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FINITELY GENERATED SUBRINGS OF R[X]

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Abstract. In this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring R in one variable x. We describe rings R such that all subalgebras of $R[x]$ are finitely generated over R.

Introduction

Let K be a field and let L be a subfield of $K(x_1, \dots, x_n)$ containing K. In 1954, Zariski in [15], proved that if $n \leqslant 2$, then the ring $L \cap K[x_1, \dots, x_n]$ is finitely generated over K. This is a result concerning the fourteenth problem of Hilbert. Today we know ([8], [9], [7]) that a similar statement for $n \geqslant 3$ is not true. Many results on this subject one can find, for example, in [4], [5], [10], [13], and also in the author articles ([11], [12]) published by University of Lodz in Materials of the Conferences of Complex Analytic and Algebraic Geometry.

We are interested in the case $n = 1$. It is well known that every K-subalgebra A of $K[x_1]$ is finitely generated over K. In this case we do not assume that A has a form $L \cap K[x_1]$. We recall it (with a proof) as Theorem 2.1. An elementary proof one can find, for example, in [6]. The assumption that K is a field is here very important. What happens in the case when K is a commutative ring and K is not a field? In this article we will give a full answer to this question.

Throughout this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring R in one variable x. We say that a ring R is an *sfg-ring*, if every R-subalgebra of $R[x]$ is finitely generated over R. We already know that if R is a field then R is an sfg-ring. We will show

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that the rings $\mathbb{Z}$ and $\mathbb{Z}_4$ are not sfg-rings. But, for instance, the rings $\mathbb{Z}_6$ and $\mathbb{Z}_{105}$ are sfg-rings.

The main result of this article states that R is an sfg-ring if and only if R is a finite product of fields. For a proof of this fact we prove, in Section 3, many various lemmas. A crucial role plays the Artin-Tate Lemma (Lemma 1.3). If R is an sfg-ring then we successively prove that R is Noetherian, reduced, that every prime ideal of R is maximal, and by this way we obtain that R is a finite product of fields. Moreover, in the last section, we present a proof that every finite product of fields is an sfg-ring.

1. Preliminary lemmas and notations

We start with the following well known lemma (see for example [2] Proposition 6.5).

Lemma 1.1. *If R is a Noetherian ring and M is a finitely generated R-module, then M is a Noetherian module.*

Let A be an algebra over a ring R. If S is a subset of A, then we denote by $R[S]$ the smallest R-subalgebra of A containing R and S. Several times we will use the following obvious lemma.

Lemma 1.2. *Let $A = R[S]$. If the algebra A is finitely generated over R, then there exists a finite subset S_0 of S such that $A = R\,[S_0]$.*

The next lemma comes from [14] (Lemma 2.4.3). This is a particular case of the Artin and Tate result published in [1]. Since this lemma plays an important role in our article, we present also its simple proof.

Lemma 1.3 (Artin, Tate, 1951). *Let R be a Noetherian ring, B a finitely generated R-algebra, and A an R-subalgebra of B. If B is integral over A, then the algebra A is finitely generated over R.*

Proof. Let $B = R[b_1, \dots, b_s]$, where $b_1, \dots, b_s$ are some elements of B. Since each b_i is integral over A, we have equalities of the form

$$b_i^{n_i} + a_{i1}b^{n_i-1} + \cdots + a_{in_i} = 0, \quad \text{for } i = 1, \dots, s,$$

where all coefficients a_{ij} belong to A, and $n_1, \dots, n_s$ are positive integers. Let $\{a_1, \dots, a_m\}$ be the set of all the coefficients a_{ij}, and put

$$A' = R[a_1, \dots, a_m]\,.$$

It is clear that A' is a Noetherian ring and B is an A'-module generated by all elements of the form $b_1^{j_1}b_2^{j_2}\cdots b_s^{j_s}$, where $0 \leqslant j_1 < n_1, \dots, 0 \leqslant j_s < n_s$. Thus, B is a finitely generated A'-module and so, by Lemma 1.1, B is a Noetherian A'-module. This means that every submodule of B is finitely generated. In particular,

A is a finitely generated A'-module. Assume that $a_{m+1}, a_{m+2}, \dots, a_n \in A$ are its generators. Then

$$A = A'a_{m+1} + \cdots + A'a_n = R[a_1, \dots, a_n],$$

and we see that the algebra A is finitely generated over R. $\square$

Let us fix some notations. For a given subset I of a ring R, we denote by $I[x]$ the set of all polynomials from $R[x]$ with the coefficients belonging to I. If I is an ideal of R, then $I[x]$ is an ideal of $R[x]$, and then the rings $R[x]/I[x]$ and $(R/I)[x]$ are isomorphic.

Let $f : S \to T$ be a homomorphism of rings. We denote by $\overline{f}$ the mapping from $S[x]$ to $T[x]$ defined by the formula

$$\overline{f}\left(\sum_j s_j x^j\right) = \sum_j \varphi(s_j) x^j$$

for all $\sum_j s_j x^j \in S[x]$. This mapping is a homomorphism of rings and $\operatorname{Ker} \overline{f} = (\operatorname{Ker} f)[x]$. We will say that $\overline{f}$ is the *homomorphism associated with* f. If f a surjection, then $\overline{f}$ is also a surjection. It is clear that if S and T are R-algebras, and $f : S \to T$ is a homomorphism of R-algebras, then $\overline{f} : S[x] \to T[x]$ is also a homomorphism of R-algebras.

In next sections we will use the following two lemmas.

Lemma 1.4. *Let I be an ideal of a ring R, and let $A = R[ax;\ a \in I]$. If the ideal I is not finitely generated, then the algebra A is not finitely generated over R.*

Proof. Assume that I is not finitely generated and suppose that A is finitely generated over R. Then, by Lemma 1.2, there exists a finite subset $\{a_1, \dots, a_n\}$ of I such that $A = R[a_1x, \dots, a_nx]$. Then of course $(a_1, \dots, a_n) \neq I$ so, there exists $b \in I \smallsetminus (a_1, \dots, a_n)$. Since $bx \in A = R[a_1x, \dots, a_nx]$, we have $bx = F(a_1x, \dots, a_nx)$, where F is a polynomial belonging to $R[t_1, \dots, t_n]$. Let

$$F = r_0 + r_1t_1 + r_2t_2 + \cdots + r_nt_n + G$$

where $r_0, r_1, \dots, r_n \in R$ and $G \in R[t_1, \dots, t_n]$ is a polynomial in which the degrees of all nonzero monomials are greater than 1. Then, in the ring $R[x]$ we have

$$bx = F(a_1x, \dots, a_nx) = r_0 + r_1a_1x + \cdots + r_na_nx + hx^2,$$

where h is some element of $R[x]$. This implies that $b = r_1a_1 + \cdots + r_na_n \in (a_1, \dots, a_n)$, but it is a contradiction, because $b \notin (a_1, \dots, a_n)$. $\square$

Lemma 1.5. *Let $A = R\left[bx, bx^2, \dots, bx^n\right]$, where $n \geqslant 1$, $0 \neq b \in R$ and $b^2 = 0$. Then every element u of A is of the form $u = r_0 + r_1bx + r_2bx^2 + \cdots + r_nbx^n$ for some $r_0, r_1, \dots, r_n \in R$.*

Proof. Let $u \in A$. Then $u = F(bx, bx^2, \dots, bx^n)$ for some n, where F is a polynomial in n variables belonging to the polynomial ring $R[t_1, \dots, t_n]$. Let

$$F(t_1, \dots, t_n) = r_0 + r_1 t_1 + r_2 t_2 + \dots + r_n t_n + G(t_1, \dots, t_n),$$

where $r_0, \dots, r_n \in R$ and $G \in R[t_1, \dots, t_n]$ is a polynomial such that the degrees of all nonzero monomials of F are greater than 1. Then $G(bx, \dots, bx^n) = b^2 H(x)$, gdzie $H(x) \in R[x]$. But $b^2 = 0$, so $u = r_0 + r_1 bx + r_2 bx^2 + \dots + r_n bx^n$. $\square$

2. Subalgebras of K[x]

Let us start with the following consequence of Lemma 1.3.

Theorem 2.1. *If $K[x]$ is the polynomial ring in one variable over a field K, then every K-subalgebra of $K[x]$ is finitely generated over K.*

Proof. Let $A \subset K[x]$ be a K-subalgebra. If $A = K$ then of course A is finitely generated over K. Assume that $A \neq K$ and let $f \in A \smallsetminus K$. Multiplying f by the inverse of its initial coefficient, we may assume that f is monic. Let $f = x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$, where $n \geqslant 1$ and $a_1, \dots, a_n \in K$. It follows from the equality

$$x^n + a_1 x^{n-1} + \dots + a_{n-1} x + (a_n - f) = 0,$$

that the variable x is integral over A. This implies that the ring $K[x]$ is integral over A and, by Lemma 1.3, the algebra A is finitely generated over K. $\square$

For the polynomial rings in two or bigger number of variables, a similar assertion is not true.

Example 2.2. *Let $K[x, y]$ be the polynomial ring in two variables over a field K, and*

$$A = K\left[xy, xy^2, xy^3, \dots\right].$$

The algebra A is not finitely generated over K.

Proof. For every positive integer n, consider the ideal I_n of A, generated by the monomials $xy, xy^2, \dots, xy^n$. Observe that $xy^{n+1} \notin I_n$. Indeed, suppose $xy^{n+1} = F_1 xy + F_2 xy^2 + \dots + F_n xy^n$, where $F_1, \dots, F_n \in A$. Every element of A is of the form $a + Gxy$ with $a \in K$ and $G \in K[x, y]$. In particular $F_j = a_j + G_j xy$, where $a_j \in K$, $G_j \in K[x, y]$ for all $j = 1, \dots, n$. Thus, in $K[x, y]$ we have

$$y^{n+1} = a_1 y + a_2 y^2 + \dots + a_n y^n + \left(G_1 y^2 + G_2 y^3 + \dots + G_n y^n\right) x.$$

Let $\varphi : K[x, y] \to K[y]$ be the homomorphism of K-algebras defined by $x \mapsto 0$ and $y \mapsto y$. Then in the ring $K[y]$, we have the false equality $y^{n+1} = \varphi\left(y^{n+1}\right) = a_1 y + a_2 y^2 + \dots + a_n y^n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. The ring A is not Noetherian. In particular, the algebra A is not finitely generated over K. $\square$

In Theorem 2.1 we assumed that K is a field. This assumption is here very important. For instance, if K is the ring of integers $\mathbb{Z}$, then a similar assertion is not true.

Example 2.3. *Let $A = \mathbb{Z}\left[2x, 2x^2, 2x^3, \dots\right]$. Then A is a subalgebra of $\mathbb{Z}[x]$ and A is not finitely generated over $\mathbb{Z}$.*

Proof. For every positive integer n, consider the ideal I_n of A, generated by the monomials $2x, 2x^2, \dots, 2x^n$. Observe that $2x^{n+1} \notin I_n$. Indeed, suppose $2x^{n+1} = 2xF_1 + 2x^2F_2 + \cdots + 2x^nF_n$, where $F_1, \dots, F_n \in A$. Every element of A is of the form $a + 2xG$ with $a \in \mathbb{Z}$ and $G \in \mathbb{Z}[x]$. In particular, $F_j = a_j + 2xG_j$, where $a_j \in \mathbb{Z}$, $G_j \in \mathbb{Z}[x]$ for all $j = 1, \dots, n$. Thus, in $\mathbb{Z}[x]$ we have the equality

$$x^{n+1} = a_1x + a_2x^2 + \cdots + a_nx^n + 2\left(G_1x^2 + G_2x^3 + \cdots + G_nx^{n+1}\right).$$

For an integer u, denote by $\overline{u}$ the element u modulo 2. Then, in the ring $\mathbb{Z}_2[x]$ we have the false equality $x^{n+1} = \overline{a_1}x + \overline{a_2}x^2 + \cdots + \overline{a_n}x^n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. The ring A is not Noetherian. In particular, the algebra A is not finitely generated over $\mathbb{Z}$. □

3. Properties of sfg-rings

Let us recall that a ring R is said to be an *sfg-ring*, if every R-subalgebra of $R[x]$ is finitely generated over R. We already know (by Theorem 2.1) that if R is a field then R is an sfg-ring. Moreover we know (by Example 2.3) that $\mathbb{Z}$ is not an sfg-ring. In this section we will prove that every sfg-ring is a finite product of fields. For a proof of this fact we need the following 9 successive lemmas. In all the lemmas we assume that R is an sfg-ring.

Lemma 3.1. *R is Noetherian.*

Proof. Suppose R is not Noetherian. Then there exists an ideal I of R which is not finitely generated. Consider the R-algebra $A = R\left[ax;\ a \in I\right]$. It follows from Lemma 1.4 that this algebra is not finitely generated over R. But this contradicts our assumption that R is an sfg-ring. □

Now we know, by this lemma, that if R is an sfg-ring, then every R-subalgebra of $R[x]$ is a Noetherian ring.

Lemma 3.2. *If I is an ideal of R, then R/I is also an sfg-ring.*

Proof. Put $\overline{R} := R/I$. Let $\varphi : R \to \overline{R}$, $r \mapsto r+I$ be the natural ring homomorphism, and let $\overline{\varphi} : R[x] \to \overline{R}[x]$ be the homomorphism associated with φ. Let B be an $\overline{R}$-subalgebra of $\overline{R}[x]$. We need to show that B is finitely generated over $\overline{R}$. For this aim consider the R-algebra $A := \overline{\varphi}^{-1}(B)$. It is an R-subalgebra of $R[x]$. Since R is an sfg-ring, the algebra A is finitely generated over R. Let $W \subset A$ be a finite set of generators of A. Then it is easy to check that $\overline{\varphi}(W)$ is a finite set of generators of B over $\overline{R}$. □

Lemma 3.3. *Every non-invertible element of R is a zero divisor.*

Proof. Suppose there exists a non-invertible element $b \in R$ such that b is not a zero divisor of R. Then $b \neq 0$ and b is not a zero divisor of $R[x]$. Consider the R-subalgebra $A = R\left[bx, bx^2, bx^3, \dots\right]$. For every positive integer n, let I_n be the ideal of A, generated by the monomials $bx, bx^2, \dots, bx^n$. Observe that $bx^{n+1} \notin I_n$. Indeed, suppose $bx^{n+1} = bxF_1 + bx^2F_2 + \cdots + bx^nF_n$, where $F_1, \dots, F_n \in A$. Every element of A is of the form $a + bxG$ with $a \in R$ and $G \in R[x]$. In particular, $F_j = a_j + bxG_j$, where $a_j \in \mathbb{R}$, $G_j \in R[x]$ for all $j = 1, \dots, n$. Since the element b is not a zero divisor of $R[x]$, we have in $R[x]$ the following equality

$$x^{n+1} = a_1x + a_2x^2 + \cdots + a_nx^n + b\left(G_1x^2 + G_2x^3 + \cdots + G_nx^n\right).$$

Consider the factor ring $R/(b)$. Let $\varphi : R \to R/(b)$, $r \mapsto r + (b)$, be the natural homomorphism and $\overline{\varphi} : R[x] \to R/(b)[x]$ be the homomorphism associated with φ. Using $\overline{\varphi}$, from the above equality we obtain that $x^{n+1} = \varphi(a_1)x + \varphi(a_2)x^2 + \cdots + \varphi(a_n)x^n$. This is a false equality in the polynomial ring $\mathbb{R}/(b)[x]$. Therefore, $bx^{n+1} \notin I_n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. This means that the ring A is not Noetherian. In particular, by Lemma 3.1, the algebra A is not finitely generated over R. But this contradicts our assumption that R is an sfg-ring. □

It follows from the above lemma that every ring without zero divisors, which is not a field, is not an sfg-ring. Thus, we see again, for instance, that $\mathbb{Z}$ is not an sfg-ring.

Lemma 3.4. *R is a reduced ring, that is, R is without nonzero nilpotent elements.*

Proof. Suppose that there exists $c \in R$ such that $c \neq 0$ and $c^m = 0$ for some $m \geqslant 2$. Assume that m is minimal and put $b := c^{m-1}$. Then $0 \neq b \in R$ and $b^2 = 0$. Consider the R-algebra $A = R[bx, bx^2, bx^3, \dots]$. It is an R-subalgebra of $R[x]$. Since R is an sfg-ring, this algebra is finitely generated over R. Hence, by Lemma 1.2, $A = R\left[bx, bx^2, \dots, bx^n\right]$ for some fixed n. But $bx^{n+1} \in A$ so, by Lemma 1.5,

$$bx^{n+1} = r_0 + r_1bx + r_2bx^2 + \cdots + r_nbx^n,$$

where $r_0, r_1, \dots, r_n \in R$. It is an equality in the polynomial ring $R[x]$. This implies that $b = 0$ and we have a contradiction. Therefore, the algebra A is not finitely generated over R, and this contradicts our assumption that R is an sfg-ring. □

Lemma 3.5. $(b) = (b^2)$ *for all* $b \in R$.

Proof. It is clear when R is a field. Assume that R is not a field. Let $b \in R$ and suppose $(b^2) \neq (b)$. Then $b \notin (b^2)$. Consider the ideal $I := (b^2)$ and the factor ring $\overline{R} := R/I$. Let $\overline{b} = b + I$. Then $0 \neq \overline{b} \in \overline{R}$ and $\overline{b}^2 = 0$, so the ring $\overline{R}$ has a nonzero nilpotent. Hence, by Lemma 3.4, $\overline{R}$ is not an sfg-ring. However, by Lemma 3.2, this is an sfg-ring. Thus, we have a contradiction. □

Lemma 3.6. *The Jacobson radical $J(R)$ is equal to zero.*

Proof. Put $J := J(R)$. It follows from Lemma 3.1 that J is a finitely generated R-module. If $b \in J$ then, by Lemma 3.5, $b = ub^2$ for some $u \in R$, and so, $b \in J^2$. Thus, we have the equality $J^2 = J$. Now, by Nakayama's Lemma, $J = 0$. □

Lemma 3.7. *If R is local, then R is a field.*

Proof. Assume that R is local and M is the unique maximal ideal of R. Then M is the Jacobson radical of R. It follows from Lemma 3.6 that $M = 0$. Thus R is a field. □

Lemma 3.8. *Every prime ideal of R is maximal.*

Proof. Let P be a prime ideal of R and suppose P is not maximal. Then there exists a maximal ideal M such that $P \subset M$ and $M \neq P$. Let $b \in M \smallsetminus P$. It follows from Lemma 3.5 that $b = ub^2$ for some $u \in R$. Then

$$b(1 - ub) = 0 \in P.$$

But $b \notin P$, so $1 - ub \in P \subset M$. Hence, $b \in M$ and $1 - ub \in M$. This implies that $1 \in M$, that is, $M = R$. However $M \neq R$, so we have a contradiction. □

Lemma 3.9. *R is Artinian.*

Proof. We already know by Lemma 3.1 that R is Noetherian. Moreover we know, by Lemma 3.8 that the Krull dimension of R is equal to 0. Using a basic fact of commutative algebra (see for example [2] or [3] 99) we deduce that R is Artinian. □

Now we are ready to prove the mentioned proposition which is the main result of this section.

Proposition 3.10. *Every sfg-ring is a finite product of fields.*

Proof. Let R be an sfg-ring. We already know (by Lemma 3.9) that R is Artinian. It is known (see for example [2] or [3]) that every Artinian ring is a finite product of some local Artinian rings. Hence,

$$R = R_1 \times R_2 \times \cdots \times R_s,$$

where $R_1, \ldots, R_s$ are local Artinian rings. Since all projections $\pi_j : R \to R_j$ (for $j = 1, \ldots, s$) are surjections of rings, it follows from Lemma 3.2 that all the rings $R_1, \ldots, R_s$ are sfg-rings. Moreover, they are local so, by Lemma 3.7, they are fields. □

According to the above proposition we know that if R is an sfg-ring, then R is a finite product of fields. In the next sections we will prove that the opposite implication is also true.

4. Initial coefficients

Let us assume that R is a ring which is not a field, and A is an R-subalgebra of the R-algebra $R[x]$. Let us denote by $\mathcal{W}_A$ the set of all nonzero initial coefficients of polynomials of positive degree belonging to A. Note three lemmas concerning this set.

Lemma 4.1. *Let $a \in \mathcal{W}_A$. Then the polynomial ax is integral over A.*

Proof. There exists a polynomial $f(x) = ax^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$, with $n \geqslant 1$ and $r_0, \ldots, r_{n-1} \in R$. Let $g(x) = a^{n-1}f(x)$. Then

$$g(x) = (ax)^n + r_{n-1}(ax)^{n-1} + ar_{n-2}(ax)^{n-2} + \cdots + r_1a^{n-2}(ax) + r_0a^{n-1}$$

is also a polynomial belonging to A. Consider the polynomial

$$H(t) = t^n + r_{n-1}t^{n-1} + ar_{n-2}t^{n-2} + \cdots + r_1a^{n-2}t + r_0a^{n-1} - g(x)\,.$$

It is a monic polynomial in the variable t and all its coefficients are in A. Since $H(ax) = g(x) - g(x) = 0$, the element ax is integral over A. $\square$

Lemma 4.2. *If R is Noetherian and $\mathcal{W}_A$ contains an invertible element, then the algebra A is finitely generated over R.*

Proof. Let $a \in \mathcal{W}_A$ be invertible in R. Then, by Lemma 4.1, the variable x is integral over A and this means that the ring $R[x]$ is integral over A. Hence, by Lemma 1.3, the algebra A is finitely generated over R. $\square$

Lemma 4.3. *Let $a, r \in R$. If $a \in \mathcal{W}_A$ and $ra \neq 0$, then $ra \in \mathcal{W}_A$.*

Proof. Assume that $f = ax^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A$ with $n \geqslant 1$. Then rf is a polynomial belonging to A and the initial coefficient equals $ra \neq 0$. Hence, $ra \in \mathcal{W}_A$. $\square$

Consider for example the ring $\mathbb{Z}_6$. Using the above lemmas we will show that $\mathbb{Z}_6$ is an sfg-ring. Let $R = \mathbb{Z}_6$, and let $A \subset R[x]$ be an R-subalgebra. We need to show that A is finitely generated over R. It is clear if $\mathcal{W}_A = \emptyset$, because in this case $A = R$. If $\mathcal{W}_A$ contains an invertible element of R (in our case 1 or 5) then, by Lemma 4.2, it is also clear.

Let us assume that $\mathcal{W}_A \subset \{2, 3, 4\}$. Since $2 \cdot 2 = 4$ and $2 \cdot 4 = 2$ in $\mathbb{Z}_6$, we have $4 \in \mathcal{W}_A \iff 2 \in \mathcal{W}_A$. If $3 \in \mathcal{W}_A$ and $4 \in \mathcal{W}_A$ then, by Lemma 4.1, the polynomials $4x$ and $3x$ are integral over A, and then $R[x]$ is integral over A, because $x = 4x - 3x$, and in this case, by Lemma 1.3, the algebra A is finitely generated over R.

Assume that $\mathcal{W}_A = \{2, 4\}$, and let $f(x) = 4x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$ where $n \geqslant 1$ and $r_0, \ldots, r_{n-1} \in \mathbb{Z}_6$. Since $r_0 = r_0 \cdot 1 \in A$, we may assume that $r_0 = 0$. The polynomial $3f(x)$ also belongs to A. Hence, $3r_{n-1}x^{n-1} + \cdots + 3r_1x \in A$.

Suppose that for some $j \in \{1, \dots, n-1\}$ we have $3r_j \neq 0$. Let us take the maximal j. Then $3r_j \in \mathcal{W}_A = \{2, 4\}$, so $r_j = 0, 2$ or 4 and in every case we have a contradiction, because $3r_j \neq 0$. Therefore, all the elements $3r_1, \dots, 3r_{n-1}$ are zeros. This means that $r_i = 4b_i$ with $b_i \in \mathbb{Z}_6$, for all $i = 1, \dots, n-1$. Observe that 4 is an idempotent in $\mathbb{Z}_6$. We have $4 = 4^m$ for every positive integer m. Hence,

$$\begin{aligned} f(x) &= 4x^n + 4b_{n-1}x^{n-1} + 4b_{n-2}x^{n-2} + \cdots + 4b_1 x \\ &= (4x)^n + b_{n-1}(4x)^{n-1} + \cdots + b_1(4x)^1 \end{aligned}$$

and hence, A is a $\mathbb{Z}_6$-subalgebra of the $\mathbb{Z}_6$-algebra $\mathbb{Z}_6[4x]$. In this case $4 \in \mathcal{W}_A$ so, by Lemma 4.1, the monomial $4x$ is integral over A and so, the ring $\mathbb{Z}_6[4x]$ is integral over A. Therefore, by Lemma 1.3, the algebra A is finitely generated over $R = \mathbb{Z}_6$.

Now let us assume that $\mathcal{W}_A = \{3\}$. In this case we use a similar way, as in the previous case. We show that A is a subalgebra of $\mathbb{Z}_6$-algebra $\mathbb{Z}_6[3x]$ and, using again Lemma 1.3, we see that A is finitely generated over $\mathbb{Z}_6$. Therefore we proved that $\mathbb{Z}_6$ is an sfg-ring.

5. Finite products of fields

In this section we prove that every finite product of fields is an sfg-ring. Throughout this section

$$R = K_1 \times K_2 \times \cdots \times K_n,$$

where $K_1, \dots, K_n$ are fields. It is clear that the ring R is Noetherian, and even Artinian. Let A be an R-subalgebra of $R[x]$. We will show that A is finitely generated over R. We know, by Theorem 2.1, that it is true for $n = 1$. Now we assume that $n \geqslant 2$.

Let us fix the following notations:

$$\begin{aligned} &N = \{1, 2, \dots, n\}\,; \\ &e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \quad \dots, \quad e_n = (0, 0, \dots, 1)\,; \\ &I = \{i \in N;\ e_i \in \mathcal{W}_A\}\,; \\ &J = N \smallsetminus I\,; \\ &\varepsilon = \sum_{i \in I} e_i\,. \end{aligned}$$

Observe that if $I = \emptyset$, then $A = R$ and nothing to prove. We know, by Lemma 4.1, that if $i \in I$, then $e_i x$ is an integral element over A. If $I = N$, then the variable x is integral over A, because $x = (1, 1, \dots, 1)x = \sum_{i=1}^{n} e_i x$, and in this case, by Lemma 1.3, the algebra A is finitely generated over R. Hence, we will assume that $I \neq \emptyset$ and $I \neq N$. Without loss of generality we may assume that

$$I = \{1, 2, \dots, s\}, \quad J = \{s+1, \dots, n\}, \quad \text{where } 1 \leqslant s < n\,,$$

and $\varepsilon = e_1 + \cdots + e_s$. Note two simple lemmas. The first one is obvious.

Lemma 5.1. *Let u be an element of R such that $ue_j = 0$ for all $j \in J$. Then $u = \varepsilon u$.*

Lemma 5.2. *Let $u \in R$. If $u \in \mathcal{W}_A$, then $u = \varepsilon u$.*

Proof. Let $u = (u_1, \dots, u_n)$ and assume that $u \in \mathcal{W}_A$. Suppose there exists $j \in J$ such that $ue_j \neq 0$. Then u_j is a nonzero element of the field K_j, and $vu = e_j$, where $v = (0, \dots, 0, u_j^{-1}, 0, \dots, 0)$. Hence, $e_j = v \cdot ue_j$ and so, by Lemma 4.3, the element e_j belongs to $\mathcal{W}_A$. But it is a contradiction, because $j \in J = N \setminus I$. Therefore, $ue_j = 0$ for all $j \in J$ and so, by Lemma 5.1, we have $u = \varepsilon u$. □

Now consider the R-subalgebra B of $R[x]$, defined by

$$B = R\left[e_1x, e_2x, \dots, e_sx\right] .$$

We will prove that $A \subset B$, that is, that B is a subalgebra of A.

Let f be an arbitrary element of A. If $\deg f = 0$, then obviously $f \in B$. Assume that $\deg f \geqslant 1$ and $u \in R$ is the initial coefficient of f. Since $R \subset A$, we may assume that the constant term of f is equal to zero, Then we have

$$f = ux^n + d_1x^{n_1} + d_2x^{n_2} + \cdots + d_px^{n_p},$$

where $d_1, \dots, d_p$ are nonzero elements of R, and $n > n_1 > n_2 > \cdots > n_p \geqslant 1$. It follows from Lemma 5.2 that $u = \varepsilon u$.

Let $j \in J$. Then $ue_j = u\left(\varepsilon e_j\right) = u0 = 0$ and then

$$e_jf = e_jd_1x^{n_1} + e_jd_2x^{n_2} + \cdots + e_jd_px^{n_p} \in A\,.$$

Suppose $e_jd_q \neq 0$ for some $q \in \{1, \dots, p\}$. Let us take the minimal q. Then $0 \neq e_jd_q \in \mathcal{W}_A$. Put $d_q = (c_1, \dots, c_n)$ with $c_i \in K_i$ for all $i = 1, \dots, n$. Since $e_jd_q \neq 0$, we have $c_j \neq 0$ and so, $vd_q = e_j$, where $v = \big(0, \dots, 0, c_j^{-1}, 0, \dots, 0\big)$. This implies that $e_j = v(e_jd_q) \in \mathcal{W}_A$. But $e_j \notin \mathcal{W}_A$, because $j \in J = N \setminus I$. Hence, we have a contradiction.

Therefore, all the elements $e_jd_1, \dots, e_jd_p$ are zeros, and such situation is for all $j \in J$. This means, by Lemma 5.1, that $d_1 = \varepsilon d_1,\ \dots,\ d_p = \varepsilon d_p$. Observe that the element ε is an idempotent of R, so $\varepsilon = \varepsilon^m$ for $m \geqslant 1$. Hence,

$$\begin{array}{rcl} f & = & ux^n + d_1x^{n_1} + d_2x^{n_2} + \cdots + d_px^{n_p} \\ & = & u\varepsilon x^n + d_1\varepsilon x^{n_1} + d_2\varepsilon x^{n_2} + \cdots + d_p\varepsilon x^{n_p} \\ & = & u\varepsilon^n x^n + d_1\varepsilon^{n_1}x^{n_1} + d_2\varepsilon^{n_2}x^{n_2} + \cdots + d_p\varepsilon^{n_p}x^{n_p} \\ & = & u(\varepsilon x)^n + d_1(\varepsilon x)^{n_1} + d_2(\varepsilon x)^{n_2} + \cdots + d_p(\varepsilon x)^{n_p}\,, \end{array}$$

and hence, the polynomial f belongs to the ring $R[\varepsilon x]$. But

$$R[\varepsilon x] \subset R[e_1x, e_2x, \dots, e_sx] = B\,,$$

so $f \in B$. Thus, we proved that A is an R-subalgebra of B. Let us recall that all the monomials $e_1x, \dots, e_sx$ are integral over A. Hence, the ring B is integral over A. It follows from Lemma 1.3 that A is finitely generated over R. Therefore, we proved the following proposition.

Proposition 5.3. *Every finite product of fields is an sfg-ring.*

Immediately from this proposition and Proposition 3.10 we obtain the following main result of this article.

Theorem 5.4. *A ring R is an sfg-ring if and only if R is a finite product of fields.*

Now, by this theorem and the Chinese Remainder Theorem, we have

Colorary 5.5. *The ring $\mathbb{Z}_m$ is an sfg-ring if and only if m is square-free.*

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