(\Omega, \Sigma, \nu)$, they must agree. But in the context of the von Neumann algebra $\mathcal{R}$, these characteristic functions are just the spectral projections $\chi_{(1,\infty)}(\Psi(g)f)$ and $\chi_{(1,\infty)}(\mathbf{f}^\Psi(f)g)$. This suffices to prove the lemma. $\qquad\square$

THEOREM 7.2. *Let $a\,\eta\,\mathcal{M}$ be given. Let Ψ be an Orlicz function and let $\mathbf{f}^\Psi$ be the fundamental function of $L^\Psi(\mathbb{R})$ equipped with the Luxemburg norm. For any $\epsilon > 0$ we then have that*

$$d_{(a\otimes\mathbf{f}^\Psi(\exp))}(\epsilon) = \tau_\mathcal{M}(\Psi(|a|/\epsilon)).$$

PROOF. Observe that the operators $(\mathbb{1} \otimes \exp)$ and $|a \otimes 1| = |a| \otimes 1$ are commuting positive operators affiliated to $\mathcal{M}\overline{\otimes}L^\infty(\mathbb{R})$. We may now apply the Lemma to this pair, with $(\mathbb{1} \otimes \exp)$ playing the role of f, and $|a| \otimes 1$ the role of g, to see that

$$
\begin{aligned}
\chi_{(1,\infty)}(|a \otimes \mathbf{f}^\Psi(\exp)|) &= \chi_{(1,\infty)}((|a| \otimes 1)(\mathbb{1} \otimes \mathbf{f}^\Psi(\exp))) \\
&= \chi_{(1,\infty)}((|a| \otimes 1)\mathbf{f}^\Psi(\mathbb{1} \otimes \exp)) \\
&= \chi_{(1,\infty)}(\Psi(|a| \otimes 1)(\mathbb{1} \otimes \exp)) \\
&= \chi_{(1,\infty)}((\Psi(|a|) \otimes 1)(\mathbb{1} \otimes \exp)) \\
&= \chi_{(1,\infty)}(\Psi(|a|) \otimes \exp)
\end{aligned}
$$

By Proposition 6.70 we have that

$$\tau_\mathfrak{M}(\chi_{(1,\infty)}(|a \otimes \mathbf{f}^\Psi(\exp)|)) = \psi(\mathbb{1})$$

where ψ is the weight $f \mapsto \tau_\mathcal{M}(\Psi(|a|)^{1/2}f\Psi(|a|)^{1/2})$. (Note that Proposition 6.67 ensures that Proposition 6.70 is applicable to $\Psi(|a|) \otimes \exp$.) In other words

$$\tau_\mathfrak{M}(\chi_{(1,\infty)}(|a \otimes \mathbf{f}^\Psi(\exp)|)) = \tau_\mathcal{M}(\Psi(|a|)).$$

Given $\epsilon > 0$, we therefore have that

$$
\begin{aligned}
d_{a\otimes \mathbf{f}^\Psi(\exp)}(\epsilon) &= \tau_{\mathfrak{M}}(\chi_{(\epsilon,\infty)}(|a\otimes \mathbf{f}^\Psi(\exp)|)) \\
&= \tau_{\mathfrak{M}}(\chi_{(\epsilon,\infty)}(|a|\otimes \mathbf{f}^\Psi(\exp))) \\
&= \tau_{\mathfrak{M}}(\chi_{(1,\infty)}((|a|/\epsilon)\otimes \mathbf{f}^\Psi(\exp))) \\
&= \tau_{\mathcal{M}}(\Psi(|a|/\epsilon))
\end{aligned}
$$

as required.   □

The first consequence of this theorem gives us some clue as to how $L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ may be realised inside $\widetilde{\mathfrak{M}}$.

COROLLARY 7.3 (Luxemburg norm). *Let Ψ be an Orlicz function. Given $a\,\eta\,\mathcal{M}$, we have that $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if $a \otimes \mathbf{f}^\Psi(\exp)$ belongs to $\widetilde{\mathfrak{M}}$. Moreover for any $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ we have the following formula for the Luxemburg norm:*

$$
\|a\|_\Psi = \mathbf{m}_{(a\otimes \mathbf{f}_\Psi(\exp))}(1).
$$

PROOF. Recall that by Remark 5.33 we have that $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if $\tau_{\mathcal{M}}(\Psi(\alpha|a|)) < \infty$ for some $\alpha > 0$. But by the theorem, this is the same as saying that $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if $d_{(a\otimes \mathbf{f}^\Psi(\exp))}(1/\alpha) < \infty$ for some $\alpha > 0$. From the basic theory of τ-measurable operators we further have that $d_{(a\otimes \mathbf{f}^\Psi(\exp))}(\epsilon) < \infty$ for some $\epsilon > 0$ if and only if $(a \otimes \mathbf{f}_\Psi(\exp))$ is $\tau_{\mathfrak{M}}$-measurable. Combining these facts, leads to the conclusion that $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if $a \otimes \mathbf{f}^\Psi(\exp)$ is $\tau_{\mathfrak{M}}$-measurable.

Now let $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ be given. To see the second claim we use the theorem to conclude that

$$
\begin{aligned}
\|a\|_\Psi &= \inf\{\epsilon > 0 \colon \tau_{\mathcal{M}}(\Psi(|a|/\epsilon) \le 1\} \\
&= \inf\{\epsilon > 0 \colon d_{(a\otimes \mathbf{f}^\Psi(\exp))}(\epsilon) \le 1\} \\
&= \inf\{\epsilon \ge 0 \colon d_{(a\otimes \mathbf{f}^\Psi(\exp))}(\epsilon) \le 1\} \\
&= \mathbf{m}_{(a\otimes \mathbf{f}^\Psi(\exp))}(1).
\end{aligned}
$$

(Here the second to last equality follows from the fact that the function $s \mapsto d_{(a\otimes \mathbf{f}^\Psi(\exp))}(s)$ is right-continuous.)   □

The preceding theorem and corollary describe how one may use the simple tensors in $\widetilde{\mathfrak{M}}$, to realise an isometric copy of $L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$ inside $\widetilde{\mathfrak{M}}$. What remains to be done is to find a reliable test for identifying those elements of $\widetilde{\mathfrak{M}}$ that are of the form $a \otimes \mathbf{f}^\Psi(\exp)$ for some $a \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$.

THEOREM 7.4. *Let a Young function* Ψ, *and* $\widetilde{a} \in \widetilde{\mathfrak{M}}$ *be given. Then* $\widetilde{a}$ *is of the form* $\widetilde{a} = a \otimes \mathbf{f}^{\Psi}(\exp)$ *for some* $a \in L^{\Psi}(\mathcal{M}, \tau_{\mathcal{M}})$ *if and only if* $\widetilde{\theta}_s(\widetilde{a}) = v_s\widetilde{a}$ *for all* $s \in \mathbb{R}$, *where* $v_s = \mathbf{f}^{\Psi}(e^{-s}h)\mathbf{f}^{\Psi}(h)^{-1}$. *(Here h is the density* $\frac{d\widetilde{\tau}_{\mathcal{M}}}{d\tau_{\mathfrak{M}}} = \mathbb{1} \otimes \exp$.)*

We pause to point out that the operators v_s are all bounded. This fact will be verified in a more general context in Lemma 7.7.

PROOF. The "if" part is easy to see. If indeed $a \otimes \mathbf{f}^{\Psi}(\exp) \in \widetilde{\mathfrak{M}}$, then for any $s \in \mathbb{R}$,

$$
\begin{aligned}
\widetilde{\theta}_s(a \otimes \mathbf{f}^{\Psi}(\exp)) &= a \otimes \ell_s \mathbf{f}^{\Psi}(\exp) \\
&= a \otimes \mathbf{f}^{\Psi}(e^{-s}\exp) \\
&= v_s(a \otimes \mathbf{f}^{\Psi}(\exp)).
\end{aligned}
$$

Now suppose that we are given some $\widetilde{a} \in \widetilde{\mathfrak{M}}$ such that $\widetilde{\theta}_s(\widetilde{a}) = v_s\widetilde{a}$ for all $s \in \mathbb{R}$. Recall that $h = \mathbb{1} \otimes \exp$ commutes with every element of $\widetilde{\mathfrak{M}}$, and therefore also each operator v_s. (See Theorem 6.74.) On the basis of this fact it is now fairly easy to see that then $\widetilde{\theta}_s(|\widetilde{a}|^2) = \widetilde{\theta}_s(\widetilde{a}^*)\widetilde{\theta}_s(\widetilde{a}) = v_s^2|\widetilde{a}|^2$, and hence that $\widetilde{\theta}_s(|\widetilde{a}|) = v_s|\widetilde{a}|$ for all $s \in \mathbb{R}$. Let $\widetilde{a} = \widetilde{u}|\widetilde{a}|$ be the polar decomposition of $\widetilde{a}$. Now for a given $s \in \mathbb{R}$, it is clear that $\widetilde{\theta}_s(\widetilde{u})\widetilde{\theta}_s(|\widetilde{a}|)$ is a polar decomposition of $\widetilde{\theta}_s(|\widetilde{a}|)$. But since $\widetilde{\theta}_s(|\widetilde{a}|) = v_s|\widetilde{a}|$, we have $\widetilde{\theta}_s(\widetilde{a}) = v_s\widetilde{a} = uv_s|\widetilde{a}| = u\widetilde{\theta}_s(|\widetilde{a}|)$. Thus $\widetilde{\theta}_s(\widetilde{a}) = u\widetilde{\theta}_s(|\widetilde{a}|)$ is then also a polar decomposition of $\widetilde{a}$. The uniqueness of the polar decomposition then ensures that $\widetilde{\theta}_s(\widetilde{u}) = \widetilde{u}$. This holds for every $s \in \mathbb{R}$, which by Proposition 6.69, ensures that $\widetilde{u} \in \mathcal{M} \otimes 1$. That is $\widetilde{u} = u \otimes 1$ for some partial isometry $u \in \mathcal{M}$. Thus if we can prove that the claim holds for $|\widetilde{a}|$, it will also hold for $\widetilde{a}$. We therefore may, and do, assume that $\widetilde{a} \geq 0$.

By Theorem 6.74, $h = \mathbb{1} \otimes \exp$ and $\widetilde{a}$, and hence also $\mathbf{f}^{\Psi}(h)^{-1}$ and $\widetilde{a}$, are commuting affiliated operators. So by the Borel functional calculus the strong product $\widetilde{b} = \mathbf{f}^{\Psi}(h)^{-1}\widetilde{a}$ is a densely defined positive operator affiliated to $\mathfrak{M}$. For this operator, we have that

$$
\begin{aligned}
\widetilde{\theta}_s(\widetilde{b}) &= \mathbf{f}^{\Psi}(\widetilde{\theta}_s h)^{-1} v_s \widetilde{a} \\
&= \mathbf{f}^{\Psi}(e^{-s}h)^{-1}[\mathbf{f}^{\Psi}(e^{-s}h)\mathbf{f}^{\Psi}(h)^{-1}]\widetilde{a} \\
&= \mathbf{f}^{\Psi}(h)^{-1}\widetilde{a} \\
&= \widetilde{b}
\end{aligned}
$$

for each $s \in \mathbb{R}$. Since $\widetilde{b} = \mathbf{f}^\Psi(h)^{-1}\widetilde{a}$ is positive, the operator $e^{-\widetilde{b}}$ is bounded. For this operator, we still have that $\widetilde{\theta}_s(e^{-\widetilde{b}}) = e^{-\widetilde{\theta}_s(\widetilde{b})} = e^{-\widetilde{b}}$ for all $s \in \mathbb{R}$. So by Proposition 6.69, there must then exist some $b \in \mathcal{M}_+$ such that $e^{-\widetilde{b}} = b \otimes 1$. But then $\mathbf{f}^\Psi(h)^{-1}\widetilde{a} = \widetilde{b} = -\log(b \otimes 1) = -(\log(b) \otimes 1)$. Given that $\mathbf{f}^\Psi(h)^{-1} = \mathbf{f}^\Psi(\mathbb{1} \otimes \exp)^{-1} = (\mathbb{1} \otimes \mathbf{f}^\Psi(\exp))^{-1}$, we therefore have that $\widetilde{a} = -(\log(b) \otimes \mathbf{f}^\Psi(\exp))$. By Theorem 7.2, the $\tau_{\mathfrak{M}}$-measurability of $\widetilde{a}$ now ensures that $\log(b) \in L^\Psi(\mathcal{M}, \tau_{\mathcal{M}})$, and that $\widetilde{a}$ is of the required form. $\square$

We now gather the preceding analysis in the following theorem:

THEOREM 7.5. *Let Ψ be a Young function, and let $h = \dfrac{d\widetilde{\tau_{\mathcal{M}}}}{d\tau_{\mathfrak{M}}}$. Then the quantity $\widetilde{a} \mapsto \mathbf{m}_{\widetilde{a}}(1)$ is a norm on the space*

$$L^\Psi(\mathcal{M}) = \{\widetilde{a} \in \widetilde{\mathfrak{M}} \colon \widetilde{\theta}_s(\widetilde{a}) = v_s \widetilde{a} \text{ for all } s \in \mathbb{R}\},$$

where the operators v_s are defined by $v_s = \mathbf{f}^\Psi(e^{-s}h)\mathbf{f}^\Psi(h)^{-1}$ for each s. The mapping $(L^\Psi(\mathcal{M}, \tau_{\mathcal{M}}), \|\cdot\|_\Psi) \to (L^\Psi(\mathcal{M}), \mathbf{m}_{(\cdot)}(1)) : a \mapsto a \otimes \mathbf{f}^\Psi(\exp)$ is then a surjective isometric isomorphism.

We have one final task to perform in this section, and that is to describe the topology on $(L^\Psi(\mathcal{M}), \mathbf{m}_{(\cdot)}(1))$ more fully.

THEOREM 7.6. *For a Young function Ψ the norm topology on the space $L^\Psi(\mathcal{M})$ is homeomorphic to the relative topology induced by the topology of convergence in measure on $\widetilde{\mathfrak{M}}$.*

Note that this result also holds for L^∞!

PROOF. We remind the reader that the basic neighbourhoods of zero for the topology of convergence in measure on $\widetilde{\mathfrak{M}}$ are of the form

$$\mathcal{N}(\epsilon, \delta) = \{\widetilde{a} \colon d_{\widetilde{a}}(\epsilon) \leq \delta\}.$$

Let $1 > \epsilon > 0$ be given, and suppose that $\mathbf{m}_{\widetilde{a}}(1) < \epsilon$ for some $\widetilde{a} = a \otimes \mathbf{f}^\Psi(\exp) \in L^\Psi(\mathcal{M})$. But then there must exist an $0 < \alpha < \epsilon$ so that $d_{\widetilde{a}}(\alpha) \leq 1$. By Theorem 7.2, this ensures that $\tau_{\mathcal{M}}(\Psi(|a|/\alpha)) \leq 1$. Next notice that $\frac{1}{\sqrt{\alpha}} \geq 1$, since by assumption $0 < \alpha < \epsilon < 1$. We may therefore use the convexity of Ψ to conclude that $\frac{1}{\sqrt{\alpha}}\tau_{\mathcal{M}}(\Psi(|a|/\sqrt{\alpha})) \leq \tau_{\mathcal{M}}(\Psi(|a|/\alpha)) \leq 1$; in other words $\tau_{\mathcal{M}}(\Psi(|a|/\sqrt{\alpha})) \leq \sqrt{\alpha}$. On once again using Theorem 7.2, this can be reformulated as the claim that $d_{(a \otimes \mathbf{f}^\Psi(\exp))}(\sqrt{\alpha}) \leq \sqrt{\alpha}$. This ensures that $\widetilde{a} = a \otimes \mathbf{f}^\Psi(\exp) \in \mathcal{N}(\sqrt{\alpha}, \sqrt{\alpha})$.

Conversely suppose that for some $\epsilon, \delta > 0$ with $\delta \leq 1$, we have that $\widetilde{a} = a \otimes \mathbf{f}^{\Psi}(\exp) \in \mathcal{N}(\epsilon, \delta)$. Then $d_{(a \otimes \mathbf{f}^{\Psi}(\exp))}(\epsilon) \leq \delta \leq 1$. It then follows that $\mathbf{m}_{(a \otimes \mathbf{f}^{\Psi}(\exp))}(1) \leq \epsilon$. Thus (a_n) converges to 0 in the norm topology on $L^{\Psi}(\mathcal{M}, \tau_{\mathcal{M}})$ if and only if $((a_n \otimes \mathbf{f}^{\Psi}(\exp)))$ converges to 0 in the topology of convergence in measure on $\widetilde{\mathfrak{M}}$.

The norm topology must therefore be homeomorphic to the relative topology induced by the topology of convergence in measure on $\widetilde{\mathfrak{M}}$. $\qquad\square$

7.2 Definition and normability of general L^p and Orlicz spaces

Throughout the rest of this chapter $\mathcal{M}$ will be a von Neumann algebra and φ the "reference" faithful normal semifinite weight on $\mathcal{M}$. The crossed product $\mathfrak{M} = \mathcal{M} \rtimes_{\varphi} \mathbb{R}$ will play a crucial role in the development of the theory of Orlicz and L^p-spaces for general von Neumann algebras. Readers are therefore advised to make sure that they are familiar with the basic structural theory of this algebra as presented in Chapter 6 before attempting to make sense of the theory presented here. In view of the repeated use of $\mathfrak{M}$ in developing this theory, to simplify notation we will identify $\mathcal{M}$ with the copy $\pi(\mathcal{M})$ living inside $\mathfrak{M}$.

On the basis of the previous section, we are now ready to rigorously define L^p and Orlicz spaces for general von Neumann algebras in a manner which is a natural extension of the definition of these spaces for semifinite algebras equipped with a faithful normal semifinite trace. Before doing that, we need the following lemma:

LEMMA 7.7. *Let φ be a faithful normal semifinite weight on a von Neumann algebra $\mathcal{M}$, and let $h = \frac{d\widetilde{\varphi}}{d\tau}$ be the density of the dual weight $\widetilde{\varphi}$ on the crossed product $\mathfrak{M}$. Let $\mathbf{f} : [0, \infty) \to [0, \infty)$ be any quasi-concave function. Then the operators $v_s = \mathbf{f}(e^{-s}h)\mathbf{f}(h)^{-1}$ ($s \in \mathbb{R}$) are always bounded, with norm between 1 and e^{-s}.*

PROOF. This can be proven by noting that the facts that $t \mapsto \mathbf{f}(t)$ is increasing and $t \mapsto \frac{\mathbf{f}(t)}{t}$ decreasing, ensure that for any $t > 0$, the number $\frac{\mathbf{f}(e^{-s}t)}{\mathbf{f}(t)}$ lies between 1 and e^{-s}. The continuous functional calculus does the rest. $\qquad\square$

DEFINITION 7.8. Let φ be a fixed (canonical) faithful normal semifinite weight on the von Neumann algebra $\mathcal{M}$, and let $h = \frac{d\widetilde{\varphi}}{d\tau}$ be the density of the dual weight $\widetilde{\varphi}$ on the crossed product $\mathfrak{M}$. Given a Young function

Ψ, the Orlicz space $L^\Psi(\mathcal{M})$ associated with $\mathcal{M}$ (corresponding to the Luxemburg-Nakano norm) is defined to be

$$L^\Psi(\mathcal{M}) = \{a \in \widetilde{\mathfrak{M}} \colon \theta_s(a) = v_s^{1/2} a v_s^{1/2} \text{ for all } s \in \mathbb{R}\},$$

where $v_s = \mathbf{f}^\Psi(e^{-s}h)\mathbf{f}^\Psi(h)^{-1}$. We formally define the Luxemburg-Nakano "norm" on $L^\Psi(\mathcal{M})$ to be the quantity $\|a\|_\Psi = \mathbf{m}_a(1)$. In the case where $\Psi(t) = t^p$ $(1 \leq p)$, we will denote this quantity by $\|\cdot\|_p$. In this case we have that $\mathbf{f}^\Psi(e^{-s}h)\mathbf{f}^\Psi(h)^{-1} = e^{-s/p}\mathbb{1}$ for all $s \in \mathbb{R}$. So by analogy with the above, we may for all $0 < p < \infty$ define $L^p(\mathcal{M})$ to be the space

$$L^p(\mathcal{M}) = \{a \in \widetilde{\mathfrak{M}} \colon \theta_s(a) = e^{-s/p}a \text{ for all } s \in \mathbb{R}\}.$$

We note that the definition of Orlicz spaces for general von Neumann algebras given in [**Lab13**], differs from the one given above. However in [**LM**, Lemma 4.10] the two definitions were shown to be equivalent. For the sake of completeness we present the alternative definition:

DEFINITION 7.9 (Alternative definition). Let Ψ be an Orlicz function. We define the Orlicz space $L^\Psi(\mathcal{M})$ corresponding to the Luxemburg norm to be the space of all operators $a \in \widetilde{\mathfrak{M}}$ for which $[e\mathbf{f}_{\Psi^*}(h)^{1/2}]a[\mathbf{f}_{\Psi^*}(h)^{1/2}f] \in L^1(\mathcal{M})$ for all projections $e, f \in \mathfrak{n}_\varphi$.

REMARK 7.10. Having defined Orlicz spaces for general von Neumann algebras, our task now is to further justify the choice of $\mathbf{m}_f(1)$ as the natural Luxemburg-Nakano norm for these spaces. We saw in Theorem 4.6, that $\rho(f) = d_f(1)$ is the natural modular on $\widetilde{\mathfrak{M}}$. So the obvious candidate for a Luxemburg-Nakano "norm" on an Orlicz space $L^\Psi(\mathcal{M})$, would be

$$\inf\{\epsilon > 0 \colon \rho(\epsilon^{-1}|f|) \leq 1\} = \inf\{\epsilon > 0 \colon d_f(\epsilon) \leq 1\} = \mathbf{m}_f(1).$$

A fact which strongly supports this proposal is the fact that the analysis in the first section of this chapter, shows that in the case where φ is a faithful normal semifinite trace, the quantity $\mathbf{m}_{(\cdot)}(1)$ is indeed a norm on $L^\Psi(\mathcal{M})$, and that when equipped with this norm, the space $(L^\Psi(\mathcal{M}), \mathbf{m}_{(\cdot)}(1))$ is an isometric copy of the space $(L^\Psi(\mathcal{M}, \varphi), \|\cdot\|_\Psi)$. We now show that the quantity $\mathbf{m}_{(\cdot)}(1)$ is indeed a quasi-norm on each of the spaces introduced above, and describe the quasi-norm topology on those spaces. For this the following result is crucial.

PROPOSITION 7.11. *Let Ψ be a Young function. For any $a \in L^\Psi(\mathcal{M})$, we have that*

$$t\mathbf{m}_a(t) \leq \mathbf{m}_a(1) \qquad \text{for all} \qquad 0 < t \leq 1.$$

PROOF. Let $a \in L^\Psi(\mathcal{M})$ and $0 < t$ be given. We may of course write such a t as $t = e^s$ where $s \in \mathbb{R}$. Observe that for any $r > 0$, we then have that

$$
\begin{aligned}
\frac{1}{t} d_a(r) &= e^{-s} \tau(\chi_{(r,\infty)}(|a|)) \\
&= \tau(\theta_s(\chi_{(r,\infty)}(|a|))) \\
&= \tau(\chi_{(r,\infty)}(\theta_s|a|)) \\
&= \tau(\chi_{(r,\infty)}(|\theta_s(a)|)) \\
&= d_{\theta_s(a)}(r).
\end{aligned}
$$

With v_s as in Definition 7.8, it now follows that

$$
\begin{aligned}
\mathbf{m}_a(t) &= \inf\{r > 0 \colon d_a(r) \leq t\} \qquad\qquad (7.1) \\
&= \inf\{r > 0 \colon d_{\theta_s(a)}(r) \leq 1\} \\
&= \mathbf{m}_{\theta_s(a)}(1) \\
&= \mathbf{m}_{v_s^{1/2} a v_s^{1/2}}(1)
\end{aligned}
$$

for all $t = e^s$.

For any $c \in \widetilde{\mathfrak{M}}$ and $b \in \mathfrak{M}$, it is a simple matter to see that

$$
\mathbf{m}_{bc}(1) = \mathbf{m}_{c^*|b|^2 c}(1)^{1/2} \leq \|b\| \cdot \mathbf{m}_{c^* c}(1)^{1/2} = \|b\| \cdot \mathbf{m}_c(1)
$$

and similarly that $\mathbf{m}_{cb}(1) \leq \|b\| \cdot \mathbf{m}_c(1)$. With s and t as before, we now pass to the case where $s \leq 0$, or equivalently where $0 < t \leq 1$. In this case $\|v_s\| \leq e^{-s}$ by Lemma 7.7. As required, it therefore follows from equation (7.1) and the above computations that $\mathbf{m}_a(t) \leq \|v_s\| \cdot \mathbf{m}_a(1) \leq \frac{1}{t} \mathbf{m}_a(1)$ for all such t. $\qquad\square$

The above proposition now yields the following important theorem.

THEOREM 7.12. *The quantity $\mathbf{m}_a(1)$ ($a \in L^\Psi(\mathcal{M})$) is a quasinorm for $L^\Psi(\mathcal{M})$. The topology induced on $L^\Psi(\mathcal{M})$ by this quasinorm is complete and is homeomorphic to the topology of convergence in measure inherited from $\widetilde{\mathfrak{M}}$.*

PROOF. First suppose that we are given $a \in L^\Psi(\mathcal{M})$ with $\mathbf{m}_a(1) = 0$. By the preceding proposition, we then clearly have that $\mathbf{m}_a(t) = 0$ for all $0 < t \leq 1$. Since the decreasing rearrangement is right-continuous, this then yields the fact that $\|a\| = \lim_{t \searrow 0} \mathbf{m}_a(t) = 0$, which can only be the case if $a = 0$.

The properties of the decreasing rearrangement ensures that $\mathbf{m}_{\lambda a}(1) = |\lambda|\mathbf{m}_a(1)$ for any scalar λ and any $a \in L^\Psi(\mathcal{M})$. It remains to show that $\mathbf{m}_{(\cdot)}(1)$ satisfies a generalised triangle inequality. Given $a, b \in L^\Psi(\mathcal{M})$, we may conclude from the properties of the decreasing rearrangement and the preceding theorem that

$$\mathbf{m}_{(a+b)}(1) \leq 2\left(\frac{1}{2}\mathbf{m}_a(1/2) + \frac{1}{2}\mathbf{m}_b(1/2)\right) \leq 2(\mathbf{m}_a(1) + \mathbf{m}_b(1)).$$

We now show that $L^\Psi(\mathcal{M})$ is a closed subspace of $\widetilde{\mathfrak{M}}$ with respect to the topology of convergence in measure. Then once we have established that the topology on $L^\Psi(\mathcal{M})$ induced by $\mathbf{m}_{(\cdot)}(1)$ agrees with the topology of convergence in measure, this closedness will suffice to prove the completeness of $L^\Psi(\mathcal{M})$. It is clear from the definition of $L^\Psi(\mathcal{M})$, that membership of an element a of $\widetilde{\mathfrak{M}}$ to $L^\Psi(\mathcal{M})$ can be rephrased as the claim that a belongs to the intersection of the kernels of the operators

$$T_s : \widetilde{\mathfrak{M}} \to \widetilde{\mathfrak{M}} : a \mapsto \theta_s(a) - v_s^{1/2}av_s^{1/2} \quad s \in \mathbb{R},$$

where v_s is as in Definition 7.8. The operation $a \mapsto v_s^{1/2}av_s^{1/2}$ is clearly continuous with respect to the topology of convergence in measure, whereas the operation $a \mapsto \theta_s(a)$ was shown to be similarly continuous in Corollary 6.63. Thus by continuity, the kernels must be closed as claimed.

It remains to prove that the topology induced on $L^\Psi(\mathcal{M})$ by $\mathbf{m}_a(1)$ is precisely the topology of convergence in measure. The fact that convergence in measure implies convergence in the quasi-norm $a \mapsto \mathbf{m}_a(1)$, follows from Proposition 4.23. For the converse fix $0 < \epsilon \leq 1$, and suppose that we are given some $a \in L^\Psi(\mathcal{M})$ with $\mathbf{m}_a(1) < \epsilon^2$. By the preceding theorem, we have that $\mathbf{m}_a(\epsilon) \leq \frac{1}{\epsilon}\mathbf{m}_a(1) < \epsilon$. We may now once again use Proposition 4.23 to conclude that then a belongs to the basic neighbourhood of zero $\mathcal{N}(\epsilon, \epsilon)$ of $\widetilde{\mathfrak{M}}$. Thus any sequence in $L^\Psi(\mathcal{M})$ that converges to zero in the quasinorm $a \mapsto \mathbf{m}_a(1)$, converges to zero in measure. $\qquad \square$

Inspired by the above theorem we now make the following definition.

DEFINITION 7.13. For each $L^\Psi(\mathcal{M})$ we define $\mathbf{m}_{(\cdot)}(1)$ to be the Luxemburg-Nakano quasinorm on $L^\Psi(\mathcal{M})$, and will henceforth denote this quasinorm by $\|\cdot\|_\Psi$ where appropriate. In the case where $\Psi(t) = t^p$ for some $p \geq 1$, we will write $\|\cdot\|_p$ for this quasinorm.

A fact worth noting at this stage is the following uniqueness theorem.

PROPOSITION 7.14. *Let φ_1 and φ_2 be two faithful normal semifinite weights on $\mathcal{M}$, and let $\mathfrak{M}_1 = \mathcal{M} \rtimes_{\varphi_1} \mathbb{R}$ and $\mathcal{M}_2 = \mathcal{M} \rtimes_{\varphi_2} \mathbb{R}$ Then the *-isomorphism $\mathscr{I}$ from $\widetilde{\mathfrak{M}}_1$ onto $\widetilde{\mathfrak{M}}_2$ constructed in Theorem 6.65, will for each $1 \leq p < \infty$ restrict to a linear isometry from $L^p(\mathcal{M}, \varphi_1)$ onto $L^p(\mathcal{M}, \varphi_1)$.*

PROOF. Since $\mathscr{I}$ *-isomorphically identifies from $\widetilde{\mathfrak{M}}_1$ with $\widetilde{\mathfrak{M}}_2$, it is clear that for any $a \in \widetilde{\mathfrak{M}}_1$ we have that $\mathbf{m}_a(1) = \mathbf{m}_{\mathscr{I}(a)}(1)$. So all that needs to be checked is that $\mathscr{I}(L^p(\mathcal{M}, \varphi_1)) = L^p(\mathcal{M}, \varphi_2)$. This in turn follows from Definition 7.8 and the fact that $\mathscr{I} \circ \theta_t = \theta_t \circ \mathscr{I}$. $\square$

EXAMPLE 7.15. We remind the reader that $L^\infty(\mathcal{M}) = \mathcal{M}$ itself is an Orlicz space generated by the Young function

$$\Psi_\infty(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1 \\ \infty & \text{if } 1 < t \end{cases}.$$

This then raises the question of how the Luxemburg-Nakano quasinorm computed in the preceding theorem compares to the operator norm on $\mathcal{M}$. We therefore proceed to compute $\|\cdot\|_{\Psi_\infty}$ for this space. Firstly notice that for Ψ_∞ as above,

$$\mathbf{f}^{\Psi_\infty}(t) = \frac{1}{(\Psi_\infty)^{-1}(1/t)} = \begin{cases} 0 & \text{if } t = 0 \\ 1 & \text{if } 0 < t \end{cases}.$$

We leave the verification of this fact as an exercise. However what is clear from this fact is that in this setting $v_s = \mathbf{f}^{\Psi_\infty}(e^{-s}h).\mathbf{f}^{\Psi_\infty}(h)^{-1} = \mathbb{1}$. So by equation (7.1) we will in this setting have that $\mathbf{m}_a(t) = \mathbf{m}_a(1)$ for all $t \in (0,1]$ and all $a \in \mathcal{M}$. We may now finally use the left-continuity of $t \mapsto \mathbf{m}_a(t)$ to conclude that $\|a\| = \lim_{t \searrow 0} \mathbf{m}_a(t) = \mathbf{m}_a(1) = \|a\|_{\Psi_\infty}$ for all $a \in \mathcal{M}$.

REMARK 7.16. Despite the elegance of the preceding example, it is at this stage not clear that these Orlicz spaces are in general normable. For now the strongest statement we may conclude from Theorem 7.12, is that in this generality, Orlicz spaces may only admit a quasinorm instead of a norm. To understand why this might be the case, we look at the modular approach to Orlicz spaces (see Definition 4.4). Given some Young function Ψ, in its action on $L^\Psi(X, \Sigma, \nu)$, the quantity $\rho(|f|) = \int \Psi(|f|) \, d\nu$ can be shown to be a convex modular with $L^\Psi(X, \Sigma, \nu)$ a modular space. The

prescription $\|f\| = \inf\{\epsilon > 0: \rho(\epsilon^{-1}x) \leq 1\}$ then yields the Luxemburg-Nakano norm on $L^\Psi(X, \Sigma, \nu)$. In the theory of modular spaces the so-called Amemiya norm is given by the formula $\|f\|_\Psi^A = \inf\{k^{-1} > 0(1 + \rho(kf)): k > 0\}$. It was only fairly recently that Hudzik and Maligranda showed that in the measure space setting described above, this norm corresponds to the Orlicz norm whenever the measure space is σ-finite [**HM00**]. These facts suggest that if in the type III case we can identify the correct modular, we may be able to construct both a Luxemburg-Nakano and Orlicz norm. Since the quasinorm topology on $L^\Psi(\mathcal{M})$ agrees with the topology of convergence in measure, it is clear from Theorem 4.6 that the appropriate modular to use in this setting is nothing but $\rho(f) = d_f(1)$ – the semi-modular that determines the topology of $\widetilde{\mathfrak{M}}$. It is precisely here that the problem lies. With the structure, currently at our disposal, there is no obvious way of proving that in this generality $\rho(f) = d_f(1)$ is indeed a convex modular. The best we can do at this stage is to show that it is a semi-modular. Without such convexity, the theory regarding the Amemiya norm fails, and the Luxemburg-Nakano "norm" cannot be expected to be a norm (see for example the prerequisites for [**Mus83**, Theorem I.10]). For algebras equipped with a faithful normal semifinite trace the situation is more regular. We know from the previous chapter that in this setting, we will for any $f \in L^\Psi(\mathcal{M}, \tau)$ have that $d_{f \otimes \mathbf{f}^\Psi(f)}(1) = \tau(\Psi(|f|))$ — a fact which ensures that in its action on $\{f \otimes \mathbf{f}^\Psi(f): f \in L^\Psi(\mathcal{M}, \tau)\} \subseteq (\mathcal{M} \widetilde{\overline{\otimes} L^\infty}(\mathbb{R}))$, the quantity $d_{(\cdot)}(1)$ is in this case indeed a convex modular.

The negative tone of the previous remark aside, there is in fact a remnant of the equivalence of the Orlicz and Luxemburg-Nakano norms that survives the transition to the general setting, and also a class of more regular Orlicz spaces which do turn out to be normed. We discuss each of these issues in turn. To achieve this objective, we will need the following classical fact.

PROPOSITION 7.17 ([**BS88**, Lemma IV.8.16]). *Let Ψ be a Young function. For any $t \geq 0$ we will then have that $t \leq \Psi^{-1}(t)(\Psi^*)^{-1}(t) \leq 2t$.*

COROLLARY 7.18. *Given a Young function Ψ, let $L_\Psi(\mathcal{M})$ denote the space defined by*

$$L_\Psi(\mathcal{M}) = \{a \in \widetilde{\mathfrak{M}}: \theta_s(a) = \widetilde{v}_s^{1/2} a \widetilde{v}_s^{1/2} \text{ for all } s \leq 0\},$$

where $\widetilde{v}_s = \mathbf{f}_\Psi(e^{-s}h)\mathbf{f}_\Psi(h)^{-1}$. (Here $\mathbf{f}_\Psi$ is the fundamental function corresponding to the Orlicz norm on $L^\Psi(0,\infty)$.) The quantity $\|a\|_\Psi^O = \mathbf{m}_a(1)$ is a quasinorm for $L_\Psi(\mathcal{M})$, with the quasinormed topology homeomorphic to the topology of convergence in measure. Moreover there is a canonical bijection ι from $L^\Psi(\mathcal{M})$ onto $L_\Psi(\mathcal{M})$ for which we have that $\|a\|_\Psi \le \|\iota(a)\|_\Psi^O \le 2\|a\|_\Psi$ for all $a \in L^\Psi(\mathcal{M})$.

PROOF. The first claim may be proved using similar arguments as those used for $L^\Psi(\mathcal{M})$. To see the second, observe that on replacing t with $\frac{1}{u}$, the inequalities in the preceding proposition may be rephrased as the claim that

$$\frac{1}{\Psi^{-1}(1/u)} \le u(\Psi^*)^{-1}(1/u) \le 2\frac{1}{\Psi^{-1}(1/u)} \text{ for all } u > 0,$$

or equivalently that

$$\mathbf{f}^\Psi(u) \le \mathbf{f}_\Psi(u) \le 2\mathbf{f}^\Psi(u) \text{ for all } u > 0.$$

Next consider the mapping $\iota : a \mapsto w_\Psi^{1/2}aw_\Psi^{1/2}$ ($a \in \widetilde{\mathfrak{M}}$) where $w_\Psi = \mathbf{f}_\Psi(h)\mathbf{f}^\Psi(h)^{-1}$. The previously centred inequality ensures that $\mathbb{1} \le w_\Psi \le 2\mathbb{1}$. Now observe that for v_s as in Definition 7.8 and $\widetilde{v}_s$ as in the hypothesis, we have that

$$
\begin{aligned}
\theta_s(w_\Psi)v_s &= \mathbf{f}_\Psi(\theta_s(h))^{1/2}\mathbf{f}^\Psi(\theta_s(h))^{-1}.\mathbf{f}^\Psi(e^{-s}h)\mathbf{f}^\Psi(h)^{-1} \\
&= \mathbf{f}_\Psi(e^{-s}h)^{1/2}\mathbf{f}^\Psi(e^{-s}h)^{-1}.\mathbf{f}^\Psi(e^{-s}h)\mathbf{f}^\Psi(h)^{-1} \\
&= \mathbf{f}_\Psi(e^{-s}h)^{1/2}\mathbf{f}^\Psi(h)^{-1} \\
&= \widetilde{v}_s w_\Psi.
\end{aligned}
$$

This ensures that for any $a \in L^\Psi(\mathcal{M})$ we have that

$$\theta_s(w_\Psi^{1/2}aw_\Psi^{1/2}) = \theta_s(w_\Psi^{1/2})v_s^{1/2}av_s^{1/2}\theta_s(w_\Psi^{1/2}) = \widetilde{v}_s^{1/2}(w_\Psi^{1/2}aw_\Psi^{1/2})\widetilde{v}_s^{1/2}.$$

In other words the prescription $a \mapsto w_\Psi^{1/2}aw_\Psi^{1/2}$ maps $L^\Psi(\mathcal{M})$ into $L_\Psi(\mathcal{M})$. A similar argument now shows that the prescription $a \mapsto w_\Psi^{-1/2}aw_\Psi^{-1/2}$ maps $L_\Psi(\mathcal{M})$ into $L^\Psi(\mathcal{M})$, and hence that the original map ι is a bijection from $L^\Psi(\mathcal{M})$ to $L_\Psi(\mathcal{M})$. Given $f \in L^\Psi(\mathcal{M})$ we have that $\|\iota(f)\|_\Psi^O = \mathbf{m}_{w_\Psi^{1/2}aw_\Psi^{1/2}}(1) \le \|w_\Psi\|\mathbf{m}_f(1) \le 2\|f\|_\Psi$ and that

$$\|f\|_\Psi = \mathbf{m}_f(1) \le \|w_s^{-1}\|\mathbf{m}_{w_\Psi^{1/2}aw_\Psi^{1/2}}(1) \le \|\iota(f)\|_\Psi^O.$$

$\square$

For the next result we need the concept of a fundamental index of an Orlicz space. This definition makes sense for all rearrangement invariant Banach function spaces, but that is outside the scope of these notes. For the spaces under consideration, these indices may be defined as below.

DEFINITION 7.19. Let Ψ be a Young function and let

$$M_\Psi(t) = \sup_{s>0} \frac{\mathbf{f}^\Psi(st)}{\mathbf{f}^\Psi(s)}.$$

Then the lower and upper fundamental indices of $L^\Psi(\mathcal{M})$ are defined to be

$$\underline{\beta}_{L^\Psi} = \lim_{s\to 0+} \frac{\log M_\Psi(s)}{\log s} \quad \text{and} \quad \overline{\beta}_{L^\Psi} = \lim_{s\to\infty} \frac{\log M_\Psi(s)}{\log s}$$

respectively.

PROPOSITION 7.20. *Let $L^\Psi(\mathcal{M})$ be an Orlicz space with upper fundamental index strictly less than 1. Then $L^\Psi(\mathcal{M}) \subseteq (L^\infty + L^1)(\mathfrak{M}, \tau_{\mathfrak{M}})$. Moreover the canonical topology on $L^\Psi(\mathcal{M})$ then agrees with the subspace topology inherited from $(L^\infty + L^1)(\mathfrak{M}, \tau_{\mathfrak{M}})$.*

PROOF. If indeed the upper fundamental index of $L^\Psi(\mathcal{M})$ is strictly less than 1, then there must exist some $t_0 > 1$ and some $0 < \delta < 1$ such that $\frac{\log M_\Psi(t)}{\log t} \leq \delta$ for all $t > t_0$. Since for $t \geq 1$ we have that $M_\Psi(t) \geq 1$, this can be shown to be equivalent to the claim that $M_\Psi(t) \leq t^\delta$ for all $t \geq t_0$, or equivalently that $M_\Psi(\frac{1}{t}) \leq (\frac{1}{t})^\delta$ for all $0 < t \leq \frac{1}{t_0}$.

Now recall that by equation (7.1), we have that $\mathbf{m}_a(t) = \mathbf{m}_{v_s^{1/2} a v_s^{1/2}}(1)$ for all $t = e^{s_t}$ where $s_t < 0$, and $v_{s_t} = \mathbf{f}^\Psi(e^{-s_t} h)\mathbf{f}^\Psi(h)^{-1}$. If indeed $s_t \leq -\log(t_0)$ (equivalently $0 < t < \frac{1}{t_0}$), we will have that $\frac{\mathbf{f}^\Psi(t^{-1}r)}{\mathbf{f}^\Psi(r)} \leq M_\Psi(\frac{1}{t}) \leq (\frac{1}{t})^\delta$ for all $r > 0$, and hence that $\|v_{s_t}\| \leq (\frac{1}{t})^\delta$. It therefore follows that $\mathbf{m}_a(t) = \mathbf{m}_{v_s^{1/2} a v_s^{1/2}}(1) \leq \|v_s\| \mathbf{m}_a(1) \leq (\frac{1}{t})^\delta \mathbf{m}_a(1)$ for all $0 < t \leq \frac{1}{t_0}$, with $t\mathbf{m}_a(t) \leq \mathbf{m}_a(1)$ on $[t_0^{-1}, 1]$.

We may now use the fact that $t \mapsto \mathbf{m}_t(a)$ is non-increasing alongside this fact, to see that

$$\mathbf{m}_a(1) \leq \int_0^1 \mathbf{m}_a(t)\, dt \leq \left[\int_0^{1/t_0} (\frac{1}{t})^\delta\, dt + \int_{1/t_0}^1 \frac{1}{t}\, dt\right] \mathbf{m}_a(1).$$

The result now follows from Theorem 5.58.   □

EXAMPLE 7.21. We show that the upper fundamental index of the space $L^{\cosh -1}(0,\infty)$ is $\frac{1}{2}$. This space is therefore one of the Orlicz spaces for which the canonical topology is a norm topology. The first step in proving this claim is to note that isomorphic Orlicz spaces share the same indices. We leave the verification of this fact as an exercise. So it is sufficient to prove this for a space isomorphic to $L^{\cosh -1}(0,\infty)$. We show how to construct such a space before proving the claim. It is easy to see that the graphs of e^t and $\frac{e^2}{4}t^2$ are tangent at $t = 2$. This fact ensures that

$$\Psi_e(t) = \begin{cases} \frac{e^2}{4}t^2 & \text{if} \quad 0 \le t \le 2 \\[2mm] e^t & \text{if} \quad 2 < t \end{cases}$$

is a Young function. Using Maclaurin series it is easy to see that

$$\lim_{t\to 0+} \frac{\cosh(t) - 1}{\Psi_e(t)} = \frac{2}{e^2}.$$

Since we also have that $\lim_{t\to\infty} \frac{\cosh(t)-1}{\Psi_e(t)} = \lim_{t\to\infty} \frac{e^t+e^{-t}-2}{2e^t} = \frac{1}{2}$, it is clear that $\Psi_e \approx \cosh -1$. (To see this note that the limit formulae ensure that we may find $0 < \alpha < \beta < \infty$ so that $\frac{1}{e^2} < \frac{\cosh(t)-1}{\Psi_e(t)} < \frac{3}{e^2}$ on $[0, \alpha]$, and $\frac{1}{4} < \frac{\cosh(t)-1}{\Psi_e(t)} < \frac{3}{4}$ on $[\beta, \infty)$. Since the function $\frac{\cosh(t)-1}{\Psi_e(t)}$ has both a minimum and maximum on the interval $[\alpha, \beta]$, a combination of these facts ensures that we can find positive constants $0 < m < M < \infty$ so that $m\Psi_e(t) < \cosh(t) - 1 < M\Psi_e(t)$ for all $t \in [0, \infty)$.) This is clearly enough to ensure that $L^{\Psi_e}(0,\infty) \equiv L^{\cosh -1}(0,\infty)$.

It remains to compute the fundamental indices of $L^{\Psi_e}(0,\infty)$, where we assume that $L^{\Psi_e}(0,\infty)$ is equipped with the Luxemburg-Nakano norm. Since

$$\Psi_e^{-1}(t) = \begin{cases} \frac{2}{e}t^{1/2} & \text{if} \quad 0 \le t \le e^2 \\ \log(t) & \text{if} \quad e^2 < t \end{cases} \quad ,$$

it now follows that the fundamental function $\frac{1}{\Psi_e^{-1}(1/t)}$ of $L^{\Psi_e}(0,\infty)$ is given by

$$\mathbf{f}_e(t) = \begin{cases} \frac{e}{2}t^{1/2} & \text{if} \quad t \ge e^{-2} \\ \frac{1}{-\log(t)} & \text{if} \quad t < e^{-2} \end{cases}$$

We proceed to compute the function $M_{\Psi_e}(s) = \sup_{t>0} \frac{\mathbf{f}_e(st)}{\mathbf{f}_e(t)}$. In computing this function, we first consider the case where $0 < s \le 1$. Since $\mathbf{f}_e$ is

increasing, we then have that

$$\frac{\mathbf{f}_e(st)}{\mathbf{f}_e(t)} \le \frac{\mathbf{f}_e(t)}{\mathbf{f}_e(t)} = 1$$

for any $t > 0$. Since we also have that

$$\lim_{t \to 0} \frac{\mathbf{f}_e(st)}{\mathbf{f}_e(t)} = \lim_{t \to 0} \frac{\log(t)}{\log(st)} = \lim_{t \to 0} \frac{\log(t)}{\log(s) + \log(t)} = 1,$$

it is clear that $M_{\Psi_e}(s) = 1$ in this case.

Now let s be given with $s > 1$. We then have that

$$\frac{\mathbf{f}_e(st)}{\mathbf{f}_e(t)} = \begin{cases} s^{1/2} & \text{if } t > \frac{1}{e^2} \\ -\frac{e}{2}s^{1/2}t^{1/2}\log(t) & \text{if } \frac{1}{e^2} > t > \frac{1}{se^2} \\ \frac{\log(t)}{\log(st)} & \text{if } t < \frac{1}{se^2} \end{cases}$$

It is not too difficult to see that on the interval $(0, 1)$, the function $t \mapsto -\frac{e}{2}s^{1/2}t^{1/2}\log(t)$ has a maximum of $s^{1/2}$ at $t = e^{-2}$. So for $t \in (\frac{1}{se^2}, \frac{1}{e^2})$, the supremum of the above quotient is $s^{1/2}$. Finally consider the function

$$t \mapsto \frac{\log(t)}{\log(st)} = \frac{\log(t)}{\log(s)+\log(t)} = 1 - \frac{\log(s)}{\log(s)+\log(t)} = 1 - \frac{\log(s)}{\log(st)}.$$

It is easy to see that

$$\frac{d}{dt}\left(1 - \frac{\log(s)}{\log(s)+\log(t)}\right) = \frac{\log(s)}{t(\log(st))^2} > 0$$

on $t \in (0, \frac{1}{se^2})$. Hence on $(0, \frac{1}{se^2}]$, $t \mapsto \frac{\log(t)}{\log(st)}$ attains a maximum of $1 + \frac{1}{2}\log(s)$ at $t = \frac{1}{se^2}$. Using the fact that $1 + \log(t) \le t$, it is now easy to see that $1 + \frac{1}{2}\log(s) = 1 + \log(s^{1/2}) \le s^{1/2}$. Putting all these facts together leads to the conclusion that $M_{\Psi_e}(s) = \sup_{t>0} \frac{\mathbf{f}_e(st)}{\mathbf{f}_e(t)} = s^{1/2}$ in this case. We therefore have that

$$\overline{\beta}_{\Psi_e} = \lim_{s \to \infty} \frac{\log M_{\Psi_e}(s)}{\log s} = \lim_{s \to \infty} \frac{\log s^{1/2}}{\log s} = \frac{1}{2}$$

as claimed. Similarly

$$\underline{\beta}_{\Psi_e} = \lim_{s \to 0+} \frac{\log M_{\Psi_e}(s)}{\log s} = \lim_{s \to 0+} \frac{\log 1}{\log s} = 0.$$

We conclude this discussion of Orlicz spaces for general von Neumann algebras by showing that as in the semifinite setting, the spaces $L^{1 \cap \infty}(\mathcal{M})$

and $L^{1+\infty}(\mathcal{M})$ (constructed using the Young functions described in Proposition 5.55) are in a very concrete sense respectively the smallest and largest of all the Orlicz spaces. With $\Psi_{1\cap\infty}$ and $\Psi_{1+\infty}$ as in Proposition 5.55, it is an exercise to conclude from Proposition 5.51 that the fundamental functions corresponding to these spaces, are respectively

$$\mathbf{f}^{1\cap\infty}(t) = \max(1,t) \text{ and } \mathbf{f}^{1+\infty}(t) = \min(1,t).$$

Now recall that for an arbitrary Young function Ψ, the fundamental function $\mathbf{f}^{\Psi}$ is quasi-concave; that is $\mathbf{f}^{\Psi}$ is increasing, continuous on $(0,\infty)$, zero valued at precisely zero, and with $t \mapsto \frac{\mathbf{f}^{\Psi}(t)}{t}$ decreasing. This ensures that $\frac{\mathbf{f}^{\Psi}(t)}{t} \geq \mathbf{f}^{\Psi}(1)$ whenever $0 \leq t \leq 1$, and that $\frac{\mathbf{f}^{\Psi}(t)}{t} \leq \mathbf{f}^{\Psi}(1)$ whenever $t \geq 1$. Put differently, this ensures that for any $t \geq 0$ we have that

$$\mathbf{f}^{\Psi}(1)\mathbf{f}^{1+\infty}(t) \leq \mathbf{f}^{\Psi}(t) \leq \mathbf{f}^{\Psi}(1)\mathbf{f}^{1\cap\infty}(t).$$

These inequalities are the cornerstone for the following result:

PROPOSITION 7.22. *Let Ψ be a Young function, and let $\zeta^{\Psi}(t)$ and $\zeta_{\Psi}(t)$ respectively be the functions defined by $\zeta^{\Psi}(t) = \frac{\mathbf{f}^{1+\infty}(t)}{\mathbf{f}^{\Psi}(t)}$ and $\zeta_{\Psi}(t) = \frac{\mathbf{f}^{\Psi}(t)}{\mathbf{f}^{1\cap\infty}(t)}$. These functions are respectively bounded by $\frac{1}{\mathbf{f}^{\Psi}(1)}$ and $\mathbf{f}^{\Psi}(1)$. The operators $\zeta^{\Psi}(h)$ and $\zeta_{\Psi}(h)$ are therefore bounded operators (with the same bounds). The prescriptions $\iota_{\Psi} : a \mapsto \zeta_{\Psi}(h)^{1/2}a\zeta_{\Psi}(h)^{1/2}$ and $\iota^{\Psi} : a \mapsto \zeta^{\Psi}(h)^{1/2}a\zeta^{\Psi}(h)^{1/2}$ where $a \in \widetilde{\mathfrak{M}}$, are then continuous maps on $\widetilde{\mathfrak{M}}$ which respectively restrict to continuous embeddings of $L^{1\cap\infty}(\mathcal{M})$ into $L^{\Psi}(\mathcal{M})$, and $L^{\Psi}(\mathcal{M})$ into $L^{1+\infty}(\mathcal{M})$.*

PROOF. Except for the final claim, all statements follow fairly immediately from the discussion preceding the proposition. We prove the final

claim. First assume that $a \in L^\Psi(\mathcal{M})$. For any $s \in \mathbb{R}$ we will then have

$$\theta_s(\zeta^\Psi(h)^{1/2}a\zeta^\Psi(h)^{1/2})$$

$$= \theta_s(\zeta^\Psi(h)^{1/2})\theta_s(a)\theta_s(\zeta^\Psi(h)^{1/2})$$

$$= \theta_s(\mathbf{f}^{1+\infty}(h)^{1/2}\mathbf{f}^\Psi(h)^{-1/2}) \cdot \theta_s(a) \cdot \theta_s(\mathbf{f}^\Psi(h)^{-1/2}\mathbf{f}^{1+\infty}(h)^{1/2})$$

$$= [\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}\mathbf{f}^\Psi(e^{-s}h)^{-1/2}]$$

$$\quad \cdot [\mathbf{f}^\Psi(e^{-s}h)^{1/2}\mathbf{f}^\Psi(h)^{-1/2}a\mathbf{f}^\Psi(h)^{-1/2}\mathbf{f}^\Psi(e^{-s}h)^{1/2}]$$

$$\quad \cdot [\mathbf{f}^\Psi(e^{-s}h)^{-1/2}\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}]$$

$$= [\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}\mathbf{f}^\Psi(h)^{-1/2}]a[\mathbf{f}^\Psi(h)^{-1/2}\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}]$$

$$= [\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}\mathbf{f}^{1+\infty}(h)^{-1/2}] \cdot [\mathbf{f}^{1+\infty}(h)^{1/2}\mathbf{f}^\Psi(h)^{-1/2}] \cdot a$$

$$\quad \cdot [\mathbf{f}^\Psi(h)^{-1/2}\mathbf{f}^{1+\infty}(h)^{1/2}] \cdot [\mathbf{f}^{1+\infty}(h)^{-1/2}\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}]$$

$$= [\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}\mathbf{f}^{1+\infty}(h)^{-1/2}][\zeta^\Psi(h)^{1/2}a\zeta^\Psi(h)^{1/2}]$$

$$\quad [\mathbf{f}^{1+\infty}(h)^{-1/2}\mathbf{f}^{1+\infty}(e^{-s}h)^{1/2}]$$

By definition, $\zeta^\Psi(h)^{1/2}a\zeta^\Psi(h)^{1/2}$ must then be an element of $L^{1+\infty}(\mathcal{M})$. The proof of the other case runs along similar lines. $\square$

Even in this generality, the L^p spaces of $\mathcal{M}$ have a much more regular structure than the class of Orlicz spaces. In the rest of this chapter, we will therefore focus on refining the theory of L^p spaces for general von Neumann algebras. We already noted in Example 7.15 that the Luxemburg-Nakano norm on $L^\infty(\mathcal{M})$ agrees with the operator norm on $\mathcal{M}$. We now pass to analysing the $(p\text{-})$norm on $L^p(\mathcal{M})$ where $0 < p < \infty$. For $0 < p < 1$, t^p is of course not convex and hence $L^p(\mathcal{M})$ not an Orlicz space. However by analogy with the theory already developed, we will here also write $\|\cdot\|_p$ for the action of $\mathbf{m}_{(\cdot)}(1)$ on $L^p(\mathcal{M})$. The following lemma provides useful technical information for the analysis of the $(p\text{-})$norm $\|\cdot\|_p$.

LEMMA 7.23. *Let $0 < p < \infty$ be given.*

(1) *For any $a \in \widetilde{\mathfrak{M}}$ with polar decomposition $a = u|a|$, the following are equivalent:*
  - $a \in L^p(\mathcal{M})$;
  - $a^* \in L^p(\mathcal{M})$;
  - $|a| \in L^p(\mathcal{M})$ and $u \in \mathcal{M}$;
  - $|a|^p \in L^1(\mathcal{M})$ and $u \in \mathcal{M}$.

(2) *For any $0 \neq a \in L^p(\mathcal{M})$ we have that*
  - $\frac{1}{t}d_a(r) = d_a(t^{1/p}r)$ for all $t, r > 0$.

- $\|a\|_p = \mathbf{m}_a(1) = d_a(1)^{1/p}$;
- $t^{-1/p}\mathbf{m}_a(1) = \mathbf{m}_a(t)$ *for all* $t > 0$.

PROOF. First consider part(1). Recall that a being an element of $L^p(\mathcal{M})$ means that $\theta_s(a) = e^{-s/p}a$ for any $s \in \mathbb{R}$. But then we must clearly have that $\theta_s(|a|) = |\theta_s(a)| = |e^{-s/p}a| = e^{-s/p}|a|$ for any $s \in \mathbb{R}$. So $|a| \in L^p(\mathcal{M})$ as claimed. Similarly $a^* \in L^p(\mathcal{M})$. As far as u is concerned, it is clear that for any $s \in \mathbb{R}$, $\theta_s(u)\theta_s(|a|)$ is the polar decomposition of $\theta_s(a)$. But from what has already been verified, this means that $\theta_s(u)|a|$ is the polar decomposition of a. From the uniqueness of the polar decomposition, it follows that $\theta_s(u) = u$ for all $s \in \mathbb{R}$. But then $u \in \mathcal{M}$. That then proves the equivalence of the first three bullets. The proof that the fourth is equivalent to the third, follows on noting that the definition ensures that $|a| \in L^p(\mathcal{M})$ if and only if $|a|^p \in L^1(\mathcal{M})$.

Now consider part (2). These claims may be verified by modifying the technique used in the proof of Proposition 7.11. For the sake of clarity, we repeat the essentials of that argument. Let $a \in L^p(\mathcal{M})$ and $0 < t$ be given. We may then select $s \in \mathbb{R}$ so that $t = e^s$. For any $r > 0$, we then have that

$$
\begin{aligned}
\frac{1}{t}d_a(r) &= e^{-s}\tau(\chi_{(r,\infty)}(|a|)) \\
&= \tau(\theta_s(\chi_{(r,\infty)}(|a|))) \\
&= \tau(\chi_{(r,\infty)}(\theta_s|a|)) \\
&= \tau(\chi_{(r,\infty)}(|\theta_s(a)|)) \\
&= d_{\theta_s(a)}(r) \\
&= d_{e^{-s/p}a}(r) \\
&= d_{t^{-1/p}a}(r) \\
&= d_a(t^{1/p}r).
\end{aligned}
$$

Note that since $a \neq 0$, we must have that $d_a(s) \neq 0$ for some $s > 0$. If we combine this with the equality just verified, it follows that in fact $d_a(s) \neq 0$ for all $s > 0$. Now observe that the above equality on the one hand gives $r^p d_a(r) = d_a(1)$, and on the other that $d_a(d_a(1)^{1/p}) = \frac{1}{d_a(1)}d_a(1) = 1$ with $d_a((d_a(1) - \epsilon)^{1/p}) = \frac{1}{d_a(1)-\epsilon}d_a(1) > 1$ for any $0 < \epsilon < d_a(1)$. Hence $\mathbf{m}_a(1) = \inf\{r > 0 : d_a(r) \leq 1\} = d_a(1)^{1/p}$. We may now use the first bullet of part (2) to conclude that $t^{1/p}\mathbf{m}_a(t) = t^{1/p}\inf\{r > 0 : d_a(r) \leq t\} = t^{1/p}\inf\{r > 0 : d_a(t^{1/p}r) \leq 1\} = \inf\{s > 0 : d_a(s) \leq 1\} = \mathbf{m}_a(1)$. $\qquad\square$

Armed with the above lemma, we are now able to prove a Hölder and Minkowski inequality for the present context.

PROPOSITION 7.24. *Let $p, q, r > 0$ be given such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. For any $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$, we will have that $ab \in L^r(\mathcal{M})$ with $\|ab\|_r \leq \|a\|_p \|b\|_p$.*

PROOF. Let $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$ be given. For any $s \in \mathbb{R}$ we will have that $\theta_s(ab) = e^{-s/p}a \cdot e^{-s/q}b = e^{-s/r}ab$. So by definition $ab \in L^r(\mathcal{M})$. It now follows from Theorem 5.2, that

$$\exp \int_0^\infty \log(\mathbf{m}_{ab}(t))\, dt \leq \exp \int_0^\infty \log(\mathbf{m}_a(t))\, dt \cdot \exp \int_0^\infty \log(\mathbf{m}_b(t))\, dt$$

By Lemma 7.23 this may be rewritten as

$$\exp \int_0^\infty \log(t^{-1/r}\mathbf{m}_{ab}(1))\, dt \leq \exp \int_0^\infty \log(t^{-1/p}\mathbf{m}_a(1))\, dt$$
$$\cdot \exp \int_0^\infty \log(t^{-1/q}\mathbf{m}_b(1))\, dt,$$

which yields

$$e^{1/r}\|ab\|_r = e^{1/r}\mathbf{m}_{ab}(1) \leq e^{1/p}\mathbf{m}_a(1) \cdot e^{1/q}\mathbf{m}_b(1) = e^{1/r}\|a\|_p \|b\|_q.$$

$\square$

PROPOSITION 7.25. *For any $a, b \in L^p(\mathcal{M})$, we have that*

- $\|a + b\|_p \leq \|a\|_p + \|b\|_p$ *whenever $1 < p \leq \infty$,*
- *and $\|a + b\|_p^p \leq \|a\|_p^p + \|b\|_p^p$ whenever $0 < p \leq 1$.*

The quantity $\| \cdot \|_p = \mathbf{m}_{(\cdot)}(1)$ is therefore a norm when $1 \leq p \leq \infty$, and a p-norm when $0 < p < 1$.

PROOF. The case $p = \infty$ follows from Example 7.15. Now consider the case where $1 < p < \infty$. It then follows from Theorem 4.22, that $\int_0^1 \mathbf{m}_{a+b}(t)\, dt \leq \int_0^1 \mathbf{m}_a(t)\, dt + \int_0^1 \mathbf{m}_b(t)\, dt$. The final bullet in Lemma 7.23 now ensures that this inequality corresponds to the claim that

$$\frac{p}{p-1}\mathbf{m}_{a+b}(1) \leq \frac{p}{p-1}(\mathbf{m}_a(1) + \mathbf{m}_b(1)).$$

The claim follows.

Now let $0 < p \leq 1$. For any $0 < r < p$, it then follows from Theorem 5.9 that

$$\int_0^1 \mathbf{m}_{|a+b|^r}(t)\, dt = \int_0^1 \mathbf{m}_{a+b}(t)^r\, dt$$
$$\leq \int_0^1 \mathbf{m}_a(t)^r\, dt + \int_0^1 \mathbf{m}_b(t)^r\, dt$$
$$= \int_0^1 \mathbf{m}_{|a|^r}(t)\, dt + \int_0^1 \mathbf{m}_{|b|^r}(t)\, dt.$$

Now if say $f \in L^p(\mathcal{M})$, then $\theta_s(|f|^r) = |\theta_s(f)|^r = |e^{-s/p}f|^r = e^{-s/(p/r)}|f|$ for all $s \in \mathbb{R}$. In other words we then have that $|f| \in L^{p/r}(\mathcal{M})$. That then ensures that $\mathbf{m}_{|f|^r}(t) = t^{-r/p}\mathbf{m}_{|f|^r}(t) = t^{-r/p}\mathbf{m}_f(t)^r$ for all $t > 0$. If we apply this fact to the inequality

$$\int_0^1 \mathbf{m}_{|a+b|^r}(t)\, dt \leq \int_0^1 \mathbf{m}_{|a|^r}(t)\, dt + \int_0^1 \mathbf{m}_{|b|^r}(t)\, dt,$$

we get $\frac{p-r}{p}\mathbf{m}_{a+b}(1)^r \leq \frac{p-r}{p}(\mathbf{m}_a(1)^r + \mathbf{m}_b(1)^r)$. On dividing throughout by $\frac{p-r}{p}$ and letting $r \nearrow p$, we obtain $\mathbf{m}_{a+b}(1)^p \leq \mathbf{m}_a(1)^p + \mathbf{m}_b(1)^p$ as required.

Regarding the final claim, recall that in the case $1 \leq p < \infty$ we already know that $\|\cdot\|_p$ is a quasi-norm. The validity of the triangle inequality now ensures that it is a norm. In the case $0 < p < 1$, the one fact regarding a p-norm that is not immediately obvious is the fact that here too $\|\cdot\|_p$ is non-degenerate. To see this, note that given some $a \in L^p(\mathcal{M})$ for which $\mathbf{m}_a(1) = 0$, Lemma 7.23 informs us that then $\mathbf{m}_a(t) = 0$ for all $t > 0$. Therefore $\|a\|_\infty = \lim_{t \searrow 0} \mathbf{m}_t(a) = 0$, or equivalently $a = 0$. $\qquad\square$

REMARK 7.26. It is clear from the above result, that the $L^p(\mathcal{M})$-spaces are in fact Banach spaces whenever $p \geq 1$.

7.3 The trace functional and tr-duality for L^p-spaces

Although we have defined L^p-spaces for general von Neumann algebras and have even proved a Hölder and Minkowski inequality for these spaces, at present they bear little resemblance to their classical counterparts. We now remedy this by introducing the so-called trace functional on $L^1(\mathcal{M})$. This functional is a crucial tool for the development of duality theory in this context.

DEFINITION 7.27. We define tr to be the linear functional on $L^1(\mathcal{M})$ given by $\mathrm{tr}(a) = \omega_a(\mathbb{1})$, where $\omega_a \in \mathcal{M}_*$ is the normal functional on $\mathcal{M}$ corresponding to a by means of the bijection described in Theorem 6.72.

PROPOSITION 7.28. *Let $0 < p < \infty$ be given. For any $a \in L^p(\mathcal{M})$, the quantity $\|a\|_p = \mathbf{m}_a(1)$ agrees with $\mathrm{tr}(|a|^p)^{1/p}$.*

PROOF. First let $a \in L^1(\mathcal{M})$. Let $\omega_{|a|}$ be the functional in $\mathcal{M}_*$ corresponding to $|a|$ by means of the bijection in Theorem 6.72. Since $\omega_{|a|}(\mathbb{1}) = d_{|a|}(1)$ by Theorem 6.72, it then follows from Lemma 7.23 that $\mathrm{tr}(|a|) = \mathbf{m}_a(1)$ in this setting. Now suppose that $a \in L^p(\mathcal{M})$. It is easy to see that then $|a|^p \in L^1(\mathcal{M})$. We may then use what we have noted regarding $L^1(\mathcal{M})$ to conclude that $\mathrm{tr}(|a|^p)^{1/p} = \mathbf{m}_{|a|^p}(1)^{1/p} = \mathbf{m}_{|a|}(1) = \mathbf{m}_a(1)$. $\square$

The above proposition now enables us to show that $L^1(\mathcal{M})$ is an isometric copy of $\mathcal{M}_*$, and to show that $L^2(\mathcal{M})$ is in fact a Hilbert space.

PROPOSITION 7.29. *For any $a \in L^p(\mathcal{M})$, we have that $\overline{\mathrm{tr}(a)} = \mathrm{tr}(a^*)$ and $|\mathrm{tr}(a)| \leq \mathrm{tr}(|a|) = \|\omega_a\|$, where ω_a is the normal functional corresponding to a (Theorem 6.72). The quantity $\mathrm{tr}(|\cdot|) = \|\cdot\|_1$ is a norm on $L^1(\mathcal{M})$, and when equipped with this norm, $L^1(\mathcal{M})$ is isometrically isomorphic to $\mathcal{M}_*$.*

PROOF. Let a and ω_a be as in the hypothesis. All the claims follow fairly immediately from Theorem 6.72. For the sake of the reader, we provide suitable details. For the first claim note that $\overline{\mathrm{tr}(a)} = \overline{\omega_a(\mathbb{1})} = \omega_a^*(\mathbb{1}) = \omega_{a^*}(\mathbb{1}) = \mathrm{tr}(a^*)$. For the second claim note that $|\mathrm{tr}(a)| = |\omega_a(\mathbb{1})| \leq \|\omega_a\| = \|\,|\omega_a|\,\| = \|\omega_{|a|}\| = |\omega_{|a|}(\mathbb{1})| = \mathrm{tr}(|a|)$. The equality $\mathrm{tr}(|\cdot|) = \|\cdot\|_1$ was proved in the preceding proposition. Given $a, b \in L^1(\mathcal{M})$, we may use this equality to see that $\|a + b\|_1 = \mathrm{tr}(|a + b|) = \|\omega_{a+b}\| = \|\omega_a + \omega_b\| \leq \|\omega_a\| + \|\omega_b\| = \mathrm{tr}(|a|) + \mathrm{tr}(|b|) = \|a\|_1 + \|b\|_1$. The final claim is now an immediate consequence of Theorem 6.72. $\square$

PROPOSITION 7.30. *The prescription $\langle a, b \rangle = \mathrm{tr}(b^*a)$ defines an inner product on $L^2(\mathcal{M})$ for which $\langle a, a \rangle = \|a\|_2^2$. The space $L^2(\mathcal{M})$ is therefore a Hilbert space.*

PROOF. For any $f \in L^1(\mathcal{M})$ we will again write ω_f for the normal functional corresponding to f by means of the bijection described in Theorem 6.72. For any $a_1, a_2, b \in L^2(\mathcal{M})$ and any $\gamma \in \mathbb{R}$, we have that $\langle a_1 + a_2, b \rangle = \mathrm{tr}(b^*(a_1 + \gamma a_2)) = \omega_{b^*(a_1 + \gamma a_2)} = \omega_{b^*a_1} + \gamma \omega_{b^*a_2} =$

$\text{tr}(b^* a_1) + \gamma \text{tr}(b^* a_2) = \langle a_1, b \rangle + \gamma \langle a_2, b \rangle$. The remaining properties of an inner product may be proved by similar techniques. Given $a \in L^2(\mathcal{M})$, the claim regarding the norm follows by applying Proposition 7.28 to the equality $\langle a, a \rangle = \text{tr}(|a|^2)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We proceed to show that tr satisfies a trace-like property. Once that is done, L^p-duality will follow fairly quickly.

LEMMA 7.31. *For any* $f \in \widetilde{\mathfrak{M}}_+$*, the mapping* $z \mapsto f^z$ *is a differentiable* $\widetilde{\mathfrak{M}}$*-valued map on the open right half-plane* $\mathbb{C}_+^o = \{z \colon \text{Re}(z) > 0\}$*.*

PROOF. Let $z_0 \in \mathbb{C}_+^o$ and $f \in \widetilde{\mathfrak{M}}_+$ be given. We will show that $f^{z_0} \log(f) \in \widetilde{\mathfrak{M}}$ with $\frac{d}{dz} f^z \big|_{z_0} = f^{z_0} \log(f)$.

First consider the case where $f \in \mathfrak{M}_+$. Notice that $t \mapsto t^{z_0} \log(t)$ extends to a function which is continuous on $[0, \|f\|]$, and 0-valued at 0. Since $\text{sp}(f) \subseteq [0, \|f\|]$, it therefore follows from the continuous functional calculus that we also have that $f^{z_0} \log(f) \in \mathfrak{M}$. It is an exercise to see that as $z \to z_0$, the expression $\frac{t^z - t^{z_0}}{z - z_0} - t^{z_0} \log(t)$ will converge to 0 uniformly on $[0, \|f\|]$. We may then once again apply the continuous functional calculus to see that $\frac{1}{z - z_0}(f^z - f^{z_0}) - f^{z_0} \log(f)$ converges to 0 in norm. Thus in this case $\frac{d}{dz} f^z \big|_{z_0} = f^{z_0} \log(f)$ as required.

Now suppose that $f \in \widetilde{\mathfrak{M}}_+$. It then follows from what we have just proven that $f^{z_0} \log(f) \chi_{[0,\gamma]}(f) \in \mathfrak{M}$ for any $\gamma > 0$. If therefore we can show that $\tau(\chi_{(\epsilon, \infty)}(f^{x_0} |\log(f)|)) \to 0$ as $\epsilon \to \infty$, it will follow that $f^{z_0} \log(f) \in \widetilde{\mathfrak{M}}$. Let x_0 denote $\text{Re}(z_0)$, and notice that $|f^{z_0} \log(f)| = f^{x_0} |\log(f)|$. The function $t \mapsto t^{x_0} \log(t)$ has a minimum of $-\frac{1}{ex_0}$ on $(0, \infty)$. So if we choose $\gamma > 0$ large enough so that $\gamma^{x_0} \log(\gamma) > \frac{1}{ex_0}$, that would ensure that all of the statements

$$t^{x_0} \log(t) > \gamma^{x_0} \log(\gamma), \quad t^{x_0} \log(t) > \gamma^{x_0} \log(\gamma), \quad t > \gamma$$

are equivalent. So for such a γ, the Borel functional calculus ensures that $\chi_{(\gamma, \infty)}(f) = \chi_{(\gamma^{x_0} \log(\gamma), \infty)}(f^{x_0} |\log(f)|)$. We may now use this equality to conclude from the known fact that $\tau(\chi_{(\gamma, \infty)}(f)) \to 0$ as $\gamma \to \infty$, that also $\tau(\chi_{(\gamma, \infty)}(f_0^x |\log(f)|)) \to 0$ as $\gamma \to \infty$.

Now let $\epsilon > 0$ be given and select γ so that $\tau(\chi_{(\gamma, \infty)}(f)) < \epsilon$. With e denoting $e = \chi_{[0,\gamma]}(f)$, the operator fe is of course bounded. It therefore follows from the first part of the proof that $\|\frac{1}{z - z_0}((fe)^z - (fe)^{z_0}) - (fe)^{z_0} \log(fe)\| \leq \epsilon$ for z close enough to z_0. But this means that

$\frac{1}{z-z_0}(f^z - f^{z_0}) - f^{z_0}\log(f) \in N(\epsilon, \epsilon)$ for z close enough to z_0. So by definition $\frac{1}{z-z_0}(f^z - f^{z_0}) - f^{z_0}\log(f)$ converges to 0 in measure as $z \to z_0$, which then proves the lemma. $\qquad\square$

LEMMA 7.32. *Let S^o be the open strip $S^o = \{z \in \mathbb{C}\colon 0 < \mathrm{Re}(z) < 1\}$. Let $f, g \in L^1(\mathcal{M})$ be given. For any $z \in S^o$ we have that $f^z g^{1-z} \in L^1(\mathcal{M})$. Moreover the map $S^o \to L^1(\mathcal{M}) : z \mapsto f^z g^{1-z}$ is analytic.*

PROOF. Let $z \in S^o$ be given. It is not difficult to see that each of f^z and g^{1-z} belongs to $\widetilde{\mathfrak{M}}$. Now observe that for each $s \in \mathbb{R}$ we have that $\theta_s(f^z g^{1-z}) = (\theta_s(f))^z(\theta_s(g))^{1-z} = (e^{-sz}f^z)(e^{-s(1-z)}g^{1-z}) = e^{-s}f^z g^{1-z}$. Thus by definition $f^z g^{1-z} \in L^1(\mathcal{M})$. We know from the previous lemma that as maps into $\widetilde{\mathfrak{M}}$, each of $z \mapsto f^z$ and $z \mapsto g^{1-z}$ is analytic on S^o. (Here we used the fact that $1 - z \in \mathbb{C}_+^o$ if $z \in S^o$.) It is not difficult to show that the product rule holds in this context, from which we may then conclude that $z \mapsto f^z g^{1-z}$ is analytic as a map into $\widetilde{\mathfrak{M}}$. But we know that this map is $L^1(\mathcal{M})$-valued. So since by Theorem 7.12 the norm topology on $L^1(\mathcal{M})$ agrees with the topology of convergence in measure inherited from $\widetilde{\mathfrak{M}}$, we are done. $\qquad\square$

LEMMA 7.33. *Let $t \in \mathbb{R}$ be given and let $L^{1/((1/2)+it)}(\mathcal{M})$ denote the vector space*

$$L^{1/((1/2)+it)}(\mathcal{M}) = \{a \in \widetilde{\mathfrak{M}}\colon \theta_s(a) = e^{-s((1/2)+it)}a \text{ for all } s \in \mathbb{R}\}.$$

For all $a, b \in L^{1/((1/2)+it)}(\mathcal{M})$ we then have that $b^*a, ab^* \in L^1(\mathcal{M})$ with $\mathrm{tr}(b^*a) = \mathrm{tr}(ab^*)$.

PROOF. Given any $a, b \in L^{1/((1/2)+it)}(\mathcal{M})$ and any $s \in \mathbb{R}$, it is easy to check that

$$\theta_s(b^*a) = (\theta_s(b))^*\theta_s(a) = e^{-s((1/2)-it)}b^* \cdot e^{-s((1/2)+it)}a = e^{-s}b^*a.$$

Similarly $\theta_s(ab^*) = e^{-s}ab^*$ for any $s \in \mathbb{R}$. So by definition $b^*a, ab^* \in L^1(\mathcal{M})$. We may now apply Lemmas 7.23 and 7.28 to see that $\mathrm{tr}(a^*a) = d_{a^*a}(1) = d_{aa^*}(1) = \mathrm{tr}(aa^*)$ for any $a \in L^{1/((1/2)+it)}(\mathcal{M})$. Given $a, b \in L^{1/((1/2)+it)}(\mathcal{M})$, we may then apply this fact to the polarisation identities

$$b^*a = (1/4)\sum_{k=0}^{3}(a + i^k b)^*(a + i^k b) \text{ and } ab^* = (1/4)\sum_{k=0}^{3}(a + i^k b)(a + i^k b)^*$$

to see that $\mathrm{tr}(b^*a) = \mathrm{tr}(ab^*)$. $\qquad\square$

THEOREM 7.34. *Let $p, q \geq 1$ be given with $1 = \frac{1}{p} + \frac{1}{q}$. For any $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$, we have that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.*

PROOF. Let p, q and a, b be as in the hypothesis. We already noted in Proposition 7.24 that then $ab, ba \in L^1(\mathcal{M})$. So tr is well-defined on both these products. First consider the case where say $p = \infty$. Let $\omega_b \in \mathcal{M}_*$ be the functional corresponding to $b \in L^1(\mathcal{M})$ by means of the bijection described in Theorem 6.72. It then follows from that theorem and the definition of tr, that $\operatorname{tr}(ab) = \omega_{ab}(\mathbb{1}) = a \cdot \omega_b(\mathbb{1}) = \omega_b(\mathbb{1}) \cdot a = \omega_{ba}(\mathbb{1}) = \operatorname{tr}(ba)$.

Next suppose that $1 < p, q < \infty$. We know from Lemma 7.23 that if $a \in L^p(\mathcal{M})$, then each of a^* and $|a|$, also belong to $L^p(\mathcal{M})$. Using these facts it is easy to see that a can then be written as a linear combination of positive elements of $L^p(\mathcal{M})$, specifically $a = (|\operatorname{Re}(a)| + \operatorname{Re}(a)) - (|\operatorname{Re}(a)| - \operatorname{Re}(a)) + i(|\operatorname{Im}(a)| + \operatorname{Im}(a)) - i(|\operatorname{Im}(a)| - \operatorname{Im}(a))$. If therefore we can show that in the case where a and b are positive we have that $\operatorname{tr}(ab) = \operatorname{tr}(ba)$, the same equality will then by linearity hold for general elements a and b. We may therefore without loss of generality assume that $a, b \geq 0$. Since then $a^p, b^q \in L^1(\mathcal{M})$, Lemma 7.32 ensures that the functions $F(z) = \operatorname{tr}(a^{pz} b^{q(1-z)})$ and $G(z) = \operatorname{tr}(b^{q(1-z)} a^{pz})$ are analytic on S^o. Now notice that for any fixed $t \in \mathbb{R}$ we have that $\theta_s(a^{p((1/2)+it)}) = (\theta_s(a))^{p((1/2)+it)} = (e^{-s/p}a)^{p((1/2)+it)} = e^{-s((1/2)+it)} a^{p((1/2)+it)}$ for all $s \in \mathbb{R}$. Hence $a^{p((1/2)+it)} \in L^{((1/2)+it)^{-1}}(\mathcal{M})$. We similarly have that $b^{q((1/2)+it)} \in L^{((1/2)+it)^{-1}}(\mathcal{M})$. Now consider the functions F, G defined on the open strip $S^o = \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}$, by $F(z) = \operatorname{tr}(a^{pz} b^{q(1-z)})$, and $G(z) = \operatorname{tr}(b^{q(1-z)} a^{pz})$. It then follows from Lemma 7.32 that F and G are well-defined analytic functions on the domain S^o. Now observe that by Lemma 7.33,

$$
\begin{aligned}
F((1/2) + it) &= \operatorname{tr}(a^{p((1/2)+it)} b^{q(1-[(1/2)+it])}) \\
&= \operatorname{tr}(a^{p((1/2)+it)} b^{q((1/2)-it)}) \\
&= \operatorname{tr}(a^{p((1/2)+it)} (b^{q((1/2)+it)})^*) \\
&= \operatorname{tr}((b^{q((1/2)+it)})^* a^{p((1/2)+it)}) \\
&= \operatorname{tr}(b^{q((1/2)-it)} a^{p((1/2)+it)}) \\
&= \operatorname{tr}(b^{q(1-[(1/2)+it])} a^{p((1/2)+it)}) \\
&= G((1/2) + it)
\end{aligned}
$$

for any $t \in \mathbb{R}$. Thus F and G are analytic functions on the domain S^o, which agree on the line $(1/2)+it$ ($t \in \mathbb{R}$). That ensures that F and G agree on all of S^o, and in particular that $\mathrm{tr}(ab) = F(1/p) = G(1/p) = \mathrm{tr}(ba)$. $\square$

We are now ready to start the development of a duality theory for the general case. We follow essentially the same strategy as in chapter 5. Most of the proofs are minor modifications of the earlier ones. For the sake of the reader we provide occasional details.

LEMMA 7.35. *Suppose $1 \leq p < \infty$. For any $a \in L^p(\mathcal{M})$, we have that $\|a\|_p = \sup\{|\mathrm{tr}(ab)| \colon b \in L^q(\mathcal{M}), \|b\|_q \leq 1\}$, where $1 = \frac{1}{p} + \frac{1}{q}$.*

PROOF. Hölder's inequality combined with the fact that $|\mathrm{tr}(ab)| \leq \mathrm{tr}(|ab|)$ for each $a \in L^p$, $b \in L^q$ ensures that

$$\sup\{|\mathrm{tr}(ab)| \colon b \in L^q(\mathcal{M}), \|b\|_q \leq 1\} \leq \|a\|_p.$$

For the converse let $0 \neq a \in L^p$ be given. In the case where $p = 1$, the inequality $\sup\{|\mathrm{tr}(ab)| \colon b \in L^\infty(\mathcal{M}), \|b\|_\infty \leq 1\} \leq \|a\|_1$ is obvious. We simply choose $b \in L^\infty$ to be $b = u^*$ where u is the partial isometry in the polar decomposition $a = u|a|$ of a, to see that $\mathrm{tr}(ab) = \mathrm{tr}(ba) = \mathrm{tr}(|a|) = \|a\|_1$. In the case where $1 < p < \infty$, we set $b = \|a\|_p^{-p/q}|a|^{p-1}u^*$, where $a = u|a|$ is the polar decomposition of a. For any $s \in \mathbb{R}$ we have that $\theta_s(b) = \|a\|_p^{-p/q}|\theta_s(a)|^{p-1}\theta_s(u^*) = \|a\|_p^{-p/q}|e^{-s/p}a|^{p-1}u^* = e^{-s/q}\|a\|_p^{-p/q}|a|^{p-1}u^* = e^{-s/q}b$, ensuring that $b \in L^q(\mathcal{M})$. Observe that $\|b\|_q^q = \mathbf{m}_b(1)^q = \mathbf{m}_{b^*}(1)^q = \mathrm{tr}(|b^*|^q) = \|a\|_p^{-p}\mathrm{tr}(|a|^{qp-q}) = \|a\|_p^{-p}\mathrm{tr}(|a|^p) = 1$, and hence that $b \in L^q$ with $\|b\|_q = 1$. By construction $\mathrm{tr}(ba) = \|a\|_p^{-p/q}\tau(|a|^p) = \|a\|_p^{p-pq} = \|a\|_p$. Hence equality must hold. $\square$

We will now prove that part of the Clarkson-McCarthy inequalities also hold in this context. For this we need the following lemma:

LEMMA 7.36. *Let $1 \leq p < \infty$, and let $f, g \in L_+^p(\mathcal{M})$ be given. Then*

$$2^{1-p}\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \leq \|f + g\|_p^p.$$

PROOF. The first inequality may be proven in exactly the same way as in Lemma 5.24. To prove the second we need to modify the proof of part (ii) of Proposition 5.8, to ensure that it goes through for tr instead of τ. Now recall that in the proof of Proposition 5.8, we showed that there exist contractions $a, b \in \mathfrak{M}_+$ so that $a(f + g)^{1/2} = f^{1/2}$, $b(f + g)^{1/2} = g^{1/2}$ and $a^*a + b^*b = \mathfrak{s}(f+g)$. We claim that both these contractions are in $\mathcal{M}$. To see

this note that for any $s \in \mathbb{R}$ we have that $e^{-2s/p}a(f+g)^{1/2} = e^{-2s/p}f^{1/2} = \theta_s(f^{1/2}) = \theta_s(a(f+g)^{1/2}) = \theta_s(a)\theta_s((f+g)^{1/2}) = e^{-2s/p}\theta_s(a)(f+g)^{1/2}$. This can of course only be the case if $\theta_s(a) = a$ for all $s \in \mathbb{R}$, in which case $a \in \mathcal{M}$. The proof that $b \in \mathcal{M}$ is entirely analogous. It then easily follows from Lemma 5.7 that

$$
\begin{aligned}
\|f\|_p + \|g\|_p^p &= \mathbf{m}_1((a(f+g)a^*))^p + \mathbf{m}_1((b(f+g)b^*))^p \\
&= \mathbf{m}_1((a(f+g)a^*)^p) + \mathbf{m}_1((b(f+g)b^*)^p) \\
&\le \mathbf{m}_1(a(f+g)^p a^*) + \mathbf{m}_1(b(f+g)^p b^*)
\end{aligned}
$$

Now recall that for any $p \in \widetilde{\mathfrak{M}}$, we have that $\mathbf{m}_{p^*p}(1) = \mathbf{m}_p(1)^2 = \mathbf{m}_{p^*}(1)^2 = \mathbf{m}_{pp^*}(1)$. In addition it can easily be shown that both $(f+g)^{p/2}a^*a(f+g)^{p/2}$ and $(f+g)^{p/2}b^*b(f+g)^{p/2}$ are in $L^1(\mathcal{M})$ We may now use these two facts to conclude from the above inequality that

$$
\begin{aligned}
\|f\|_p^p + \|g\|_p^p &\le \mathbf{m}_1(a(f+g)^p a^*) + \mathbf{m}_1(b(f+g)^p b^*) \\
&= \mathbf{m}_1((f+g)^{p/2}a^*a(f+g)^{p/2}) + \mathbf{m}_1((f+g)^{p/2}b^*b(f+g)^{p/2})) \\
&= \mathrm{tr}((f+g)^{p/2}a^*a(f+g)^{p/2}) + \mathrm{tr}((f+g)^{p/2}b^*b(f+g)^{p/2})) \\
&= \mathrm{tr}((f+g)^{p/2}(a^*a + b^*b)(f+g)^{p/2}) \\
&= \mathrm{tr}((f+g)^p) \\
&= \|f+g\|_p^p.
\end{aligned}
$$

$\square$

Armed with the above lemma, we are now able to prove a general version of Proposition 5.25. The proof is entirely analogous to the former proof, the only difference being that we use the above lemma, in place of the earlier semifinite version.

PROPOSITION 7.37. *Let* $2 \le p < \infty$, *and let* $a, b \in L^p(\mathcal{M})$ *be given. Then*

$$
\|a+b\|_p^p + \|a-b\|_p^p \le 2^{p-1}(\|a\|_p^p + \|b\|_p^p).
$$

We are now finally ready to prove tr-duality for $L^p(\mathcal{M})$-spaces.

THEOREM 7.38. *Let* $1 < p \le \infty$ *and* $1 \le q < \infty$ *be given with* $1 = \frac{1}{p} + \frac{1}{q}$. *The bilinear form*

$$
L^p(\mathcal{M}) \times L^q(\mathcal{M}) \to \mathbb{C} : (b, a) \mapsto \mathrm{tr}(ba)
$$

defines a dual action of $L^p(\mathcal{M})$ on $L^q(\mathcal{M})$ with respect to which $L^p(\mathcal{M}) = (L^q(\mathcal{M}))^*$. Specifically for each $a \in L^p(\mathcal{M})$, the prescription $b \mapsto \mathrm{tr}(ba)$ defines a bounded linear functional ω_a on $L^q(\mathcal{M})$. Moreover the mapping $a \mapsto \omega_a$ is a surjective isometry from $L^p(\mathcal{M})$ onto $(L^q(\mathcal{M}))^*$. In addition $\omega_a \geq 0$ if and only if $a \geq 0$.

PROOF. By Proposition 7.29 the case where $p = \infty$ is just a restatement of the well-known duality of $\mathcal{M}$ and $\mathcal{M}_*$. Hence we may assume that $1 < p < \infty$. For this case the proof is almost identical to the proof of Theorem 5.27. The only changes needed are to replace τ with tr, $L^p(\mathcal{M}, \tau)$ and $L^p(\mathcal{M}, \tau)$ with $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$, and references to Lemma 5.23, with references to Lemma 7.35. The one aspect that gets used in the last part of the proof which may be less obvious is the fact that given $a \in L^p(\mathcal{M})$ with $a = a^*$, we will have that $|a|^{p-1}\chi_{(0,-\infty)}(a)$ is a positive element of $L^q(\mathcal{M}, \tau)$. The positivity of this element is clear. The membership of $L^q(\mathcal{M})$ can be seen by noting that for every $s \in \mathbb{R}$, $\theta_s(|a|^{p-1}) = |\theta_s(a)|^{p-1} = e^{-s(p-1)/p}|a| = e^{-s/q}|a|$ and $\theta_s(\chi_{(0,-\infty)}(a)) = \chi_{(0,-\infty)}(\theta_s(a)) = \chi_{(0,-\infty)}(e^{-s/p}a) = \chi_{(0,-\infty)}(a)$. Hence $|a|^{p-1} \in L^q$ and $\chi_{(0,-\infty)}(a) \in L^\infty(\mathcal{M})$, which ensures that $|a|^{p-1}\chi_{(0,-\infty)}(a) \in L^q(\mathcal{M})$. $\square$

7.4 Dense subspaces of L^p-spaces

One of our main objectives in this section is to show that the noncommutative analogue of the simple functions is dense in each $L^p(\mathcal{M})$ ($1 \leq p < \infty$). Formally these simple functions are linear combinations of terms of the form $h^{1/2p}eh^{1/2p}$, where $h = \frac{d\widetilde{\varphi}}{d\tau}$ is the density of the dual weight, and $e \in \mathcal{M}$ a projection with $\varphi(e) < \infty$. The main challenge that we need to overcome here is the fact that in general h is not τ-measurable! We therefore need to be extremely careful when working with these operators, and for this reason will in this section depart from the notational conventions we have been using for τ-measurable operators. Given two affiliated operators a and b, we shall denote the operator product with ab. If the product is in fact closed, we shall where necessary indicate this by writing (ab). If the product is closable, the minimal closed extension will be denoted by $[ab]$.

Our first task is to describe those elements of $\mathcal{M}$ for which products of the form $h^{1/q}a$ are τ-measurable.

LEMMA 7.39. *Let $a \in \mathcal{M}$ be given, and let $h = h_\varphi = \frac{d\widetilde{\varphi}}{d\tau}$ be the density of the dual weight $\widetilde{\varphi}$ of the canonical faithful normal semifinite weight φ*

on $\mathcal{M}$. Recall that we may then make sense of the form product $a\,\hat{}\,h\,\hat{}\,a^*$ as an element of the extended positive part $\widehat{\mathfrak{M}}_+$ of $\mathfrak{M}$ (see the discussion preceding Proposition 3.26). Then the following holds:

(a) *The partially defined operator $h_\varphi^{1/2}a^*$ is densely defined if and only if $a\,\hat{}\,h_\varphi\,\hat{}\,a^*$ is a (non-negative self-adjoint) operator. In this case $a\,\hat{}\,h_\varphi\,\hat{}\,a^* = \left|h_\varphi^{1/2}a^*\right|^2$*

(b) *$a \in \mathfrak{n}_\varphi$ if and only if $a\,\hat{}\,h_\varphi\,\hat{}\,a^* \in L^1(\mathcal{M})$, in which case $\varphi(a^*a) = \mathrm{tr}(|h^{1/2}a^*|^2)$.*

PROOF. Let H be the Hilbert space for which $\mathfrak{M} \subseteq B(H)$. For any $\xi \in H$, let ω_ξ be the positive functional defined by $a \mapsto \langle a\xi, \xi \rangle$. For such a $\xi \in H$, we will then in the notation of Proposition 6.67 have that

$$(a\,\hat{}\,h_\varphi\,\hat{}\,a^*)(\omega_\xi) = h_\varphi(a\omega_\xi a^*) = h_\varphi(\omega_{a^*\xi})$$

$$= \begin{cases} \left\|h_\varphi^{1/2}a^*\xi\right\|^2 & \text{if } a^*\xi \in \mathrm{dom}(h_\varphi^{1/2}), \\ \infty & \text{otherwise.} \end{cases}$$

Therefore $h_\varphi^{1/2}a^*$ is clearly densely defined if and only if $a\,\hat{}\,h_\varphi\,\hat{}\,a^*$ corresponds to an operator, in the sense that the projection p_∞ in the spectral resolution $a\,\hat{}\,h_\varphi\,\hat{}\,a^* = \int_0^\infty \lambda\, de_\lambda + \infty.p_\infty$ is 0. The above equality therefore also shows that $a\,\hat{}\,h_\varphi\,\hat{}\,a^* = |h_\varphi^{1/2}a^*|^2$.

Now recall that the action of the automorphisms θ_s extend to the extended positive part $\widehat{\mathfrak{M}}_+$. In their action on $\widehat{\mathfrak{M}}_+$ we have that

$$\theta_s(a\,\hat{}\,h_\varphi\,\hat{}\,a^*) = \theta_s(a)\,\hat{}\,\theta_s(h_\varphi)\,\hat{}\,\theta_s(a)^* = e^{-s}a\,\hat{}\,h_\varphi\,\hat{}\,a^*.$$

It is therefore clear that $(a\,\hat{}\,h_\varphi\,\hat{}\,a^*) \in L^1(\mathcal{M})$ if and only if $a\,\hat{}\,h_\varphi\,\hat{}\,a^* = h_{a^*\varphi a}$ is τ-measurable. (Here we used part (2) of Theorem 3.24.) By Corollary 6.71 this is in turn equivalent to the assertion that $a^*\varphi a \in \mathcal{M}_*$. We therefore have that

$$(a\,\hat{}\,h_\varphi\,\hat{}\,a^*) \in L^1(\mathcal{M}) \Leftrightarrow \varphi(a^*a) < \infty \Leftrightarrow a \in \mathfrak{n}_\varphi$$

as required. Notice that we then also have that $|h_\varphi^{1/2}a^*| \in L^2(\mathcal{M})$. It then follows from the definition of tr, that

$$\mathrm{tr}(|h_\varphi^{1/2}a^*|^2) = \mathrm{tr}(a\,\hat{}\,h_\varphi\,\hat{}\,a^*) = \mathrm{tr}(h_{a^*\varphi a}) = (a^*\varphi a)(\mathbb{1}) = \varphi(a^*a).$$

$\square$

PROPOSITION 7.40. *Let $q \in [2, \infty)$. If $a \in \mathfrak{n}_\varphi$ then $ah^{1/q}$ is closable with $[ah^{1/q}]$, $h^{1/q}a^* \in L^q(\mathcal{M})$, and $[ah^{1/q}] = (h^{1/q}a^*)^*$.*

PROOF. Let $q \in [2, \infty)$ be given. By Lemma 7.39, the closed operator $h^{1/2}a^*$ is densely defined, with $|h^{1/2}a^*| \in L^2(\mathcal{M})$. Hence $|h^{1/2}a^*|$, and therefore also $h^{1/2}a^*$ is τ-measurable. In the case where $q > 2$ we may write $h^{1/2}a^*$ as $h^{1/2}a^* = h^{1/r}.h^{1/q}a^*$ where $r > 2$ is given such that $\frac{1}{2} = \frac{1}{q} + \frac{1}{r}$. This clearly shows that $\mathrm{dom}(h^{1/2}a^*) \subseteq \mathrm{dom}(h^{1/q}a^*)$. The τ-measurability of $h^{1/2}a^*$, ensures that $\mathrm{dom}(h^{1/2}a^*)$ is τ-dense. But then the same must be true of $\mathrm{dom}(h^{1/q}a^*)$. Therefore the closed operator $h^{1/q}a^*$ is also τ-measurable. We therefore need only confirm that $\theta_s(h^{1/q}a^*) = e^{-s/q}$ for each $s \in \mathbb{R}$ to prove that $h^{1/q}a^* \in L^q(\mathcal{M})$.

Given $\gamma > 0$, we have that $\chi_{[0,\gamma]}(h)(h^{1/q}a^*) \subseteq [\chi_{[0,\gamma]}(h)h^{1/q}]a^*$, and hence that $[\chi_{[0,\gamma]}(h)(h^{1/q}a^*)] = [\chi_{[0,\gamma]}(h)h^{1/q}]a^*$ since all the (bracketed) operators in the product are τ-measurable. So for any $s \in \mathbb{R}$ and any $n \in \mathbb{N}$, we have

$$
\begin{aligned}
\chi_{[0,e^s n]}(h)\theta_s(h^{1/q}a^*) &= \chi_{[0,n]}(e^{-s}h)\theta_s(h^{1/q}a^*) \\
&= \theta_s(\chi_{[0,n]}(h))\theta_s(h^{1/q}a^*) \\
&= \theta_s(\chi_{[0,n]}(h)(h^{1/q}a^*)) \\
&\subseteq \theta_s([\chi_{[0,n]}(h)h^{1/q}]a^*) \\
&= \theta_s([\chi_{[0,n]}(h)h^{1/q}])\theta_s(a^*) \\
&= e^{-s/q}[\chi_{[0,n]}(e^{-s}h)h^{1/q}]a^* \\
&= e^{-s/q}[\chi_{[0,e^s n]}(h)h^{1/q}]a^* \\
&= e^{-s/q}[\chi_{[0,e^s n]}(h)(h^{1/q}a^*)]
\end{aligned}
$$

The τ-measurability of $h^{1/q}a^*$ ensures that the τ-measurable extension of $\chi_{[0,e^s n]}(h)\theta_s(h^{1/q}a^*)$ agrees with $e^{-s/q}[\chi_{[0,e^s n]}(h)(h^{1/q}a^*)]$ (see Proposition 2.51). We may therefore combine Proposition 4.12 with the fact that $\chi_{[0,e^s n]}(h) \nearrow \mathbb{1}$ as $n \nearrow \infty$, to see that

$$
\begin{aligned}
&\mathbf{m}_{\theta_s(h^{1/q}a^*)-e^{-s/q}(h^{1/q}a^*)}(t) \\
&= \mathbf{m}_{|\theta_s(h^{1/q}a^*)-e^{-s/q}(h^{1/q}a^*)|^2}(t)^{1/2} \\
&= \sup_{n\in\mathbb{N}} \mathbf{m}_{|\chi_{[0,e^s n]}(h)(\theta_s(h^{1/q}a^*)-e^{-s/q}(h^{1/q}a^*))|^2}(t)^{1/2} \\
&= 0
\end{aligned}
$$

for all $t > 0$, and hence that

$$\|\theta_s(h^{1/q}a^*) - e^{-s/q}(h^{1/q}a^*)\|_\infty = \lim_{t \searrow 0} \mathbf{m}_{\theta_s(h^{1/q}a^*) - e^{-s/q}(h^{1/q}a^*)}(t) = 0.$$

To see the claims regarding $[ah^{1/q}]$, we firstly note that $(ah^{1/q})^* = h^{1/q}a^*$. Hence the second adjoint of $(ah^{1/q})^*$ exists and agrees with $(h^{1/q}a^*)^*$. This shows that $ah^{1/q}$ is closable with $[ah^{1/q}] = (h^{1/q}a^*)^*$. $\qquad\square$

REMARK 7.41. If we apply a suitable polar formula to the equality in part (b) of Lemma 7.39, then the added technology provided by Proposition 7.40, enables us to conclude that $\varphi(b^*a) = \text{tr}((h^{1/2}b^*)[ah^{1/2}])$ for all $a, b \in \mathfrak{n}_\varphi$.

We need one more piece of mathematical technology before we are able to prove our first density result. Since $\mathfrak{n}_\varphi$ is a left-ideal, we know from Theorem 1.21 that it admits a right approximate identity. The semifiniteness of φ ensures that the projection to which such a right approximate identity converges, must be the identity $\mathbb{1}$. However elegant this fact may seem, we shall need a right approximate identity of $\mathfrak{n}_\varphi$, with more refined properties. In particular we shall need a net inside $\mathfrak{n}_\varphi$ which increases to $\mathbb{1}$ and which consists of analytic elements. In the development of noncommutative integraton theory, similar techniques have been used by many authors [**PT73, Ter82, Vae01**]. We however require some very particular facts. With Proposition 6.12 as starting point, we pause to give some hints on how the approximate identity we need is constructed.

PROPOSITION 7.42 ([**Vae01**, Lemma 3.1], [**Ter82**, Lemma 9]). *There exists a net (f_λ) of positive entire analytic elements in $\mathfrak{n}_\varphi$ converging strongly to $\mathbb{1}$, and for which*

(a) $\sigma_z^\varphi(f_\lambda) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ *for each $z \in \mathbb{C}$ and each λ,*
(b) $\|\sigma_z^\varphi(f_\lambda)\| \leq e^{\delta(\text{Im}(z))^2}$ *for each $z \in \mathbb{C}$ and each λ,*
(c) $(\sigma_z^\varphi(f_\lambda))$ *is σ-weakly convergent to $\mathbb{1}$ for each $z \in \mathbb{C}$.*

OUTLINE OF PROOF. We will give details as appropriate, but merely sketch some parts of the proof.

One starts by selecting any right approximate identity (g_λ) of $\mathfrak{n}_\varphi$. Recall that (as we noted above) in this case g_λ must increase σ-strongly to $\mathbb{1}$ as λ increases. Fixing some $\delta > 0$, one now defines the net (f_λ) by means of the prescription $f_\lambda = \sqrt{\dfrac{\delta}{\pi}} \int \sigma_t^\varphi(g_\lambda)e^{-\delta t^2}\, dt$.

The next step is to show that the function

$$F : \mathbb{C} \to \mathcal{M} : z \mapsto \sqrt{\frac{\delta}{\pi}} \int \sigma_t^\varphi(g_\lambda) e^{-\delta(t-z)^2} \, dt$$

fulfils the criteria of Definition 6.11. (Details of this part may be found in the proof of [**BR87a**, Proposition 2.5.22].) Having verified this fact, the values $\sigma_z^\varphi(f_\lambda)$, are then given by $\sigma_z^\varphi(f_\lambda) = \sqrt{\frac{\delta}{\pi}} \int \sigma_t^\varphi(g_\lambda) e^{-\delta(t-z)^2} \, dt$. This then enables us to conclude that

$$\|\sigma_z^\varphi(f_\lambda)\| \leq \sqrt{\frac{\delta}{\pi}} \int \|\sigma_t^\varphi(g_\lambda)\| |e^{-\delta(t-z)^2}| \, dt$$

$$\leq \sqrt{\frac{\delta}{\pi}} \int |e^{-\delta(t-z)^2}| \, dt$$

$$= e^{\delta(\mathrm{Im}(z))^2}.$$

The quick way to see that $\sigma_z^\varphi(f_\lambda) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ for each $z \in \mathbb{C}$, is to appeal to the technology of left Hilbert algebras. The connection of $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ to left Hilbert algebras may be found in for example Theorem VII.2.6 of [**Tak03a**]. The fact that $\sigma_z^\varphi(f_\lambda) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ for each $z \in \mathbb{C}$, then follows from for example [**SZ79**, Corollary, p272]. The verification of this fact is also embedded in the proof of [**Tak03a**, Theorem VI.2.2(i)] (see p 25 of that reference). For the sake of the reader we provide the skeleton of a direct proof of this fact. Firstly let $R > 0$ be given and let $S_N = \sqrt{\frac{\delta}{\pi}} \sum_{k=1}^N \int e^{-\delta(\widetilde{t}_k - z)^2} \sigma_{t_k}^\varphi(g_\lambda) \Delta t_k$ be a Riemann-sum of $\sqrt{\frac{\delta}{\pi}} \int_{-R}^R \sigma_t^\varphi(g_\lambda) e^{-\delta(t-z)^2} \, dt$. Next recall that in its action on $\mathfrak{n}_\varphi$, φ satisfies a Cauchy-Schwarz inequality. If we combine this fact with the fact that $\varphi \circ \sigma_t^\varphi = \varphi$ for all $t \in \mathbb{R}$, then for any $s, t \in \mathbb{R}$ and any λ, we will have that

$$|\varphi(\sigma_s^\varphi(g_\lambda^*)\sigma_t^\varphi(g_\lambda))| \leq \varphi(|\sigma_s^\varphi(g_\lambda)|^2)^{1/2}.\varphi(|\sigma_t^\varphi(g_\lambda)|^2)^{1/2} = \varphi(|g_\lambda|^2) < \infty.$$

One may then use this fact to see that

$$\varphi(S_N^* S_N) \leq \frac{\delta}{\pi} \left(\sum_{k=1}^N |e^{-\delta(\widetilde{t}_k - z)^2}| \Delta t_k \right)^2 \varphi(|g_\lambda|^2).$$

Taking the limit yields

$$\frac{\delta}{\pi} \varphi \left(| \int_{-R}^R \sigma_t^\varphi(g_\lambda) e^{-\delta(t-z)^2} \, dt|^2 \right) \leq \frac{\delta}{\pi} \left(\int_{-R}^R |e^{-\delta(t-z)^2}| \, dt \right)^2 \varphi(|g_\lambda|^2).$$

Now let $R \to \infty$, to see that

$$\varphi(|\sigma_z^\varphi(f_\lambda)|^2) \leq \frac{\delta}{\pi} \left(\int |e^{-\delta(t-z)^2}| \, dt \right)^2 \varphi(|g_\lambda|^2) = (e^{\delta(\mathrm{Im}(z))^2})^2 \varphi(|g_\lambda|^2) < \infty.$$

It remains to show that (f_λ) converges strongly to $\mathbb{1}$. Let H be the Hilbert space on which $\mathcal{M}$ acts. For any $\xi \in H$, we then have that

$$\begin{aligned}
\lim_\lambda \langle f_\lambda \xi, \xi \rangle &= \lim_\lambda \left\langle \left(\sqrt{\frac{\delta}{\pi}} \int \sigma_t^\varphi(g_\lambda) e^{-\delta t^2} \, dt \right) \xi, \xi \right\rangle \\
&= \lim_\lambda \sqrt{\frac{\delta}{\pi}} \int \langle \sigma_t^\varphi(g_\lambda)\xi, \xi \rangle e^{-\delta t^2} \, dt \\
&= \sqrt{\frac{\delta}{\pi}} \int \langle \sigma_t^\varphi(\mathbb{1})\xi, \xi \rangle e^{-\delta t^2} \, dt \\
&= \sqrt{\frac{\delta}{\pi}} \int e^{-\delta t^2} \, dt. \|\xi\|^2 \\
&= \|\xi\|^2.
\end{aligned}$$

If we combine the above formula with the fact that $\|f_\lambda\| \leq 1$, that then enables us to conclude that

$$\limsup_\lambda \|f_\lambda \xi - \xi\|^2 = \limsup_\lambda \left(\|f_\lambda \xi\|^2 - \langle f_\lambda \xi, \xi \rangle - \langle \xi, f_\lambda \xi \rangle + \|\xi\|^2 \right) \leq 0,$$

which proves the claim regarding the strong convergence of (f_λ).

We pass to proving (c). For any $\xi, \zeta \in H$ we have that

$$\begin{aligned}
\lim_\lambda \langle \sigma_z^\varphi(f_\lambda)\xi, \zeta \rangle &= \lim_\lambda \left\langle \left(\sqrt{\frac{\delta}{\pi}} \int \sigma_t^\varphi(g_\lambda) e^{-\delta(t-z)^2} \, dt \right) \xi, \zeta \right\rangle \\
&= \lim_\lambda \sqrt{\frac{\delta}{\pi}} \int \langle \sigma_t^\varphi(g_\lambda)\xi, \zeta \rangle e^{-\delta(t-z)^2} \, dt \\
&= \sqrt{\frac{\delta}{\pi}} \int \langle \sigma_t^\varphi(\mathbb{1})\xi, \zeta \rangle e^{-\delta(t-z)^2} \, dt \\
&= \sqrt{\frac{\delta}{\pi}} \int e^{-\delta(t-z)^2} \, dt \cdot \langle \xi, \zeta \rangle \\
&= \langle \xi, \zeta \rangle.
\end{aligned}$$

Thus $(\sigma_z^\varphi(f_\lambda))$ converges to $\mathbb{1}$ in the Weak Operator Topology. But the σ-weak topology and the Weak Operator Topology agree on the unit ball of $\mathcal{M}$. Hence by part (b), $(\sigma_z^\varphi(f_\lambda))$ is σ-weakly convergent to $\mathbb{1}$ as claimed. $\qquad\square$

With the existence of such a net verified, we now prove the technical lemma which will unlock the first density result.

LEMMA 7.43. *Let* $a \in \mathcal{M}$ *be entire analytic with respect to the modular automorphism group* σ_t^φ. *Then, for any* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) \geq 0$, *we have that*

$$ah^z \subseteq h^z \sigma_{iz}^\varphi(a).$$

PROOF. Let $D = \bigcap_{n \in \mathbb{N}} \mathrm{dom}(h^n)$ and $\mathbb{C}_+ = \{z \in \mathbb{C}\colon \mathrm{Re}(z) \geq 0\}$. Then for each $z \in \mathbb{C}_+$, D is a core for h^z. For $\xi, \zeta \in D$, define the following two functions $\mathbb{C}_+ \to \mathbb{C}$:

$$F\colon \alpha \mapsto \langle \xi, ah^z\zeta\rangle; \qquad G\colon \alpha \mapsto \left\langle h^{\bar{z}}\xi, \sigma_{i\alpha}(a)\zeta\right\rangle.$$

These are continuous on $\mathbb{C}_+$, analytic in the interior of $\mathbb{C}_+$ and satisfy

$$F(it) = \left\langle \xi, ah^{it}\zeta\right\rangle = \left\langle \xi, h^{it}\sigma_{-t}(a)\zeta\right\rangle = \left\langle h^{-it}\xi, \sigma_{-t}(a)\zeta\right\rangle = G(it)$$

for all $t \in \mathbb{R}$. (Note that $\sigma_{-t}^\varphi(a) = h^{-it}ah^{it}$ by Proposition 6.40 and Theorem 6.62). Therefore F and G coincide. Now let $\xi \in \mathrm{dom}(h^z)$. If (ζ_n) is a sequence in D converging to ζ in the graph norm of h^z, then

$$\langle \xi, ah^z\zeta\rangle = \lim \langle \xi, ah^z\zeta_n\rangle = \lim \left\langle h^{\bar{z}}\xi, \sigma_{iz}(a)\zeta_n\right\rangle = \left\langle h^{\bar{z}}\xi, \sigma_{iz}(a)\zeta\right\rangle$$

for each $\xi \in D$. Since D is a core for $h^{\bar{z}}$, the result follows. $\qquad\square$

COROLLARY 7.44. *Let* $a \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ *be an entire analytic element for which* $\sigma_z^\varphi(a) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ *for each* $z \in \mathbb{C}$. *Given* $z \in \mathbb{C}$ *with* $0 \leq \mathrm{Re}(z) \leq 1/2$, *we have that*

$$[ah^z] = h^z \sigma_{iz}(a).$$

PROOF. Set $z = s + it$ where $s \in [0, 1/2]$ and $t \in \mathbb{R}$, and let $a \in \mathfrak{n}_\infty$. Lemma 7.43 then informs us that $ah^\alpha \subseteq h^\alpha \sigma_{i\alpha}(a)$. Moreover since $a \in \mathfrak{n}$, we know from Proposition 7.40 that $ah^\alpha = (ah^s)h^{it}$ is a product of τ-premeasurable operators. Similarly since $\sigma_{i\alpha}(a) \in \mathfrak{n}^*$, $h^\alpha \sigma_{i\alpha}(a) = h^{it}(h^s \sigma_{i\alpha}(a))$ will by the same proposition also be a product of τ-premeasurable operators. The result therefore follows from the uniqueness of the τ-measurable extension (Proposition 2.51). $\qquad\square$

THEOREM 7.45.(a) *For any $q \in [2, \infty)$, $\{[ah^{1/q}]: a \in \mathfrak{n}_\varphi\}$ is dense in $L^q(\mathcal{M})$.*

(b) *For any $p \in [1, \infty)$, $\{(h^{1/2p}a^{1/2})[a^{1/2}h^{1/2p}]: a \in \mathfrak{p}_\varphi\}$ is dense in $L^p_+(\mathcal{M})$.*

PROOF. *Part (a):* Let $r > 1$ be given so that $1 = \frac{1}{q} + \frac{1}{r}$. Suppose that $z \in L^r(\mathcal{M})$. If we can show that we must have that $z = 0$ whenever $\mathrm{tr}(z[ah^{1/q}]) = 0$ for each $a \in \mathfrak{n}_\varphi$, then by the duality theory developed in the previous section, $\{[ah^{1/q}]: a \in \mathfrak{n}_\varphi\}$ will be weakly dense in $L^q(\mathcal{M})$, and hence norm dense. So suppose that we do indeed have that $\mathrm{tr}(z[ah^{1/q}]) = 0$ for each $a \in \mathfrak{n}_\varphi$. For each $b \in \mathcal{M}$, the fact that $\mathfrak{n}_\varphi$ is a left-ideal ensures that $ba \in \mathfrak{n}_\varphi$ for each $a \in \mathfrak{n}_\varphi$, and hence that $\mathrm{tr}(z[bah^{1/q}]) = 0$ for each $a \in \mathfrak{n}_\varphi$. Now notice that for any $a \in \mathfrak{n}_\varphi$ we have that $b[ah^{1/q}] \supseteq bah^{1/q}$. By the uniqueness of the τ-measurable extension (Proposition 2.51), we have that the τ-measurable operator corresponding to the product $b \cdot \left[ah^{1/q}\right]$ is just $[bah^{1/q}]$. That means that for any $a \in \mathfrak{n}_\varphi$ and any $b \in \mathcal{M}$, we have that $0 = \mathrm{tr}(z[bah^{1/q}]) = \mathrm{tr}([bah^{1/q}]z) = \mathrm{tr}(b([ah^{1/q}]z))$. The duality between L^1 and L^∞, then ensures that this can only be the case if $[ah^{1/q}]z = 0$ for each $a \in \mathfrak{n}_\varphi$, or equivalently that $z^*(h^{1/q}a^*) = 0$ for each $a \in \mathfrak{n}_\varphi$. It therefore remains to show that $z = 0$ (equivalently $z^* = 0$) if $z^*(h^{1/q}a^*) = 0$ for each $a \in \mathfrak{n}_\varphi$.

It is easy to see that $z^*(h^{1/q}a^*) = 0$ if and only if $|z^*|(h^{1/q}a^*) = 0$. Since trivially $z^* = 0$ if and only if $|z^*| = 0$, it follows that we may assume that $z^* \geq 0$. Having made this assumption one may then further note that $z^* = 0$ if and only if $(z^*\chi_{[0,\gamma]}(z^*)) = 0$ for every $\gamma > 0$. Since in this setting the equality $z^*(h^{1/q}a^*) = 0$ ensures that $0 = \chi_{[0,\gamma]}(z^*)z^*(h^{1/q}a^*)$ and hence that $0 = (z^*\chi_{[0,\gamma]}(z^*))(h^{1/q}a^*)$, it follows that we may further assume z^* to be bounded.

Now apply Proposition 7.42 to select a net (f_λ) of positive entire analytic elements in $\mathfrak{n}_\varphi$ which increase to $\mathbb{1}$, and for which $\sigma^\varphi_{i/q}(f_\lambda) \in \mathfrak{n}_\varphi \cap \mathfrak{n}^*_\varphi$ for each λ. Assuming z^* to be bounded, we have that $0 = z^*(h^{1/q}\sigma^\varphi_{i/q}(f_\lambda))$ for each λ. But by Corollary 7.44, $[f_\lambda h^{1/q}] = (h^{1/q}\sigma^\varphi_{i/q}(f_\lambda))$. Hence given $\xi \in \mathrm{dom}(h^{1/q}) \subseteq \mathrm{dom}([f_\lambda h^{1/q}])$, we have that $\xi \in \mathrm{dom}(h^{1/q}\sigma_{i/q}(f_\lambda))$, and $h^{1/q}\sigma_{i/q}(f_\lambda)\xi = f_\lambda h^{1/q}\xi \to h^{1/q}\xi$. The fact that $0 = z^*(h^{1/q}\sigma^\varphi_{i/q}(f_\lambda))$ for each λ, therefore ensures that $z^* = 0$ on the range of $h^{1/q}$, which must be

dense by the fact that h is non-singular and positive. Hence as required, $z^* = 0$.

Part (b): This claim easily follows from part (a). To see this let $f \in L^p_+(\mathcal{M})$ be given. Then $f^{1/2} \in L^{2p}(\mathcal{M})$. So by part (a) we may select a sequence $(a_n) \subseteq \mathfrak{n}_\varphi$ so that $[a_n h^{1/2p}] \to f^{1/2}$ (or equivalently that $(h^{1/2p}a_n^*) = [a_n h^{1/2p}]^* \to f^{1/2}$). Let u_n be the partial isometry from the polar decomposition of a_n. It is not difficult to see that then $(h^{1/2p}a_n^*)[a_n h^{1/2p}] = (h^{1/2p}a_n^*)u_n[|a_n|h^{1/2p}] = (h^{1/2p}|a_n|)[|a_n|h^{1/2p}]$. To see this, note that since $\mathrm{ran}([|a_n|h^{1/2p}]) \subseteq \ker(h^{1/2p}|a_n|)^\perp = \ker(|a_n|)^\perp = \overline{\mathrm{ran}}(|a_n|)$, the operator product $u_n[|a_n|h^{1/2p}]$ is closed, hence τ-measurable. (Here we silently used the fact that $\mathfrak{n}_\varphi$, being a left ideal, is invariant under the absolute value map.) But then $u_n[|a_n|h^{1/2p}]$ and $[a_n h^{1/2p}]$ must agree, since both are τ-measurable extensions of $a_n h^{1/2p}$ (Proposition 2.51). We may now apply Hölder's inequality to see that

$$\|f - (h^{1/2p}|a_n|)[|a_n|h^{1/2p}]\|_p$$
$$= \|f - (h^{1/2p}a_n^*)[a_n h^{1/2p}]\|_p$$
$$= \|f - f^{1/2}[a_n h^{1/2p}] + f^{1/2}[a_n h^{1/2p}] - (h^{1/2p}a_n^*)[a_n h^{1/2p}]\|_p$$
$$\leq \|f^{1/2}(f^{1/2} - [a_n h^{1/2p}])\|_p + \|(f^{1/2} - (h^{1/2p}a_n^*))[a_n h^{1/2p}]\|_p$$
$$\leq \|f^{1/2}\|_{2p}\|f^{1/2} - [a_n h^{1/2p}]\|_{2p} + \|f^{1/2} - (h^{1/2p}a_n^*)\|_{2p}\|[a_n h^{1/2p}]\|_{2p},$$

from which it follows that $(h^{1/2p}|a_n|)[|a_n|h^{1/2p}] \to f$ as $n \to \infty$. $\qquad\square$

In addition to answering some questions regarding dense subspaces, the preceding theorem also raises questions.

For example given $a_1, a_2 \in \mathfrak{p}_\varphi$, how do

$$(h^{1/2p}a_1^{1/2})[a_1^{1/2}h^{1/2p}] + (h^{1/2p}a_2^{1/2})[a_2^{1/2}h^{1/2p}],$$

and

$$(h^{1/2p}(a_1 + a_2)^{1/2})[(a_1 + a_2)^{1/2}h^{1/2p}]$$

compare? We now address these issues before concluding this section with an analysis of "simple functions".

DEFINITION 7.46. For $q \in [2, \infty)$ define the map

$$\mathfrak{j}^{(q)} : \mathfrak{n}_\varphi \ni a \mapsto \left[a h^{1/q}\right] \in L^q(\mathcal{M}).$$

For $p \in [1, \infty)$, define the map

$$\mathfrak{i}^{(p)} : \mathfrak{p}_\varphi \ni a \mapsto \mathfrak{j}^{(2p)}(a^{1/2})^* \mathfrak{j}^{(2p)}(a^{1/2}) \in L^p(\mathcal{M})$$

For the task we have set for ourselves, the following lemma is crucial.

LEMMA 7.47. *Let $a, b \in \mathfrak{n}_\varphi$ and $r_i, s_i \in [2, \infty)$ be given with $r_1^{-1} + s_1^{-1} = r_2^{-1} + s_2^{-1}$. Then*

$$([ah^{1/r_1}](h^{1/s_1}b^*)) = ([ah^{1/r_2}](h^{1/s_2}b^*)).$$

PROOF. Assume without loss of generality that $s_1 \leq s_2$, so that $r_1 \geq r_2$. Then

$$h^{1/r_2}a^* = h^{(1/r_2 - 1/r_1)}(h^{1/r_1}a^*) \subseteq ([ah^{1/r_1}]h^{(1/r_2 - 1/r_1)})^*,$$

so that $[ah^{1/r_2}] \supseteq [ah^{1/r_1}]h^{(1/s_1 - 1/s_2)}$, and

$$[ah^{1/r_2}]h^{1/s_2}b^* \supseteq [ah^{1/r_1}]h^{(1/s_1 - 1/s_2)}h^{1/s_2}b^* = [ah^{1/r_1}]h^{1/s_1}b^*.$$

Since each of $h^{1/s_2}b^*$ and $h^{1/s_1}b^*$ is τ-measurable by Proposition 7.40, both sides of the formula represent τ-pre-measurable operators. The claim therefore follows from the uniqueness of the τ-measurable extension (Proposition 2.51). $\qquad\square$

PROPOSITION 7.48. *For $q \in [2, \infty)$, each of the maps $\mathfrak{j}^{(q)}$ is linear and injective. Moreover they are related through the estimate*

$$\|\mathfrak{j}^{(q)}(a)\|_q \leq \|\mathfrak{j}^{(2)}(a)\|_2^{2/q}\|a\|_\infty^{1 - 2/q} \tag{7.2}$$

for $q \in \{2^k \colon k \in \mathbb{N}\}$ and $a \in \mathfrak{n}_\varphi$.

PROOF. For $a, b \in \mathfrak{n}$ and $\lambda \in \mathbb{C}$, both $\mathfrak{j}^{(q)}(a + \lambda b)$ and the strong sum $\mathfrak{j}^{(q)}(a) + \lambda \mathfrak{j}^{(q)}(b)$ are closed extensions of the τ-premeasurable operator $ah^{1/q} + \lambda bh^{1/q}$, so linearity follows from the uniqueness of the τ-measurable extension (Proposition 2.51). Injectivity of the map $\mathfrak{j}^{(q)}$ follows from the injectivity of the operator $h^{1/q}$. By Lemma 7.47 and Hölder's Inequality, we have that

$$
\begin{aligned}
\left\|\mathfrak{j}^{(2q)}(a)\right\|_{2q}^{2q} &= \operatorname{tr}\left(\left(h^{1/2q}a^*\left[ah^{1/2q}\right]\right)^q\right) \\
&= \operatorname{tr}\left(\left(\left[ah^{1/2q}\right]h^{1/2q}a^*\right)^q\right) \\
&= \operatorname{tr}\left(\left(\left[ah^{1/q}\right]a^*\right)^q\right) \\
&= \left\|\mathfrak{j}^{(q)}(a)a^*\right\|_q^q \leq \left\|\mathfrak{j}^{(q)}(a)\right\|_q^q\|a\|_\infty^q
\end{aligned}
$$

for any $q \geq 2$, $a \in \mathfrak{n}_\varphi$. The estimate follows by iterating this inequality. $\qquad\square$

THEOREM 7.49. *For $p \in [1, \infty)$ the map*

$$\mathfrak{i}^{(p)} : \mathfrak{p}_\varphi \to L^p(\mathcal{M})$$

is additive and injective—in particular it has a unique extension to a linear map (also called $\mathfrak{i}^{(p)}$) from $\mathfrak{m}_\varphi$ into $L^p(\mathcal{M})$. The extension is injective and positivity preserving, and for any $a, b \in \mathfrak{n}_\varphi$ satisfies the formula

$$\mathfrak{j}^{(2p)}(a)^* \overline{\mathfrak{j}}^{(2p)}(b) = \mathfrak{i}^{(p)}(a^*b).$$

As a positive map it is normal in the sense that if a net $(a_\lambda) \subseteq \mathfrak{p}_\varphi$ increases to $a \in \mathfrak{p}_\varphi$, then $\mathfrak{i}^{(p)}(a_\lambda)$ increases to $\mathfrak{i}^{(p)}(a)$.

PROOF. Suppose that $a, b \in \mathfrak{p}_\varphi$ and

$$h^{1/2p} a^{1/2} \left[a^{1/2} h^{1/2p} \right] = h^{1/2p} b^{1/2} \left[b^{1/2} h^{1/2p} \right].$$

By the injectivity of $h^{1/2p}$, $a^{1/2} \left[a^{1/2} h^{1/2p} \right] = b^{1/2} \left[b h^{1/2p} \right]$ and so $a h^{1/2p} = b h^{1/2p}$. But $h^{1/2p}$ has dense range, and a and b are bounded, so $a = b$. Hence the map is injective. To prove additivity we must show that for $a, b \in \mathfrak{p}_\varphi$,

$$h^{1/2p} (a+b)^{1/2} [(a+b)^{1/2} h^{1/2p}] = h^{1/2p} a^{1/2} [a^{1/2} h^{1/2p}] + h^{1/2p} b^{1/2} [b^{1/2} h^{1/2p}]$$
$$(*)$$

where the sum on the right is in the strong sense. Since $x_1 := (a+b)^{1/2} \left[(a+b)^{1/2} h^{1/2p} \right]$ and $x_2 := a^{1/2} \left[a^{1/2} h^{1/2p} \right] + b^{1/2} \left[b^{1/2} h^{1/2p} \right]$ are closable τ-pre-measurable extensions of the densely defined operator $(a + b) h^{1/2p}$, their closures coincide by the uniqueness of the τ-measurable extension (Proposition 2.51). Since then $h^{1/2p} [x_1] \supseteq h^{1/2p} (a+b)^{1/2} \left[(a+b)^{1/2} h^{1/2p} \right]$ and

$$h^{1/2p}[x_2] \ \supseteq \ h^{1/2p} \left[a^{1/2} \left[a^{1/2} h^{1/2p} \right] + b^{1/2} \left[b^{1/2} h^{1/2p} \right] \right]$$
$$\supseteq \ h^{1/2p} a^{1/2} \left[a^{1/2} h^{1/2p} \right] + h^{1/2p} b^{1/2} \left[b^{1/2} h^{1/2p} \right],$$

$(*)$ holds on the intersection of the domains of $\mathfrak{i}^{(p)}(a+b)$, $\mathfrak{i}^{(p)}(a)$ and $\mathfrak{i}^{(p)}(b)$. But this intersection is τ-dense, and hence uniqueness of the τ-measurable extension (Proposition 2.51) demands that $(*)$ holds unreservedly. Therefore $\mathfrak{i}^{(p)}$ is additive. Since $\mathfrak{m}_\varphi$ is linearly spanned by its non-negative

elements, the prescription

$$(a_1 - a_2) + i(a_3 - a_4) \mapsto \mathfrak{i}^{(p)}(a_1) - \mathfrak{i}^{(p)}(a_2) + i\left(\mathfrak{i}^{(p)}(a_3) - \mathfrak{i}^{(p)}(a_4)\right), \quad a_k \in \mathfrak{p}_\varphi$$

then gives a well-defined extension of $\mathfrak{i}^{(p)}$ to a linear map $\mathfrak{m}_\varphi \to L^p(\mathcal{M})$. Clearly this is the promised unique linear map extension of $\mathfrak{i}^{(p)}$ to all of $\mathfrak{m}_\varphi$, which is moreover injective and positivity preserving.

We now prove the stated formula for realising the extension. First consider the case where $b = a$. This case follows from the fact that $y_1 := ((h^{1/2p}a^*)[ah^{1/2p}])$ and $y_2 := ((h^{1/2p}|a|)[|a|h^{1/2p}]$ coincide. (This was verified at the start of the proof of part (b) of Theorem 7.45.) Since each of the maps $(a, b) \mapsto \mathfrak{i}^{(p)}(a^*b)$ and $(a, b) \mapsto \mathfrak{j}^{(2p)}(a)^* \overline{\mathfrak{j}}^{(2p)}(b)$ is sesquilinear (by the linearity of $\mathfrak{i}^{(p)}$ and $\mathfrak{j}^{(2p)}$) the full result follows from the polarisation identity.

Finally let $a, a_\lambda \in \mathfrak{p}_\varphi$ ($\lambda \in \Lambda$) be given, with $a_\lambda \nearrow a$. The fact that the map $\mathfrak{i}^{(p)}$ is order preserving, ensures that $0 \leq \mathfrak{i}^{(p)}(a_\lambda) \leq \mathfrak{i}^{(p)}(a)$ for each λ, and hence that $\sup_\lambda \mathfrak{i}^{(p)}(a_\lambda) = g$ exists as an element of $\widetilde{\mathfrak{M}}_+$, for which $g \leq \mathfrak{i}^{(p)}(a)$ (see Proposition 2.61). For any $\xi \in H$, we have that

$$\langle a_\lambda^{1/2} h^{1/2p}\xi, a_\lambda^{1/2} h^{1/2p}\xi \rangle \nearrow \langle a^{1/2} h^{1/2p}\xi, a^{1/2} h^{1/2p}\xi \rangle.$$

We shall use this fact to prove that $g = \mathfrak{i}^{(p)}(a)$, but we need some technical information before we are able to do so.

Let f be a closable operator on the Hilbert space H, with minimal closure $\overline{f}$. Let $\mathscr{G}(\overline{f})$ be the graph of $\overline{f}$, and P the bounded mapping $P : \mathscr{G}(\overline{f}) \to H : (u, v) \mapsto u$. By [**KR83**, Remark 2.7.7] P^* has dense range, which is contained in $\text{dom}((\overline{f})^* \overline{f})$ and which is a core for $\overline{f}$. By the closability of f, $\mathscr{G}(f)$ is dense in $\mathscr{G}(\overline{f})$, and so by continuity, $P^*(\mathscr{G}(f))$ is dense in $P^*(\mathscr{G}(\overline{f}))$, and therefore also in H. The subspace $P^*(\mathscr{G}(f))$ is clearly contained in $P^*(\mathscr{G}(\overline{f}))$ and hence in $\text{dom}((\overline{f})^* \overline{f})$, and is by definition a core for $\overline{f}$.

By Proposition 7.40, $h^{1/2p}a^{1/2}$ is τ-measurable and $a^{1/2}h^{1/2p}$ densely defined and closable, with the τ-measurable closure given by $[a^{1/2}h^{1/2p}] = (h^{1/2p}a^{1/2})^*$. As follows from the preceding discussion, $\text{dom}(a^{1/2}h^{1/2p}) = \text{dom}(h^{1/2p})$ contains a core $\mathcal{C}$ of $[a^{1/2}h^{1/2p}]$, which is also contained in

$$\text{dom}((h^{1/2p}a^{1/2})[a^{1/2}h^{1/2p}]) = \text{dom}(\mathfrak{i}^{(p)}(a)).$$

Therefore the previously centred equation may be reformulated as the claim that

$$\langle \mathfrak{i}^{(p)}(a_\lambda)\xi, \xi \rangle \nearrow \langle \mathfrak{i}^{(p)}(a)\xi, \xi \rangle \text{ for all } \xi \in \mathcal{C}.$$

This means that $\langle \mathfrak{i}^{(p)}(a)\xi, \xi \rangle = \langle g\xi, \xi \rangle$ for all $\xi \in \mathcal{C}$. But by the polarisation identity, this in turn ensures that $\langle \mathfrak{i}^{(p)}(a)\xi, \zeta \rangle = \langle g\xi, \zeta \rangle$ for all $\xi, \zeta \in \mathcal{C}$, and hence that $\mathfrak{i}^{(p)}(a)\xi = g\xi$ for all $\xi \in \mathcal{C}$. We therefore we have two τ-measurable operators agreeing on a dense subspace of H. By Proposition 2.51 this suffices to ensure that $\mathfrak{i}^{(p)}(a) = g$.   □

REMARK 7.50. With the technicalities regarding the map $\mathfrak{i}^{(p)}$ now taken care of, we will for a given $f \in \mathfrak{m}_\varphi$ in the rest of these notes simply write $h^{1/2p} f h^{1/2p}$ for $\mathfrak{i}^{(p)}(f)$ where convenient.

PROPOSITION 7.51. *Given $p \in [1, \infty)$, let*

$$\mathfrak{s}_\varphi = \mathrm{span}\{\mathfrak{i}^{(p)}(e) \colon e \in \mathbb{P}(\mathcal{M}), \varphi(e) < \infty\}.$$

The positive cone $\mathfrak{i}^{(p)}(\mathfrak{s}_\varphi^+)$ is dense in $L^p_+(\mathcal{M})$.

PROOF. We shall prove that the closure of $\mathfrak{i}^{(p)}(\mathfrak{s}_\varphi^+)$ contains $\mathfrak{i}^{(p)}(\mathfrak{p}_\varphi)$. The claim will then follow from Theorem 7.45. So let $a \in \mathfrak{p}_\varphi$ be given. With e_λ denoting the spectral resolution of a, we of course have that $a = \int_0^{\|a\|} \lambda \, de_\lambda$. We may now select a sequence (g_n) of Riemann sums increasing to a. These Riemann sums are of course of the form $a_n = \sum_{k_n=1}^{N_n} \gamma_{k_n} e(k_n)$ where the $e(k_n)$'s are mutually orthogonal spectral projections of a, and the γ_{k_n}'s non-negative reals. Next observe that for each $\epsilon > 0$, we have that $\varphi(\chi_{(\epsilon,\infty)}(a)) \leq \epsilon^{-1}\varphi(a) < \infty$. Using this fact we now replace each g_n with $a_n = g_n \chi_{(n^{-1},\infty)}(a)$. The result is a positive "simple function" for which all the projections now have finite weight. Since $g_n \nearrow a$ and $\chi_{(n^{-1},\infty)}(a) \nearrow \mathbb{1}$, we may use the Borel functional calculus to conclude that $a_n \nearrow a$. We will show that $\mathfrak{i}^{(p)}(a_n)$ converges to $\mathfrak{i}^{(p)}(a)$. The case $p = 1$ is somewhat simpler to describe. We therefore first deal with this case, before showing how that argument may be modified to prove the general case.

Consider the case $p = 1$, and let $b \in \mathcal{M}_+$ be given. It then follows from Proposition 2.63 that $b^{1/2}\mathfrak{i}^{(p)}(a_n)b^{1/2} \nearrow b^{1/2}\mathfrak{i}^{(p)}(a)b^{1/2}$. We may therefore

use part (iii) of Proposition 4.12 to conclude that

$$
\begin{aligned}
\operatorname{tr}(b\mathfrak{i}^{(p)}(a_n)) &= \operatorname{tr}(b^{1/2}\mathfrak{i}^{(p)}(a_n)b^{1/2}) \\
&= \mathbf{m}_{b^{1/2}\mathfrak{i}^{(p)}(a_n)b^{1/2}}(1) \\
&\nearrow \mathbf{m}_{b^{1/2}\mathfrak{i}^{(p)}(a)b^{1/2}}(1) \\
&= \operatorname{tr}(b^{1/2}\mathfrak{i}^{(p)}(a)b^{1/2}) \\
&= \operatorname{tr}(b\mathfrak{i}^{(p)}(a)).
\end{aligned}
$$

Since each $b \in \mathcal{M}$ is a linear combination of four positive elements, it follows that $\operatorname{tr}(b\mathfrak{i}^{(p)}(a_n)) \to \operatorname{tr}(b\mathfrak{i}^{(p)}(a))$ for each $b \in \mathcal{M}$. Therefore $\mathfrak{i}^{(p)}(a)$ is in the weak closure of $\mathfrak{i}^{(p)}(\mathfrak{s}_\varphi^+)$, which by convexity must agree with the norm closure.

Now suppose that $1 < p < \infty$, and let $1 < q < \infty$ be given with $1 = p^{-1} + q^{-1}$. Given $b \in \mathfrak{p}_\varphi$ it will in this case follow that

$$
\mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a_n)\mathfrak{j}^{(2q)}(b^{1/2}) \nearrow \mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a)\mathfrak{j}^{(2q)}(b^{1/2}).
$$

As before we may then use part (iii) of Proposition 4.12 to conclude that

$$
\begin{aligned}
\operatorname{tr}(\mathfrak{i}^{(q)}(b)\mathfrak{i}^{(p)}(a_n)) &= \operatorname{tr}(\mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a_n)\mathfrak{j}^{(2q)}(b^{1/2})^*) \\
&= \mathbf{m}_{\mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a_n)\mathfrak{j}^{(2q)}(b^{1/2})^*}(1) \\
&\nearrow \mathbf{m}_{\mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a)\mathfrak{j}^{(2q)}(b^{1/2})^*}(1) \\
&= \operatorname{tr}(\mathfrak{j}^{(2q)}(b^{1/2})\mathfrak{i}^{(p)}(a)\mathfrak{j}^{(2q)}(b^{1/2})^*) \\
&= \operatorname{tr}(\mathfrak{i}^{(q)}(b)\mathfrak{i}^{(p)}(a)).
\end{aligned}
$$

Again as before we use the fact $b \in \mathfrak{m}_\varphi$ is a linear combination of four positive elements of $\mathfrak{m}_\varphi$, to see that $\operatorname{tr}(\mathfrak{i}^{(q)}(b)\mathfrak{i}^{(p)}(a_n)) \to \operatorname{tr}(\mathfrak{i}^{(q)}(b)\mathfrak{i}^{(p)}(a))$ for each $b \in \mathfrak{m}_\varphi$. For a general $f \in L^q(\mathcal{M})$, let $\epsilon > 0$ be given, and select $b \in \mathfrak{m}_\varphi$ so that $\|f - \mathfrak{i}^{(q)}(b)\|_q \leq \epsilon$. Notice that the fact that $0 \leq \mathfrak{i}^{(p)}(a_n) \leq \mathfrak{i}^{(p)}(a)$, ensures that $\|\mathfrak{i}^{(p)}(a_n)\|_p = \mathbf{m}_{\mathfrak{i}^{(p)}(a_n)}(1) \leq \mathbf{m}_{\mathfrak{i}^{(p)}(a)}(1) = \|\mathfrak{i}^{(p)}(a)\|_p$. It therefore follows that

$$
\begin{aligned}
&|\operatorname{tr}(f(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))| \\
&\quad \leq |\operatorname{tr}((f - \mathfrak{i}^{(q)}(b))(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))| + |\operatorname{tr}(\mathfrak{i}^{(q)}(b)(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))| \\
&\quad \leq 2\epsilon\|\mathfrak{i}^{(p)}(a)\|_p + |\operatorname{tr}(\mathfrak{i}^{(q)}(b)(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))|.
\end{aligned}
$$

By the first part of the proof there must then exist $N \in \mathbb{N}$ so that $|\mathrm{tr}(f(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))| \leq \epsilon(2\|\mathfrak{i}^{(p)}(a)\|_p + 1)$ for all $n \geq N$. Thus by definition

$$\lim_{n \to \infty} |\mathrm{tr}(f(\mathfrak{i}^{(p)}(a) - \mathfrak{i}^{(p)}(a_n)))| = 0.$$

So as in the former case, we have that $\mathfrak{i}^{(p)}(a)$ is in the weak closure of $\mathfrak{i}^{(p)}(\mathfrak{s}_\varphi^+)$, and hence by convexity in the norm closure. $\qquad\square$

In the case where φ is a state rather than a weight, one is able to obtain a much stronger result with significantly less difficulty. We close this section with a consideration of this case.

PROPOSITION 7.52. *Let φ be a faithful normal state on $\mathcal{M}$. For any $p \in (0, \infty)$, $h^{1/p} \in L^p(\mathcal{M})$ where $h = \frac{d\widetilde{\varphi}}{d\tau}$. In addition for any $p \in (0, \infty)$ and $c \in [0, 1]$, $\mathrm{span}(h^{c/p}\mathbb{P}(\mathcal{M})h^{(1-c)/p})$ is dense in $L^p(\mathcal{M})$.*

We note that it is precisely the fact that in this case h is τ-measurable, that renders this case significantly simpler to deal with.

PROOF. Notice that the first claim is a direct consequence of Theorem 6.72. We shall prove the second claim in several stages, the first of which is the claim that for any $p \in [1, \infty)$, $h^{1/p}\mathcal{M}$ is norm-dense in $L^p(\mathcal{M}, \tau)$. It is a simple matter to check that $h^{1/p}\mathcal{M} \subseteq L^p(\mathcal{M})$. Now let $q \geq 1$ be given with $1 = p^{-1} + q^{-1}$, and let $g \in L^q(\mathcal{M})$ be given with $\mathrm{tr}(gh^{1/p}a) = 0$ for all $a \in \mathcal{M}$. It is easy to check that $gh^{1/p} \in L^1(\mathcal{M})$, and hence the duality theory developed earlier shows that we must then have that $gh^{1/p} = 0$. But h is a positive non-singular element of $\widetilde{\mathfrak{M}}$, which ensures that $\mathrm{ran}(h^{1/p})$ is a dense subspace of the underlying Hilbert space. The two τ-measurable operators g and 0 agree on this dense subspace, and so by the uniqueness of the τ-measurable extension (Proposition 2.51), we must have that $g = 0$. By the theory of L^p-duality developed earlier, this means that $h^{1/p}\mathcal{M}$ is weakly dense in $L^p(\mathcal{M})$. But since $h^{1/p}\mathcal{M}$ is convex, the norm closure must agree with the weak closure, proving the claim in this case. Now suppose that $p \in [\frac{1}{2}, 1)$. It is a simple matter to use Hölder's inequality to see that for any $a \in \mathcal{M}$, the embedding $L^{2p}(\mathcal{M}) \to L^p(\mathcal{M}) : f \mapsto h^{1/2p}fa$ is well-defined and continuous. It therefore follows from the former case that $h^{1/p}\mathcal{M} = h^{1/2p}(h^{1/2p}\mathcal{M})\mathcal{M}$ is dense in $h^{1/2p}L^{2p}(\mathcal{M})\mathcal{M}$, and hence that $\overline{h^{1/p}\mathcal{M}} \supseteq h^{1/2p}L^{2p}(\mathcal{M})\mathcal{M} \supseteq h^{1/2p}(\mathcal{M}h^{1/2p})\mathcal{M}$. We may now once again use the Hölder inequality to see that the bi-linear map $L^{2p}(\mathcal{M}) \times L^{2p}(\mathcal{M}) \to L^p(\mathcal{M}) : (f, g) \mapsto fg$

is well-defined and continuous, from which it then follows that $\overline{h^{1/p}\mathcal{M}} \supseteq$ $\overline{(h^{1/2p}\mathcal{M})(h^{1/2p}\mathcal{M})} \supseteq L^{2p}(\mathcal{M}).L^{2p}(\mathcal{M}) = L^p(\mathcal{M})$. Given that $h^{1/p}\mathcal{M} \subseteq$ $L^p(\mathcal{M})$, this clearly suffices to prove the claim.

To prove the claim for $p \in (0,1)$, one simply iterates the above procedure.

Since the process of taking adjoints is continuous with respect to the L^p-topology, it now trivially follows that for each $p \in (0,\infty)$, $\mathcal{M}h^{1/p} = (h^{1/p}\mathcal{M})^*$ is dense in $L^p(\mathcal{M})$.

Next let $c \in (0,1)$ be given. As before we may use Hölder's inequality to see that the bi-linear map

$$L^{p/c}(\mathcal{M}) \times L^{p/(1-c)}(\mathcal{M}) \to L^p(\mathcal{M}) : (f,g) \mapsto fg$$

is well-defined and continuous. Using this fact, the respective density of $h^{c/p}\mathcal{M}$ and $\mathcal{M}h^{(1-c)/p}$ in $L^{p/c}(\mathcal{M})$ and $L^{p/(1-c)}(\mathcal{M})$, ensures that $h^{c/p}\mathcal{M}h^{(1-c)/p} = (h^{c/p}\mathcal{M})(\mathcal{M}h^{(1-c)/p})$ is dense in $L^{p/c}(\mathcal{M}).L^{p/(1-c)}(\mathcal{M}) = L^p(\mathcal{M})$.

We have therefore proven that for each $p \in (0,\infty)$ and each $c \in [0,1]$, $h^{c/p}\mathcal{M}h^{(1-c)/p}$ is dense in $L^p(\mathcal{M})$. To conclude the proof, we need to show that $h^{c/p}\mathcal{M}h^{(1-c)/p}$ is contained in the closure of $\mathrm{span}(h^{c/p}\mathbb{P}(\mathcal{M})h^{(1-c)/p})$. For this, it suffices to consider some $a \in \mathcal{M}_+$, and show that $h^{c/p}ah^{(1-c)/p}$ is in this closure. Given $a \in \mathcal{M}_+$, we may argue as in the proof of the previous proposition, to select a sequence $(a_n) \subseteq \mathrm{span}(\mathbb{P}(\mathcal{M}))$ of (non-commutative) Riemann-sums converging to a in the L^∞-norm. Hölder's inequality once again informs us that the embedding $\mathcal{M} \to L^p(\mathcal{M}) : b \mapsto h^{c/p}bh^{(1-c)/p}$ is well-defined and continuous. Hence, $(h^{c/p}a_nh^{(1-c)/p})$ must converge to $h^{c/p}ah^{(1-c)/p}$ in the L^p-topology.   □

7.5 $L^2(\mathcal{M})$ and the standard form of a von Neumann algebra

Let $\mathcal{M}$ be a von Neumann algebra, and φ a faithful normal semifinite weight on $\mathcal{M}$. Our primary goal in this section is to prove that the theory of $L^p(\mathcal{M})$-spaces is rich enough to allow for the realisation of a standard form of $\mathcal{M}$ in the sense of Definition 6.18 within this framework.

DEFINITION 7.53. For each $p \in [1,\infty]$, we define left λ_p and right ρ_p actions of $\mathcal{M}$ on $L^p(\mathcal{M})$, by the prescriptions

$$\lambda_p(a)f = af \qquad f \in L^p(\mathcal{M}),$$
$$\rho_p(a)f = fa \qquad f \in L^p(\mathcal{M}).$$

Given $a \in \mathcal{M}$, it follows from Proposition 7.24 that

$$\|\lambda_p(a)f\|_p \le \|a\|_\infty \|f\|_p \text{ and } \|\rho_p(a)f\|_p \le \|a\|_\infty \|f\|_p$$

for each $f \in L^p(\mathcal{M})$, So each of $\lambda_p(a)$ and $\rho_p(a)$ continuously maps $L^p(\mathcal{M})$ into $L^p(\mathcal{M})$.

PROPOSITION 7.54. *For each $p \in [1, \infty]$ the following holds:*

(a) λ_p *is faithful representation and ρ_p a faithful anti-representation of $\mathcal{M}$ on the Banach space $L^p(\mathcal{M})$.*

(b) *For all $a \in \mathcal{M}$ we have that $J_p \lambda_p(a) J_p = \rho_p(a^*)$ and $J_p \rho_p(a) J_p = \lambda_p(a^*)$, where J_p denotes the anti-linear isometric involution $f \mapsto f^*$ on $L^p(\mathcal{M})$.*

(c) *For any z in the centre of $\mathcal{M}$, we have that $\lambda_p(z) = \rho_p(z)$.*

PROOF. *Part (a):* Suppose that for some $a \in \mathcal{M}$, we have that $\lambda_p(a) = 0$. Let $1 \le q$ be given so that $1 = p^{-1} + q^{-1}$. For any $f \in L^p(\mathcal{M})$ and any $g \in L^q(\mathcal{M})$, we then have that $\text{tr}(afg) = \text{tr}((\lambda_p(a)f)g) = 0$. It is not difficult to show that $L^p(\mathcal{M}).L^q(\mathcal{M}) = L^1(\mathcal{M})$. Hence by duality we must then have that $a = 0$. To see the multiplicativity, notice that for any $a, b \in \mathcal{M}$ and $f \in L^p(\mathcal{M})$ we have that $\lambda_p(ab)f = abf = a\lambda_p(b)(f) = (\lambda_p(a) \circ \lambda_p(b))(f)$. We leave the proof of the linearity of the map $a \mapsto \lambda_p(a)$ as an exercise. To see that this map is continuous, observe that it follows from the discussion preceding this proposition that $\|\lambda_p(a)\| \le \|a\|_\infty$. The proof that $a \mapsto \rho_p(a)$ is an anti-representation runs along similar lines.

Part (b): For all $a \in \mathcal{M}$ and $f \in L^p(\mathcal{M})$ we have that $J_p \lambda_p(a) J_p(f) = (af^*)^* = fa^* = \rho_p(a^*)(f)$ and that $J_p \rho_p(a) J_p(f) = (f^*a)^* = a^*f = \lambda_p(a^*)(f)$.

Part (c): It follows from Theorem 6.62 that z is in the centre of $\mathfrak{M}$, and hence (by the density of $\mathfrak{M}$ in $\widetilde{\mathfrak{M}}$), in the centre of $\widetilde{\mathfrak{M}}$. Having noted this, it is now trivial to see that for any $f \in L^p(\mathcal{M})$, we have $\lambda_p(z)f = zf = fz = \rho_p(z)(f)$. $\qquad\square$

THEOREM 7.55. *Let $p \in [1, \infty]$. For any subset $\mathcal{S} \subseteq B(L^p(\mathcal{M}))$, we will write $\mathcal{S}'$ for the set of all bounded linear operators on $L^p(\mathcal{M})$ commuting with all the elements of $\mathcal{S}$. We have that $\lambda_p(\mathcal{M}) = \rho_p(\mathcal{M})'$ and $\rho_p(\mathcal{M}) = \lambda_p(\mathcal{M})'$.*

PROOF. For all $a, b \in \mathcal{M}$ and $f \in L^p(\mathcal{M})$, we have that $\lambda_p(a)(\rho_p(b)(f)) = afb = \rho_p(b)(\lambda_p(a)(f))$. Hence $\lambda_p(\mathcal{M}) \subseteq \rho_p(\mathcal{M})'$ and $\rho_p(\mathcal{M}) \subseteq \lambda_p(\mathcal{M})'$.

We show that $\lambda_p(\mathcal{M})' \subseteq \rho_p(\mathcal{M})$. The inclusion $\rho_p(\mathcal{M})' \subseteq \lambda_p(\mathcal{M})$ follows by a similar argument. This will then suffice to prove the theorem.

Case 1 ($p = \infty$): Let $T \in \lambda_\infty(\mathcal{M})'$ be given. For any $f \in L^\infty(\mathcal{M})$ we then have that $T(f) = T(f\mathbb{1}) = T(\lambda_\infty(f)(\mathbb{1})) = \lambda_\infty(f)(T(\mathbb{1})) = fT(\mathbb{1}) = \rho_\infty(T(\mathbb{1}))(f)$. Clearly $T = \rho_\infty(T(\mathbb{1})) \in \rho_\infty(\mathcal{M})$, which proves the claim in this case.

Case 2 ($p = 1$): Let $T \in \lambda_1(\mathcal{M})'$ be given. Let $S = T^\star \in B(L^\infty(\mathcal{M}))$ be the Banach adjoint of T. So by tr-duality we then have that $\mathrm{tr}(T(f)a) = \mathrm{tr}(fS(a))$ for every $f \in L^1(\mathcal{M})$ and $a \in L^\infty(\mathcal{M})$. For any $a, b \in L^\infty(\mathcal{M})$ and $f \in L^1(\mathcal{M})$ we have that $\mathrm{tr}(S(ab)f) = \mathrm{tr}(abT(f)) = \mathrm{tr}(a\lambda_1(b)(T(f)))$ $= \mathrm{tr}(aT(\lambda_1(b)(f))) = \mathrm{tr}(aT(bf)) = \mathrm{tr}(S(a)bf)$. It therefore follows that $S(ab) = S(a)b$ for all $a, b \in L^\infty(\mathcal{M})$. This equality may be reformulated as the claim that $S \circ \rho_\infty(b) = \rho_\infty(b) \circ S$. It therefore follows from case 1 that $S = \lambda_\infty(x)$ for some $x \in L^\infty(\mathcal{M})$. But for every $a \in L^\infty(\mathcal{M})$ and $f \in L^1(\mathcal{M})$, we then have that $\mathrm{tr}(T(f)a) = \mathrm{tr}(fS(a)) = \mathrm{tr}(f\lambda_\infty(x)(a)) = \mathrm{tr}(fxa) = \mathrm{tr}(\rho_1(x)(f)a)$. This ensures that $T = \rho_1(x)$.

Case 3 ($1 < p < \infty$): Let $T \in \lambda_p(\mathcal{M})'$ be given, and select $q > 1$ so that $1 = p^{-1} + q^{-1}$. Our strategy will be to reduce the problem to the setting of case 2. To achieve this objective, we formally define a linear map $S : L^1(\mathcal{M}) \to L^1(\mathcal{M})$ by the prescription $S(\sum_{k=1}^n b_k a_k) = \sum_{k=1}^n b_k T(a_k)$ for all n-tuples $a_1, a_2, \ldots, a_n \in L^p(\mathcal{M})$ and $b_1, b_2, \ldots, b_n \in L^q(\mathcal{M})$.

We first establish the well-definiteness of S. Assume $a_1, a_2, \ldots, a_n \in L^p(\mathcal{M})$ and $b_1, b_2, \ldots, b_n \in L^q(\mathcal{M})$ are given with $\sum_{k=1}^n b_k a_k = 0$. Consider the term $a = (\sum_{k=1}^n a_k^* a_k)^{1/2}$. Since $a_m^* a_m \leq a^2$ for each $1 \leq m \leq n$, it then follows from Proposition 2.65 that there exist contractions $c_1, c_2, \ldots, c_n \in \mathcal{M}_+$ supported on $\mathrm{s}(a)$, such that $ac_m a = a_m^* a_m$ for each $1 \leq m \leq n$. Since $a(\sum_{k=1}^n c_k)a = a^2$, we must have that $\sum_{k=1}^n c_k = \mathrm{s}(a)$. Now observe that $|c_m^{1/2}a| = |a_m|$ for each $1 \leq m \leq n$. We may therefore select partial isometries u_k ($1 \leq k \leq n$) so that for $v_k = u_k c_k^{1/2}$ ($1 \leq k \leq n$), we have that $v_k a = a_k$ ($1 \leq k \leq n$), and $\sum_{k=1}^n |v_k|^2 = \mathrm{s}(a)$. Now consider the term $\sum_{k=1}^n b_k v_k$. By construction we then have that $\mathrm{s}_r(\sum_{k=1}^n b_k v_k) \leq \mathrm{s}_r(\sum_{k=1}^n v_k) \leq \mathrm{s}(\sum_{k=1}^n |v_k|^2) = \mathrm{s}(a)$. But we also have that $(\sum_{k=1}^n b_k v_k)a = \sum_{k=1}^n b_k a_k = 0$. Together these two facts ensure that

$\sum_{k=1}^{n} b_k v_k = 0$. It therefore follows that

$$
\begin{aligned}
\sum_{k=1}^{n} b_k T(a_k) &= \sum_{k=1}^{n} b_k T(v_k a) \\
&= \sum_{k=1}^{n} b_k T(\lambda_p(v_k)a) \\
&= \sum_{k=1}^{n} b_k \lambda_p(v_k)(T(a)) \\
&= \Big(\sum_{k=1}^{n} b_k v_k\Big) T(a) \\
&= 0
\end{aligned}
$$

as required.

The linearity of S is clear. Hence we next show that S is in fact bounded. Let $0 \neq c \in L^1(\mathcal{M})$ be given, with $c = u|c|$ being the polar form of c. We then set $a = u|c|^{1/p}$ and $b = |c|^{1/q}$. It is clear that $ab = c$. However we also have that $\|a\|_p \|b\|_q = \mathrm{tr}(|a|^p)^{1/p}\mathrm{tr}(|b|^q)^{1/q} = \mathrm{tr}(|c|)^{1/p}\mathrm{tr}(|c|)^{1/q} = \mathrm{tr}(|c|) = \|c\|_1$. We may therefore use Hölder's inequality to conclude that $\|S(c)\|_1 = \|bT(a)\|_1 \leq \|b\|_q \|T(a)\|_p \leq \|T\|\|b\|_q\|a\|_p = \|T\|.\|c\|_1$.

With the existence of S verified, we now use S to prove that $T \in \rho_p(\mathcal{M})$. First note that for any $f \in \mathcal{M}$, $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$, it follows from the definition of S that $S(fba) = fbT(a) = fS(ba)$. This translates to the claim that $S \circ \lambda_1(f) = \lambda_1(f) \circ S$ for each $f \in L^\infty(\mathcal{M})$. It therefore follows from case 2, that $S = \rho_1(x)$ for some $x \in L^\infty(\mathcal{M})$. But for all $a \in L^p(\mathcal{M})$ and $b \in L^q(\mathcal{M})$, we then have that

$$
\mathrm{tr}(bT(a)) = \mathrm{tr}(S(ba)) = \mathrm{tr}((ba)x) = \mathrm{tr}(b(ax)) = \mathrm{tr}(b\rho_p(x)(a)),
$$

and hence that $T(a) = \rho_p(x)$ for all $a \in L^p(\mathcal{M})$.   □

Our primary interest is in the case $p = 2$. So for this case we shall respectively simply write λ, ρ and J, for λ_2, ρ_2 and J_2.

THEOREM 7.56.(a) *λ is a faithful normal *-representation, and ρ a faithful normal *-anti-representation of $\mathcal{M}$ on the Hilbert space $L^p(\mathcal{M})$.*
  (b) *The von Neumann algebras $\lambda(\mathcal{M})$ and $\rho(\mathcal{M})$ are commutants of each other, with $\rho(\mathcal{M}) = J\lambda(\mathcal{M})J$.*
  (c) *The quadruple $(\lambda(\mathcal{M}), L^2(\mathcal{M}), J, L^2_+(\mathcal{M}))$ is a standard form of $\mathcal{M}$ in the sense of Definition 6.18.*

PROOF. *Part (a):* Due to the similarity of the proofs, we will prove only one of the claims. We know from Proposition 7.54 that $\lambda(\mathcal{M})$ is a representation of $\mathcal{M}$. We just need to check that it preserves adjoints and that it is normal. For any $a \in \mathcal{M}$ and $f, g \in L^2(\mathcal{M})$, we have that

$$\langle \lambda(a)f, g \rangle = \langle af, g \rangle = \mathrm{tr}(g^*af) = \mathrm{tr}((a^*g)^*f) = \langle f, a^*g \rangle = \langle f, \lambda(a^*)g \rangle.$$

Hence $\lambda(a^*) = \lambda(a)^*$. Now suppose that $(a_\alpha) \subseteq \mathcal{M}_+$ increases to $a \in \mathcal{M}_+$. For any $f \in L^2(\mathcal{M})$, $(f^*a_\alpha f)$ will then increase to f^*af by Proposition 2.63. But for any $f \in L^2(\mathcal{M})$, we will then have that

$$\sup_\alpha \langle \lambda(a_\alpha)f, f \rangle = \sup_\alpha \mathrm{tr}(f^*a_\alpha f) = \sup_\alpha \mathbf{m}_{f^*a_\alpha f}(1) = \mathbf{m}_{f^*af}(1)$$
$$= \mathrm{tr}(f^*af) = \langle \lambda(a)f, f \rangle.$$

In other words $\sup_\alpha \lambda(a_\alpha) = \lambda(a)$

Part (b): To see that these claims hold, notice that on combining part (b) of Proposition 7.54 and Theorem 7.55, we have that $J\lambda(\mathcal{M})J = \rho(\mathcal{M}) = \lambda(\mathcal{M})'$ and $J\rho(\mathcal{M})J = \lambda(\mathcal{M}) = \rho(\mathcal{M})'$.

Part (c): The fact that $L^2_+(\mathcal{M})$ is a self-dual cone, follows from the final claim of Theorem 7.38. It therefore remains to show that the last three criteria mentioned in Definition 6.18, hold when this cone is used. We investigate these in turn.

- By Proposition 7.54, we have that $J\lambda(z)J = \rho(z^*) = \lambda(z^*)$ for all z in the centre of $\mathcal{M}$.
- For any $f \in L^2_+(\mathcal{M})$, we trivially have that $Jf = f^* = f$.
- For any $a \in \mathcal{M}$ and $f \in L^2_+(\mathcal{M})$, we have by part (b) of Proposition 7.54, that $\lambda(a)(J\lambda(a)J)f = \lambda(a)(\rho(a^*)f) = afa^* \in L^2_+(\mathcal{M})$.

Thus the quadruple $(\lambda(\mathcal{M}), L^2(\mathcal{M}), J, L^2_+(\mathcal{M}))$ is indeed a standard form of $\mathcal{M}$. $\qquad\square$

The identification of $L^2_+(\mathcal{M})$ as a substitute for $\mathscr{P}^\natural$, bears further comment. In the case where φ is a faithful normal state with $\mathcal{M}$ in its GNS-representation, $\mathscr{P}^\natural$ may be defined as the closure of $\{\Delta^{1/4}a\Omega : a \in \mathcal{M}\} = \{\Delta^{1/4}a\Delta^{-1/4}\Omega : a \in \mathcal{M}_+\}$, where Ω is a cyclic and seperating vector representing φ. In this setting $\mathfrak{M}_\varphi = \mathcal{M}$, with $h = \frac{d\widetilde{\varphi}}{d\tau}$ an element of $L^1(\mathcal{M})$ (see Corollary 6.71 and the definition of $L^1(\mathcal{M})$). On passing to the noncommutative L^p-picture, $h^{1/2}$ proves to be a suitable substitute for Ω, since it too is cyclic and separating. The cyclicity follows from Theorem 7.45. To see that it is separating, let $a \in \mathcal{M}_+$ be given.

Since then $\langle ah^{1/2}, h^{1/2}\rangle = \mathrm{tr}(h^{1/2}ah^{1/2}) = \|h^{1/2}ah^{1/2}\|_1$, it is clear that $\langle ah^{1/2}, h^{1/2}\rangle = 0 \Rightarrow h^{1/2}ah^{1/2} = 0 \Rightarrow a = 0$. (The last implication follows from the injectivity of $\mathfrak{i}^{(1)}$.) On the other hand by Proposition 6.40 and Theorem 6.62, h may be regarded as a substitute for Δ_φ. So the formula

$$\mathscr{P}^\natural = \overline{\{\Delta^{1/4}a\Omega \colon a \in \mathcal{M}_+\}} = \overline{\{\Delta^{1/4}a\Delta^{-1/4}\Omega \colon a \in \mathcal{M}_+\}}$$

translates to the formula $\mathscr{P} = \overline{h^{1/4}\mathcal{M}_+ h^{1/4}}$ in the $L^2(\mathcal{M})$ picture, which by Theorem 7.45 is exactly $L_+^2(\mathcal{M})$.

Epilogue: Suggestions for further reading and study

The objective of these notes was to provide the reader with a fairly self-contained introduction to the basic theory of noncommutative L^p-spaces, rather than a complete survey of the theory underlying these spaces. As such even at the level of introductory theory, there are some more advanced topics that have not been presented in these notes. We mention a few of these. Readers interested in advanced applications of the theory who therefore wish to expand their knowledge and proficiency in the theory of these spaces, are well advised to read the references mentioned hereafter.

At the level of semifinite von Neumann algebras there is a very refined and burgeoning theory of noncommutative rearrangement invariant Banach function spaces which in that context supersedes the theory of noncommutative Orlicz spaces presented in these notes. See for example the 2007 survey of Ben de Pagter [dP07] and the references therein for more information on this topic.

As with the classical theory of rearrangement invariant Banach function spaces, the noncommutative theory is similarly deeply interwoven with a matching theory of "noncommutative" real interpolation for the pair $(L^\infty(\mathcal{M}, \tau), L^1(\mathcal{M}, \tau))$ where $\mathcal{M}$ is a semifinite algebras. The seminal paper of Dodds, Dodds and de Pagter [DDdP92] represents the starting point of this theory.

At present the theory of noncommutative Banach function spaces as well as the matching theory of real interpolation is only known to hold in the semifinite setting. The only portion of this theory known to extend to the type III setting is the theory of Orlicz spaces presented in these notes. The development of a type III theory of Banach function spaces and a concomitant theory of real interpolation, is therefore one of the great challenges regarding the structural theory of noncommutative function spaces still facing researchers in the field. On this issue, we hasten to

point out that for type III algebras, a verbatim application of the real-interpolative techniques that prove so successful in the semifinite setting, are known to yield the "wrong" spaces [**PX03**]. So what is needed in the type III setting is not a mere technical modification of the semifinite proofs to ensure their validity in a more general context, but a different truly type III mode of real interpolation.

In stark contrast to this state of affairs, the complex technique of interpolation has been very successfully extended to the type III case. It was Marianne Terp who in 1982 proved validity of this technique in the setting of Connes-Hilsum L^p-spaces [**Ter82**], followed two years later by Hideki Kosaki who in more modern language effectively showed that in the setting of σ-finite algebras a "Haagerup-based" approach works equally well [**Kos84a**].

Another topic worthy of investigation is the canonical extension of linear maps from $\mathcal{M}$ to $\mathcal{M} \rtimes_\alpha G$ to crossed products. For completely bounded maps which commute with the group action, one gets a very well-developed theory. A detailed account of this theory may be found in section 4 of [**HJX10**]. On a similar note the theory of p-integrable maps seeks to investigate those criteria which guarantee the canonical extension of a positive map $T : \mathcal{M} \to \mathcal{M}$ to a map $T^{(p)} : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ in a manner which respects the complex interpolative theory of these spaces. This theory was first investigated in the σ-finite setting by Goldstein and Lindsay [**GL95**]. A more recent account of the general theory may be found in section 5 of [**HJX10**]. This theory is especially useful for the study of the action of Markov maps on L^p-spaces. See for example [**GL95, GL99**], and for those fortunate enough to have it, the unpublished preprint of Goldstein, Lindsay and Skalski [**GLS**]. Recently Majewski and Labuschagne showed that when dealing with completely Markov maps T satisfying the inequality $\varphi \circ T \leq T$ on a von Neumann algebra, the two extension techniques described above, may be combined to yield a technique enabling one to extend such maps not just to $L^p(\mathcal{M})$ spaces, but also to a large class of regular Orlicz spaces [**LM**].

The final issue worth investigating is Haagerup's reduction theorem. This is an ingenious technique enabling one to "approximate" $L^p(\mathcal{M})$ spaces with L^p-spaces corresponding to finite von Neumann algebras equipped with a faithful normal tracial state. With $\mathbb{Q}_D$ denoting the diadic rationals, in for example the σ-finite case, the theorem shows that $L^p(\mathcal{M})$ may be realised as a canonical subspace of $L^p(\mathcal{M} \rtimes_\nu \mathbb{Q}_D)$, with the latter

space up to a linear isometry appearing as the limit of an increasing sequence of $L^p(\mathcal{R}_n, \tau_n)$ spaces. This theorem is especially useful in studying noncommutative H^p spaces corresponding to σ-finite algebras. The application of this theorem to this context was pioneered by Xu [**Xu05**] and was then subsequently also successfully applied by both Ji [**Ji12, Ji14**] and Labuschagne [**Lab17**]. Readers wishing to know more about the theory of noncommutative H^p-spaces can start with reading the survey paper of Blecher and Labuschagne [**BL07**]. This paper focuses exclusively on the case of finite von Neumann algebras, but nevertheless gives a very tight readable fairly comprehensive introduction to some of the main features and techniques of the theory.

In closing we sound a word of warning: In spite of the success of the reduction theorem, L^p-spaces corresponding to type III algebras are quite different to those corresponding to semifinite algebras. This fact follows from the isometric theory of noncommutative L^p-spaces. As early as 1981 Yeadon published a result on linear isometries on L^p-spaces of semifinite algebras [**Yea81**] which faithfully captured the essence of the classical Banach-Lamperti theorem on isometries on $L^p(X, \sigma, \mu)$ spaces. Then in 2005 Junge, Ruan and Sherman [**JRS05**] showed that in the case of 2-isometries, Yeadon's result carries over to the setting of Haagerup L^p-spaces. That same year Sherman proved that even without the 2-isometric restriction, Yeadon's result holds in general in the case of surjectice isometries [**She05**]. Whether or not this result holds for the non-surjective case, is still open. The significance of this result for the present discussion is that it shows that if for any two von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ we can find some $1 \leq p < \infty$ ($p \neq 2$) for which $L^p(\mathcal{M}_1)$ and $L^p(\mathcal{M}_2)$ are linearly isometric, that will force the existence of a bijective normal Jordan *-morphism $\mathscr{J} : \mathcal{M}_1 \to \mathcal{M}_2$. Given a faithful normal semifinite trace τ_2 on $\mathcal{M}_2$, we may now use the fact that $\mathscr{J}$ may be written as the sum of a *-isomorphism and *-anti-isomorphism (see [**BR87b**, Proposition 3.2.2]) to show that $\tau_2 \circ \mathscr{J}$ is a faithful normal semifinite trace on $\mathcal{M}_1$. But type III von Neumann algebras do not admit a faithful normal semifinite traces! So if $\mathcal{M}_1$ is a type III algebra and $\mathcal{M}_2$ semifinite, there is no $1 \leq p < \infty$ ($p \neq 2$) for which $L^p(\mathcal{M}_1)$ is linearly isometric to $L^p(\mathcal{M}_2)$.

Bibliography

[AAP82]   Charles A. Akemann, Joel Anderson, and Gert K. Pedersen. Triangle inequalities in operator algebras. *Linear and Multilinear Algebra*, 11(2):167–178, 1982.

[AM82]   Huzihiro Araki and Tetsuya Masuda. Positive cones and L_p-spaces for von Neumann algebras. *Publ. Res. Inst. Math. Sci.*, 18(2):759–831 (339–411), 1982.

[Arv76]   William Arveson. *An invitation to C^*-algebras.* Springer-Verlag, New York–Heidelberg, 1976. Graduate Texts in Mathematics, No. 39.

[Arv02]   William Arveson. *A short course on spectral theory*, volume 209 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002.

[ARZ07]   Maryam H. A. Al-Rashed and Bogusław Zegarliński. Noncommutative Orlicz spaces associated to a state. *Studia Math.*, 180(3):199–209, 2007.

[BK90]   Lawrence G. Brown and Hideki Kosaki. Jensen's inequality in semi-finite von Neumann algebras. *J. Operator Theory*, 23(1):3–19, 1990.

[BL07]   D.P. Blecher and L.E. Labuschagne. Von neumann algebraic H^p theory. In *Proceedings of the Fifth Conference on Function Spaces*, volume 435 of *Contemporary Mathematics*, pages 89–114. Amer. Math. Soc., 2007.

[Bla06]   B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[BR87a]   Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 1.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987. C^*- and W^*-algebras, symmetry groups, decomposition of states.

[BR87b]   Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 1.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987. C^*- and W^*-algebras, symmetry groups, decomposition of states.

[BS88]   Colin Bennett and Robert Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[Cas13]   M. Caspers. The L^p-fourier transform on locally compact quantum groups. *J. Operator Theory*, 69(1):161–193, 2013.

[CdlS15] Martijn Caspers and Mikael de la Salle. Schur and Fourier multipliers of an amenable group acting on non-commutative L^p-spaces. *Trans. Amer. Math. Soc.*, 367(10):6997–7013, 2015.

[Cec78]   Carlo Cecchini. On two definitions of measurable and locally measurable operators. *Boll. Un. Mat. Ital. A (5)*, 15(3):526–534, 1978.

[CFK14]   F. Cipriani, U. Franz, and A. Kula. Symmetries of Lévy processes on compact quantum groups, their Markov semigroups and potential theory. *J. Funct. Anal.*, 266(5):2789–2844, 2014.

[Com71]   François Combes. Poids et espérances conditionnelles dans les algèbres de von Neumann. *Bull. Soc. Math. France*, 99:73–112, 1971.

[Con73]   Alain Connes. Une classification des facteurs de type III. *Ann. Sci. École Norm. Sup. (4)*, 6:133–252, 1973.

[Con80]   A. Connes. On the spatial theory of von Neumann algebras. *J. Functional Analysis*, 35(2):153–164, 1980.

[CPPR15]   Martijn Caspers, Javier Parcet, Mathilde Perrin, and Éric Ricard. Noncommutative de Leeuw theorems. *Forum Math. Sigma*, 3:e21, 59, 2015.

[CZ76]   Ioana Ciorănescu and László Zsidó. Analytic generators for one-parameter groups. *Tohoku Math. J. (2)*, 28(3):327–362, 1976.

[Dav96]   Kenneth R. Davidson. *C*-algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.

[DDdP89]   Peter G. Dodds, Theresa K.-Y. Dodds, and Ben de Pagter. Noncommutative Banach function spaces. *Math. Z.*, 201(4):583–597, 1989.

[DDdP92]   Peter G. Dodds, Theresa K. Dodds, and Ben de Pagter. Fully symmetric operator spaces. *Integral Equations Operator Theory*, 15(6):942–972, 1992.

[DDdP93]   Peter G. Dodds, Theresa K.-Y. Dodds, and Ben de Pagter. Noncommutative Köthe duality. *Trans. Amer. Math. Soc.*, 339(2):717–750, 1993.

[DdPS]   P.G. Dodds, B. de Pagter, and F. Sukochev. Theory of noncommutative integration. work in progress.

[DFSW16]   M. Daws, P. Fima, A. Skalski, and S. White. The Haagerup property for locally compact quantum groups. *J. Reine Angew. Math.*, 711:189–229, 2016.

[Dig75]   Trond Digernes. *Duality for weights on covariant systems and its applications*. ProQuest LLC, Ann Arbor, MI, 1975. Thesis (Ph.D.) – University of California, Los Angeles.

[Dix53]   J. Dixmier. Formes linéaires sur un anneau d'opérateurs. *Bull. Soc. Math. France*, 81:9–39, 1953.

[Dix57]   Jacques Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (Algèbres de von Neumann)*. Cahiers scientifiques, Fascicule XXV. Gauthier-Villars, Paris, 1957.

[Dix71]   P. G. Dixon. Unbounded operator algebras. *Proc. London Math. Soc. (3)*, 23:53–69, 1971.

[Dix81]   Jacques Dixmier. *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam–New York, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[Dix96a]   Jacques Dixmier. *Les algèbres d'opérateurs dans l'espace hilbertien (algèbres de von Neumann)*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.

[Dix96b] Jacques Dixmier. *Les C*-algèbres et leurs représentations*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.

[DKSS12] M. Daws, P. Kasprzak, A. Skalski, and P.M. Soltan. Closed quantum subgroups of locally compact quantum groups. *Adv. Math.*, 231(6):3473–3501, 2012.

[DL88] R. J. DiPerna and P.-L. Lions. On the Fokker-Planck-Boltzmann equation. *Comm. Math. Phys.*, 120(1):1–23, 1988.

[DL89] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2)*, 130(2):321–366, 1989.

[dP07] B. de Pagter. Non-commutative Banach function spaces. In *Positivity*, Trends Math., pages 197–227. Birkhäuser, 2007.

[Fil96] Peter A. Fillmore. *A user's guide to operator algebras*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1996. A Wiley-Interscience Publication.

[FIW14] M. Fragoulopoulou, A. Inoue, and M. Weigt. Tensor products of unbounded operator algebras. *Rocky Mountain J. Math.*, 44(3):895–912, 2014.

[FK52] Bent Fuglede and Richard V. Kadison. Determinant theory in finite factors. *Ann. of Math. (2)*, 55:520–530, 1952.

[FK86] Thierry Fack and Hideki Kosaki. Generalized s-numbers of τ-measurable operators. *Pacific J. Math.*, 123(2):269–300, 1986.

[FS09] U. Franz and A. Skalski. On idempotent states on quantum groups. *J. Algebra*, 322(5):1774–1802, 2009.

[Gar79] L. Terrell Gardner. An inequality characterizes the trace. *Canadian J. Math.*, 31(6):1322–1328, 1979.

[GL95] Stanisław Goldstein and J. Martin Lindsay. KMS-symmetric Markov semigroups. *Math. Z.*, 219(4):591–608, 1995.

[GL99] Stanisław Goldstein and J. Martin Lindsay. Markov semigroups KMS-symmetric for a weight. *Math. Ann.*, 313(1):39–67, 1999.

[GLS] Stanisław Goldstein, J. Martin Lindsay, and Adam Skalski. Nonsymmetric Dirichlet forms on nontracial von Neumann algebras. work in progress.

[GP15] Stanisław Goldstein and Adam Paszkiewicz. Infinite measures on von Neumann algebras. *Internat. J. Theoret. Phys.*, 54(12):4341–4348, 2015.

[Haa75a] Uffe Haagerup. Normal weights on W^*-algebras. *J. Functional Analysis*, 19:302–317, 1975.

[Haa75b] Uffe Haagerup. The standard form of von Neumann algebras. *Math. Scand.*, 37(2):271–283, 1975.

[Haa78a] Uffe Haagerup. On the dual weights for crossed products of von Neumann algebras. I. Removing separability conditions. *Math. Scand.*, 43(1):99–118, 1978.

[Haa78b] Uffe Haagerup. On the dual weights for crossed products of von Neumann algebras. II. Application of operator-valued weights. *Math. Scand.*, 43(1):119–140, 1978.

[Haa79a]   Uffe Haagerup. L^p-spaces associated with an arbitrary von Neumann alge-
            bra. In *Algèbres d'opérateurs et leurs applications en physique mathématique
            (Proc. Colloq., Marseille, 1977)*, volume 274 of *Colloq. Internat. CNRS*,
            pages 175–184. CNRS, Paris, 1979.

[Haa79b]   Uffe Haagerup. Operator-valued weights in von Neumann algebras. I.
            J. Functional Analysis, 32(2):175–206, 1979.

[Haa79c]   Uffe Haagerup. Operator-valued weights in von Neumann algebras. II.
            J. Functional Analysis, 33(3):339–361, 1979.

[Hil81]   Michel Hilsum. Les espaces L^p d'une algèbre de von Neumann définies par
            la derivée spatiale. *J. Functional Analysis*, 40(2):151–169, 1981.

[HJX10]   Uffe Haagerup, Marius Junge, and Quanhua Xu. A reduction method
            for noncommutative L_p-spaces and applications. *Trans. Amer. Math. Soc.*,
            362(4):2125–2165, 2010.

[HK86]   Herbert Halpern and Victor Kaftal. Compact operators in type III_λ and
            type III_0 factors. *Math. Ann.*, 273(2):251–270, 1986.

[HK87]   Herbert Halpern and Victor Kaftal. Compact operators in type III_λ and
            type III_0 factors. II. *Tohoku Math. J. (2)*, 39(2):153–173, 1987.

[HKZ91]   Herbert Halpern, Victor Kaftal, and László Zsidó. Finite weight projections
            in von Neumann algebras. *Pacific J. Math.*, 147(1):81–121, 1991.

[HLP29]   G. H. Hardy, J. E. Littlewood, and G. Pólya. Some simple inequalities sat-
            isfied by convex functions. *Messenger of Math.*, 58:145–152, 1929.

[HLP88]   G. H. Hardy, J. E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Math-
            ematical Library. Cambridge University Press, Cambridge, 1988. Reprint of
            the 1952 edition.

[HM00]   Henryk Hudzik and Lech Maligranda. Amemiya norm equals Orlicz norm
            in general. *Indag. Math. (N.S.)*, 11(4):573–585, 2000.

[Ji12]   G-X Ji. A noncommutative version of H^p and characterizations of subdiag-
            onal subalgebras. *Integr Equ Oper Theory*, 72:131–149, 2012.

[Ji14]   G-X Ji. Analytic toeplitz algebras and the Hilbert transform associated with
            a subdiagonal algebra. *Sc.i China Math.*, 57(3):579–588, 2014.

[JMP14]   Marius Junge, Tao Mei, and Javier Parcet. Smooth Fourier multipliers on
            group von Neumann algebras. *Geom. Funct. Anal.*, 24(6):1913–1980, 2014.

[JNR09]   M. Junge, M. Neufang, and Z-J Ruan. A representation theorem for locally
            compact quantum groups. *Internat. J. Math.*, 20(3):377–400, 2009.

[JRS05]   Marius Junge, Zhong-Jin Ruan, and David Sherman. A classification for 2-
            isometries of noncommutative L_p-spaces. *Israel J. Math.*, 150:285–314, 2005.

[Kaf77]   Victor Kaftal. On the theory of compact operators in von Neumann algebras.
            I. *Indiana Univ. Math. J.*, 26(3):447–457, 1977.

[Kaf78]   Victor Kaftal. On the theory of compact operators in von Neumann algebras.
            II. *Pacific J. Math.*, 79(1):129–137, 1978.

[Kap68]   Irving Kaplansky. *Rings of operators*. W. A. Benjamin, Inc., New York–
            Amsterdam, 1968.

[Kat04]   Yitzhak Katznelson. *An introduction to harmonic analysis*. Cambridge
            Mathematical Library. Cambridge University Press, Cambridge, third edi-
            tion, 2004.

[KN63]   J. L. Kelley and Isaac Namioka. *Linear topological spaces*. With the collab-
           oration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue
           Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, Kennan T. Smith.
           The University Series in Higher Mathematics. D. Van Nostrand Co., Inc.,
           Princeton, N.J., 1963.

[Kos84a]   Hideki Kosaki. Applications of the complex interpolation method to a von
           Neumann algebra: noncommutative L^p-spaces. *J. Funct. Anal.*, 56(1):29–78,
           1984.

[Kos84b]   Hideki Kosaki. On the continuity of the map $\varphi \to |\varphi|$ from the predual of
           a W^*-algebra. *J. Funct. Anal.*, 59(1):123–131, 1984.

[KPS82]   S. G. Kreĭn, Yu. Ī. Petunīn, and E. M. Semënov. *Interpolation of linear
           operators*, volume 54 of *Translations of Mathematical Monographs*. American
           Mathematical Society, Providence, R.I., 1982. Translated from the Russian
           by J. Szűcs.

[KR83]   Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of
           operator algebras. Vol. I*, volume 100 of *Pure and Applied Mathematics*.
           Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York,
           1983. Elementary theory.

[KR86]   Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of
           operator algebras. Vol. II*, volume 100 of *Pure and Applied Mathematics*.
           Academic Press, Inc., Orlando, FL, 1986. Advanced theory.

[Kun58]   R. A. Kunze. L_p Fourier transforms on locally compact unimodular groups.
           Trans. Amer. Math. Soc., 89:519–540, 1958.

[Kun90]   Wolfgang Kunze. Noncommutative Orlicz spaces and generalized Arens al-
           gebras. *Math. Nachr.*, 147:123–138, 1990.

[Lab13]   Louis E. Labuschagne. A crossed product approach to Orlicz spaces. *Proc.
           Lond. Math. Soc. (3)*, 107(5):965–1003, 2013.

[Lab17]   Louis Labuschagne. Invariant subspaces for H^2 spaces of σ-finite algebras.
           Bull. Lond. Math. Soc., 49(1):33–44, 2017.

[LM]   L.E. Labuschagne and W.A. Majewski. Dynamics on noncommutative orlicz
           spaces. Acta Math. Sci. (to appear) - see arXiv:1605.01210 [math-ph].

[LM11]   Louis E. Labuschagne and Władysław A. Majewski. Maps on noncommu-
           tative Orlicz spaces. *Illinois J. Math.*, 55(3):1053–1081 (2013), 2011.

[LM17]   L. E. Labuschagne and W. A. Majewski. Integral and differential structures
           for quantum field theory, 2017. arXiv:1702.00665.

[Maj17]   W. A. Majewski. On quantum statistical mechanics: A study guide. *Adv.
           Math. Phys*, (2):Article ID 9343717, 2017.

[MC16]   M. A. Muratov and V. I. Chilin. Topological algebras of measurable and
           locally measurable operators. *Sovrem. Mat. Fundam. Napravl.*, 61:115–163,
           2016.

[ML14]   W. Adam Majewski and Louis E. Labuschagne. On applications of Orlicz
           spaces to statistical physics. *Ann. Henri Poincaré*, 15(6):1197–1221, 2014.

[ML18]   W. A. Majewski and L. E. Labuschagne. On entropy for general quantum
           systems, 2018. arXiv:1804.05579, to appear in Adv. Theor. Math. Phys.

[Mur90]   Gerard J. Murphy. C^*-*algebras and operator theory.* Academic Press, Inc., Boston, MA, 1990.

[Mus83]   Julian Musielak. *Orlicz spaces and modular spaces*, volume 1034 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1983.

[MvN36]   F. J. Murray and J. von Neumann. On rings of operators. *Ann. of Math. (2)*, 37(1):116–229, 1936.

[MvN37]   F. J. Murray and J. von Neumann. On rings of operators. II. *Trans. Amer. Math. Soc.*, 41(2):208–248, 1937.

[MvN43]   F. J. Murray and J. von Neumann. On rings of operators. IV. *Ann. of Math. (2)*, 44:716–808, 1943.

[Naĭ72]   M. A. Naĭmark. *Normed algebras.* Wolters-Noordhoff Publishing, Groningen, third edition, 1972. Translated from the second Russian edition by Leo F. Boron, Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics.

[Nel74]   Edward Nelson. Notes on non-commutative integration. *J. Functional Analysis*, 15:103–116, 1974.

[NP06]   Constantin P. Niculescu and Lars-Erik Persson. *Convex functions and their applications*, volume 23 of *CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC.* Springer, New York, 2006. A contemporary approach.

[NR11]   S. Neuwirth and E. Ricard. Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group. *Canad. J. Math.*, 63(5):1161–1187, 2011.

[Ped89]   Gert K. Pedersen. *Analysis now*, volume 118 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1989.

[Ped18]   Gert K. Pedersen. C^*-*algebras and their automorphism groups.* Pure and Applied Mathematics (Amsterdam). Academic Press, London, 2018. Second edition of [MR0548006], Edited and with a preface by Søren Eilers and Dorte Olesen.

[Pow67]   Robert T. Powers. Representations of uniformly hyperfinite algebras and their associated von Neumann rings. *Ann. of Math. (2)*, 86:138–171, 1967.

[PS95]   Giovanni Pistone and Carlo Sempi. An infinite-dimensional geometric structure on the space of all the probability measures equivalent to a given one. *Ann. Statist.*, 23(5):1543–1561, 1995.

[PT73]   Gert K. Pedersen and Masamichi Takesaki. The Radon-Nikodym theorem for von Neumann algebras. *Acta Math.*, 130:53–87, 1973.

[PX03]   Gilles Pisier and Quanhua Xu. Non-commutative L^p-spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam, 2003.

[Rud74]   Walter Rudin. *Real and complex analysis.* McGraw-Hill Book Co., New York–Düsseldorf–Johannesburg, second edition, 1974. McGraw-Hill Series in Higher Mathematics.

[Sak56]   Shôichirô Sakai. A characterization of W^*-algebras. *Pacific J. Math.*, 6:763–773, 1956.

[Sak71]   Shôichirô Sakai. C^*-*algebras and W^*-algebras.* Springer-Verlag, New York–Heidelberg, 1971. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60.

[San59]   S. Sankaran. The *-algebra of unbounded operators. *J. London Math. Soc.*, 34:337–344, 1959.

[Sch86]   Lothar M. Schmitt. The Radon-Nikodým theorem for L^p-spaces of W^*-algebras. *Publ. Res. Inst. Math. Sci.*, 22(6):1025–1034, 1986.

[Seg53]   I. E. Segal. A non-commutative extension of abstract integration. *Ann. of Math. (2)*, 57:401–457, 1953.

[She05]   David Sherman. Noncommutative L^p structure encodes exactly Jordan structure. *J. Funct. Anal.*, 221(1):150–166, 2005.

[Str81]   Şerban Strătilă. *Modular theory in operator algebras.* Editura Academiei Republicii Socialiste România, Bucharest; Abacus Press, Tunbridge Wells, 1981. Translated from the Romanian by the author.

[Str04]   R. F. Streater. Quantum Orlicz spaces in information geometry. *Open Syst. Inf. Dyn.*, 11(4):359–375, 2004.

[STS02]   A. I. Stolyarov, O. E. Tikhonov, and A. N. Sherstnev. Characterization of normal traces on von Neumann algebras by inequalities for the modulus. *Mat. Zametki*, 72(3):448–454, 2002.

[Sun87]   V. S. Sunder. *An invitation to von Neumann algebras.* Universitext. Springer-Verlag, New York, 1987.

[SW87]   Stephen J. Summers and Reinhard Werner. Maximal violation of Bell's inequalities is generic in quantum field theory. *Comm. Math. Phys.*, 110(2):247–259, 1987.

[SW93]   Anton Ströh and Graeme P. West. τ-compact operators affiliated to a semifinite von Neumann algebra. *Proc. Roy. Irish Acad. Sect. A*, 93(1):73–86, 1993.

[SW99]   H. H. Schaefer and M. P. Wolff. *Topological vector spaces*, volume 3 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1999.

[SZ79]   Şerban Strătilă and László Zsidó. *Lectures on von Neumann algebras.* Editura Academiei, Bucharest; Abacus Press, Tunbridge Wells, 1979. Revision of the 1975 original, Translated from the Romanian by Silviu Teleman.

[Tak70]   M. Takesaki. *Tomita's theory of modular Hilbert algebras and its applications.* Lecture Notes in Mathematics, Vol. 128. Springer-Verlag, Berlin-New York, 1970.

[Tak02]   M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[Tak03a]   M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.

[Tak03b]   M. Takesaki. *Theory of operator algebras. III*, volume 127 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.

[Tar13]   Zs. Tarcsay. On form sums of positive operators. *Acta Math. Hungar.*, 140(1–2):187–201, 2013.

[Ter81]   M. Terp. L^p spaces associated with von Neumann algebras. Rapport No 3a, Københavns Universitet, Mathematisk Institut, 1981.

[Ter82]   Marianne Terp. Interpolation spaces between a von Neumann algebra and its predual. *J. Operator Theory*, 8(2):327–360, 1982.

[Ter17]   Marianne Terp. L^p Fourier transformation on non-unimodular locally compact groups. *Adv. Oper. Theory*, 2(4):547–583, 2017.

[Tom70]   J. Tomiyama. Tensor products and projections of norm one in von Neumann algebras. Lecture Notes, University of Copenhagen, 1970.

[Tru78]   N. V. Trunov. Locally finite weights on von Neumann algebras (in Russian). Dep. VINITI No. 101-79, Kazan. Gos. Univ., Kazan′, 1978. No. 101–79.

[Tru82]   N. V. Trunov. On the theory of normal weights on von Neumann algebras (in Russian). *Izv. Vyssh. Uchebn. Zaved. Mat.*, (8):61–70, 1982.

[Tru85]   N. V. Trunov. Locally finite weights on von Neumann algebras, and quadratic forms (in Russian). In *Constructive theory of functions and functional analysis, No. V*, pages 80–94. Kazan. Gos. Univ., Kazan′, 1985.

[Vae01]   Stefaan Vaes. A Radon-Nikodym theorem for von Neumann algebras. *J. Operator Theory*, 46(3, suppl.):477–489, 2001.

[vD78]   A. van Daele. *Continuous crossed products and type* III *von Neumann algebras*, volume 31 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge-New York, 1978.

[Vil02]   Cédric Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.

[vN30]   J. v. Neumann. Zur Algebra der Funktionaloperationen und Theorie der normalen Operatoren. *Math. Ann.*, 102(1):370–427, 1930.

[vN40]   J. v. Neumann. On rings of operators. III. *Ann. of Math. (2)*, 41:94–161, 1940.

[vN49]   John von Neumann. On rings of operators. Reduction theory. *Ann. of Math. (2)*, 50:401–485, 1949.

[Wei09]   Martin Weigt. Jordan homomorphisms between algebras of measurable operators. *Quaest. Math.*, 32(2):203–214, 2009.

[Xu05]   Quanhua Xu. On the maximality of subdiagonal algebras. *J. Operator Theory*, 54(1):137–146, 2005.

[Xu07]   Quanhua Xu. Operator spaces and noncommutative L_p. Online, 2007. Summer School on Banach spaces and Operator spaces, Nankia University.

[Yea73]   F. J. Yeadon. Convergence of measurable operators. *Proc. Cambridge Philos. Soc.*, 74:257–268, 1973.

[Yea75]   F. J. Yeadon. Non-commutative L^p-spaces. *Math. Proc. Cambridge Philos. Soc.*, 77:91–102, 1975.

[Yea81]   F. J. Yeadon. Isometries of noncommutative L^p-spaces. *Math. Proc. Cambridge Philos. Soc.*, 90(1):41–50, 1981.

[Yea75]   F. J. Yeadon. On a result of P. G. Dixon. *J. London Math. Soc. (2)*, 9:610–612, 1974/75.

[Yng05]   Jakob Yngvason. The role of type III factors in quantum field theory. *Rep. Math. Phys.*, 55(1):135–147, 2005.

[Zeg90]   Bogusław Zegarliński. Log-Sobolev inequalities for infinite one-dimensional lattice systems. *Comm. Math. Phys.*, 133(1):147–162, 1990.

[Zhu93]   Ke He Zhu. *An introduction to operator algebras.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1993.

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