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WHEN THE MEDIAL AXIS MEETS THE SINGULARITIES

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Abstract. In this survey we present recent results in the study of the *medial axes* of sets definable in polynomially bounded o-minimal structures. We take the novel point of view of singularity theory. Indeed, it has been observed only recently that the medial axis — i.e. the set of points with more than one closest point to a given closed set $X \subset \mathbb{R}^n$ (with respect to the Euclidean distance) — reaches some singular points of X bringing along some metric information about them.

1. Introduction

The notion of the *medial axis* or *skeleton* of a domain in the Euclidean space appeared presumably for the first time in the sixties in Blum's article [7] as a central concept for pattern recognition. The main idea was that given a plane bounded domain $D \subset \mathbb{R}^2$, the set of those points $x \in D$ for which the Euclidean distance $d(x, \partial D)$ is realized in more than one point of the boundary ∂D — and this set is often called the skeleton of D for quite obvious reasons — suffices to reconstruct the shape of D, provided we know the distance function along the skeleton. A most common illustration of the skeleton is the propagation of grassfire. If we ignite a fire on the border of a field, then, assuming the fire propagates inwards with uniform speed, at some point the different firefronts will meet and quench to form the skeleton of the field. If the boundary is smooth, then this propagation can be described by a PDE in the type of the eikonal equation.

The medial axis could be also interpreted as the projection of the 'ridge' that forms on the graph of the distance function. And indeed, already from Clarke's paper [10] we may infer that the medial axis coincides with the non-differentiability points of the distance function (see [28], [5]).

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The study of the medial axis (or its variants like the central set of a domain, or conflict sets) on the grounds of singularity theory is motivated not only by the applications in pattern recognition mentioned above, but also by its importance in tomography and robotics (cf. e.g. [12]). Although since the sixties a huge amount of results concerning the medial axis had been amassed thanks to the work of many outstanding mathematicians (cf. e.g. [24], [25], [28], [21], [4]), it is only recently that the special relation existing between the medial axis and the singularities of the set for which it is computed has been observed in [16]. Before that, people concentrated mostly on applications. Also, the results obtained up to now have always been requiring some strong smoothness assumptions (cf. [9], [25]), or, on the contrary, have been rather too general (see also the expository paper [1]). Our setting is that of subanalytic geometry and the theory of o-minimal structures that exclude any topological pathology.

In the present survey we will concentrate on the newly introduced singularity theory approach to the medial axis. Therefore, from the extremely large bibliography on the medial axis we shall extract only those few papers that concern this point of view.

Throughout this paper *definable* means *definable in some polynomially bounded o-minimal structure expanding the field of reals* $\mathbb{R}$ (for a concise presentation of tame geometry see e.g. [11]; for simplicity one can always think about semi-algebraic sets, see also [14]). It is also important to keep in mind that subanalytic sets (see [14] or [15]) do not form an o-minimal structure unless we control them at infinity (or near the boundary). When some local property is studied, this does not play any role and the results obtained for definable sets hold for subanalytic ones. However, from the global point of view the difference is significant (see [14] and [16]).

Some additional references for general results about medial axes can be found in the papers [1], [9] and [21].

2. Basic notions and preliminary results

Consider a closed set $\varnothing \neq X \subsetneq \mathbb{R}^n$ and let $d(x,X)$ denote the Euclidean distance from $x \in \mathbb{R}^n$ to X. Put

$$m(x) = m_X(x) := \{y \in X \mid ||y-x|| = d(x,X)\}.$$

Definition 2.1. We call $m\colon \mathbb{R}^n \to \mathscr{P}(X)$ the *multifunction of closest points.* Its multivaluedness set

$$M_X := \{x \in \mathbb{R}^n \mid \#m(x) > 1\} \subset \mathbb{R}^n \setminus X$$

is called the *medial axis* or *skeleton* (formally we should be adding: 'of $\mathbb{R}^n \setminus X$').

Remark 2.2. By the strict convexity of the norm, it has empty interior: $\mathrm{int} M_X = \varnothing$. If X is definable or subanalytic, then so is M_X (see Theorem 3.3) in which case we also have $\mathrm{int}\overline{M_X} \neq \varnothing$. Outside tame geometry this may not be true as is shown in [21] Example 4A.

Already when $\mathbb{R}^n \setminus X$ does not contain a half-space (i.e. a set described by $\langle x - v, v\rangle \leq 0$, or the reverse, for some $v \neq 0$), the medial axis is nonempty (cf. [5] Theorem 2.27).

We write $\mathbb{B}(x,r) = \mathbb{B}_n(x,r) \subset \mathbb{R}^n$ for the open Euclidean ball centred at x and with radius $r > 0$ and $\mathbb{S}(x,r) := \partial\mathbb{B}(x,r)$ for the sphere. When $r = d(x,X)$, we call the sphere or ball *supporting*. Note that X cannot enter a supporting ball.

Remark 2.3. Any point $x \in \mathbb{B}(a,r)$ where $a \in X$ has its distance $d(x,X)$ realized in $\mathbb{B}(a,2r)$. This is a mere student's exercise, but it plays an important role in many proofs.

There are two other notions closely related to that of the medial axis. The first is the concept of the central set.

Definition 2.4. We call $\mathbb{B}(x,r) \subset \mathbb{R}^n \setminus X$ a *maximal ball* for X, if

$$\mathbb{B}(x,r) \subset \mathbb{B}(x',r') \subset \mathbb{R}^n \setminus X \;\Rightarrow\; x = x', r = r'.$$

The set C_X consisting of of the centres of maximal balls for X is called the *central set* (formally: 'of $\mathbb{R}^n \setminus X$').

Remark 2.5. If $\mathbb{B}(x,r)$ is a maximal ball, then $r = d(x,X)$.

The relation between M_X and C_X is considered folklore ([1]). This result has a practical consequence in that we often work with C_X instead of M_X (see e.g. the proof of Proposition 3.20). The closure $\overline{M_X}$ is sometimes called *cut locus*.

Theorem 2.6. *There is always* $M_X \subset C_X \subset \overline{M_X}$.

Proof. [5] Theorem 2.25; see also [21] for a different proof. □

Both inclusions may be strict:

Example 2.7. If X is the parabola $y = x^2$, then $M_X = \{0\} \times (1/2, +\infty)$ whereas the focal point $(0, 1/2)$ belongs to C_X.

For the second inclusion an example is given in [9]: X is the boundary of the union of $\mathbb{B}_3((0,0,1);1)$ in $\mathbb{R}^3$ with $\mathbb{B}_3((1,0,1/2);1/2)$ and the cylinder $(0,1) \times \mathbb{B}_2((0,1/2);1/2)$ joining the two balls. Then $(0,0,1/2)$ lies in $\overline{M_X}$, but not in C_X.

A third notion closely related to the previous ones is that of *conflict set*. Given two nonempty, closed, disjoint sets $X_1, X_2 \subset \mathbb{R}^n$, their conflict set consists of all the points that are equidistant to X_1 and X_2. This can be extended to more than two sets:

[1] In [9] it is given without any references, though there is no straightforward proof. It is one of many examples of a property that is intuitively clear, but whose proof is quite far from being meretricious.

Definition 2.8. If $X_1, \dots, X_k \subset \mathbb{R}^n$ are closed, pairwise disjoint, nonempty sets, where $k \geq 2$, and $\varrho(x) := \min_{i=1}^{k} d(x, X_i)$, then their *conflict set* is defined as

$$\mathrm{Conf}(X_1, \dots, X_k) = \{x \in \mathbb{R}^n \mid \exists i \neq j \colon d(x, X_i) = d(x, X_j) \leq \varrho(x)\}.$$

Remark 2.9. We assumed here that the sets X_i are pairwise disjoint. This ensures, at least in the definable case, that the dimension of their conflict set does not exceed $n-1$ (cf. [4]). Otherwise, if the sets were assumed too be only pairwise distinct, we would lose some control. For instance, the conflict set of the two intersecting half-lines $\{y = x, x \geq 0\}$ and $\{y = -x, x \leq 0\}$ is the union of the half-line $\{x = 0, y \geq 0\}$ together with the oblique quadrant $\{y \leq -|x|\}$.

The definition,just as the two previous ones, makes sense also in any metric space. In particular, if all the sets X_i are contained in $E \subset \mathbb{R}^n$, we can compute the relative conflict set $\mathrm{Conf}_E(X_1, \dots, X_k)$ with respect to a given metric in E.

Remark 2.10. For two distinct closed sets X, Y with a unique common point $X \cap Y = \{a\}$, there is

$$M_{X \cup Y} \setminus (\mathrm{Terr}^o(X) \cup \mathrm{Terr}^o(Y)) = \mathrm{Conf}(X, Y) \setminus C(X, Y),$$

where $\mathrm{Terr}^o(X) = \{x \in \mathbb{R}^n \mid d(x, X) < d(x, Y)\}$ is the *open territory* of X and $C(X, Y) = \{x \in \mathrm{Conf}(X, Y) \mid m_X(x) = m_Y(x)\}$. To see '$\supset$' take a point x equidistant to X and Y (so that x does not belong to any of the open territories) but with $m_X(x) \neq m_Y(x)$; then $\# m_{X \cup Y}(x) > 1$. To see '$\subset$' pick a point x from the set on the left-hand side. Then it is equidistant to X and Y and thus it belongs to the conflict set. But $m_X(x) = m_Y(x)$ implies that this set is contained in $X \cap Y = \{a\}$ and so $m_{X \cup Y}(x) = \{a\}$, contrary to the assumptions.

If the intersection $X \cap Y$ has more than one point, there is no such a simple relation between the medial axis and the conflict set as we can see for instance from the example of X being the unit circle in $\mathbb{R}^2$ together with the point $(2, 0)$ and Y just the unit circle.

Finally, let us recall two classical cones we will be using. The *Peano tangent cone* of X at $a \in X$, i.e.

$$C_a(X) = \{v \in \mathbb{R}^n \mid \exists X \ni x_\nu \to a, t_\nu > 0 \colon t_\nu(x_\nu - a) \to v\},$$

and the *Clarke normal cone* of X at a:

$$N_a(X) = \{w \in \mathbb{R}^n \mid \forall v \in C_a(X), \langle v, w \rangle \leq 0\}.$$

Both sets are definable (respectively, subanalytic) in the definable (respectively, subanalytic) case and we have the inequalities $\dim C_a(X) \leq \dim_a X$ and $\dim N_a(X) \geq n - \dim_a X$.

When studying the multifunction $m(x)$ we shall need some notions of continuity. As $m(x)$ is compact-valued,it is natural to make use of the Hausdorff, or more generally Kuratowski limits ([2]). To be more precise, let us recall the Kuratowski

[2]Or Painlevé-Kuratowski limits. As a matter of fact it was P. Painlevé who first introduced this convergence generalizing some previous work of Hausdorff. A similar concept was later considered

upper and lower limit of a multifunction $F\colon \mathbb{R}^n \to \mathscr{P}(\mathbb{R}^m)$; for a detailed study of these limits in o-minimal geometry see [13].

Definition 2.11. If x_0 is an accumulation point of the *domain* $\mathrm{dom}F$ i.e. of the set of points x for which $F(x) \neq \varnothing$, then we define the *Kuratowski lower and upper limit* at x_0 as follows:

- $y \in \liminf_{x\to x_0} F(x)$ iff for any sequence $x_\nu \to x_0$, $x_\nu \neq x_0$, one can find a sequence $F(x_\nu) \ni y_\nu \to y$;
- $y \in \limsup_{x\to x_0} F(x)$ iff there are sequences $x_\nu \to x_0$, $x_\nu \neq x_0$, and $F(x_\nu) \ni y_\nu \to y$.

Clearly, the upper limit contains the lower one and both are closed sets that do not alter if we replace the values of F by their closures. We write $E = \lim_{x\to x_0} F(x)$ or $F(x) \xrightarrow{K} E$ $(x \to x_0)$, if both limits coincide with the set E (that could be empty).

Remark 2.12. In particular,

$$C_a(X) = \limsup_{\varepsilon\to 0+} (1/\varepsilon)(X - a).$$

By the Curve Selection Lemma, in the definable case we can replace the upper limit by the limit itself (see e.g. [20]).

A simple computation shows that

$$(\dagger) \qquad a \in m(x) \;\Rightarrow\; x - a \in N_a(X).$$

This light observation has some heavy consequences, the first one being Nash's Lemma 3.1.

Before discussing what kind of relation there is between the medial axis M_X and the singularities of X we should recall the different classes of regular and singular points:

$$\mathrm{Reg}_k X := \{x \in X \mid X \text{ is a } \mathscr{C}^k - \text{submanifold in a neighbourhood of } x\},$$

for $k \in \mathbb{N}\cup\{\infty, \omega\}$ where $\mathscr{C}^\omega$ denotes analycity (in the latter case we write $\mathrm{Reg}X := \mathrm{Reg}_\omega X$ and $\mathrm{Sng}X := X \setminus \mathrm{Reg}X$ for the singularities.) and we put $\mathrm{Sng}_k X := X \setminus \mathrm{Reg}_k X$.

Example 2.13. For a plane analytic curve $\Gamma \subset \mathbb{R}^2$ through the origin we have $0 \in \mathrm{Sng}\Gamma$ if and only if either Γ has a cusp at zero, or there is an integer $k \geq 1$ such that $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$ and all the possibilities can occur (cf. [5] Example 3.1):
Take two relatively prime integers $p > q$ with q odd and such that for a given k, we have $k < p/q < k + 1$ and consider the curve Γ defined by $y^q = x^p$. Then $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$. For instance the function $y = x^{5/3}$ has analytic graph and is $\mathcal{C}^1$ but not $\mathcal{C}^2$ smooth at the origin.

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by Vietoris. On the other hand, Kuratowski was the first to present a thorough exposition of the theory in metric spaces in his memorable book on topology.

A major role in the theory is played by the important but apparently somewhat forgotten Poly-Raby Theorem:

Theorem 2.14 ([27])**.** *Let $X \subset \mathbb{R}^n$ be a closed, nonempty set and $\delta(x) := \mathrm{dist}(x,X)^2$. Then for any $k \geq 2$ or $k \in \{\omega, \infty\}$,*

$$\mathrm{Reg}_k X = \{x \in \mathbb{R}^n \mid \delta \text{ is of class } \mathscr{C}^k \text{ in a neighbourhood of } x\} \cap X.$$

Remark 2.15. We have to assume here $k \geq 2$ as is easily seen from the example of $X = (-\infty, 0]$ in $\mathbb{R}$.

As it happens, M_X coincides with the set of non-differentiability points of $\delta(x)$ (see [5]). This can be derived from some results concerning the Clarke subdifferential from [10]. Let us recall briefly Clarke's subdifferential of a locally Lipschitz function $f\colon U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open. By the Rademacher Theorem, the set D_f of differentiability points of f is dense in U. Hence, we can define the *Clarke subdifferential* $\partial f(x)$ at any point $x \in U$ as the convex hull $\mathrm{cvx}\nabla_f(x)$ of the set $\nabla_f(x)$ of all the possible limits of the gradients $\nabla f(x_\nu)$ for sequences $D_f \ni x_\nu \to x$. It is easy to see that $\partial f(x)$ is a compact set and by [10] it reduces to a singleton $\{y\}$ iff $x \in D_f$ and $\nabla f|_{D_f}$ is continuous at y (and then $\partial f(x) = \{y\}$). In order to compute $\partial f(x)$ we may restrict ourselves to any dense subset of D_f (see [10]). A more detailed study of the multifunction $x \mapsto \partial f(x)$ in the definable setting is presented in [20].

Theorem 2.16. *We have for any point $x \in \mathbb{R}^n$,*

1. $\partial\delta(x) = \{2(x-y) \mid y \in \mathrm{cvx}\, m(x)\}$*;*
2. *The following conditions are equivalent:*
   - (a) $x \in M_X$*;*
   - (b) $\#\partial\delta(x) > 1$*;*
   - (c) $x \notin D_\delta$*;*
   - (d) $x \notin D_d \cup X$*.*
3. $\nabla\delta(x) = 0 \Leftrightarrow x \in X$*;*
4. $\nabla\delta$ *is continuous in* $D_\delta = \mathbb{R}^n \setminus M_X$*;*
5. *If $x \notin M_X \cup X$, then $x \in D_d$ and $\nabla d(x) = \frac{x - m(x)}{d(x)}$.*

Proof. (5) is a refinement of a result shown already in [10]. According to [28], (1) can be deduced from [10] Theorem 2.1, but a self-contained proof of all the point can be found in [5] Theorem 2.23 and Lemma 2.21. □

3. Medial axis and singularities

3.1. **The medial axis of singular sets.** The starting point of the new theory is an old result of J. Nash from his famous work [26].

Lemma 3.1 ([26])**.** *Let X be a $\mathscr{C}^k$-submanifold of an open set $\Omega \subset \mathbb{R}^n$ where $k \geq 2$, or $k \in \{\infty, \omega\}$. Then there exists an arbitrarily small neighbourhood $U \subset \Omega$ of X such that*

(i) *$m|_U$ is univalued i.e. each point $x \in U$ has a unique closest point $m(x) \in X$;*
(ii) *the function $m\colon U \ni x \mapsto m(x) \in X$ is of class $\mathscr{C}^{k-1}$, or, respectively, $\mathscr{C}^k$ with $k \in \{\infty, \omega\}$.*

Proof. An elementary proof can be found in [16]. It is based on the fact that in the case considered here, by (†) we have

$$a \in m(x) \ \Rightarrow\ x - a \in (T_a X)^\perp$$

where $T_a X$ denotes the tangent space of X at a. Given a local parametrization $\varphi(t)$ of X at a, its partial derivatives span the tangent space and thus the proof reduces to applying the Implicit Function Theorem to the function $(t, x) \mapsto \left(\langle x - \varphi(t), \frac{\partial \varphi}{\partial t_i}(t)\rangle\right)_{i=1}^{d}$ (where $d = \dim X$) at the point $(\varphi^{-1}(a), a)$ and then using the function $t(x)$ found to get $m(x) = \varphi(t(x))$. $\square$

Remark 3.2. As observed by S. G. Krantz and H. R. Parker, for a finite k, we cannot expect a better class than $\mathscr{C}^{k-1}$ and we have to start from $k \geq 2$. It is easy to check this using the example of $y = |x|^{3/2}$ which will also prove useful later on.

The Nash Lemma already on its own raises two natural questions:

Problem 1.

(1) *What happens when we let X have singularities?*
(2) *What is the structure of the exceptional set of points for which there is more than one closest point?*

The first question leads us naturally towards the setting of subanalytic geometry or o-minimal structures, while the second one is a natural way of introducing the medial axis M_X into the picture.

The general singular counterpart of the Nash Lemma solving Problem 1 is the following theorem with parameter for a set definable in some o-minimal structure. Given $X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ we denote by X_t its section at the point t i.e. the set $\{x \in \mathbb{R}^n \mid (t, x) \in X\}$. Let $\pi_k(t, x) = t$.

Theorem 3.3 ([16] Theorem 2.1)**.** *Let $X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ be a nonempty set with locally closed t-sections and $Y := \pi_k(X)$. Assume that the set X is definable (in a not necessarily polynomially bounded o-minimal structure). Then there exists a definable set $W \subset \mathbb{R}^k \times \mathbb{R}^n$ with open t-sections and such that $X_t \subset W_t$ is closed in W_t and $m_t(x) \neq \varnothing$ for $x \in W_t$, where*

$$m_t(x) := \{y \in X_t \colon ||x - y|| = \operatorname{dist}(x, X_t)\}, \quad (t, x) \in W,$$

and moreoever

(1) *the multifunction $m(t, x) := m_t(x)$ is definable* [3]*;*

[3] i.e. its *graph* $\{(t, x, y) \in W \times X \mid y \in m(t, x)\}$ is definable.

(2) *If* $M_t = \{x \in W_t \mid \# m(t,x) > 1\}$, *then the set*
$$M := \bigcup_{t\in Y} \{t\} \times M_t \subset W$$
is definable with nowheredense sections M_t *and in particular* $m\colon W \setminus M \to \mathbb{R}^n$ *is a definable function;*

(3) *for any integer* $p \geq 2$, *there is a definable set* $F^p \subset W$ *containing* M *and such that each* F_t^p *is closed and nowheredense; moreover,* $X_t \setminus F_t^p = \mathrm{Reg}_p X_t$ *and*
$$m(t,\cdot) \textit{ is } \mathscr{C}^{p-1} \textit{ in a neighbourhood of } x \in W_t \setminus \overline{M_t} \Leftrightarrow \ x \notin F_t^p.$$

Remark 3.4. The Poly-Raby Theorem 2.14 is most useful for the proof. On the other hand, since the Rolin-Le Gal result on the existence of o-minimal structures that do not admit $\mathscr{C}^\infty$ cellular cell decompositions we know that we cannot expect to take $p = \infty$ in the theorem above.

When the parameter t is fixed we recognize here the multifunction $x \mapsto m(t,x)$ of the closest points to the set X_t. The section M_t of M is the set of non-unicity (multivaluedness) of this multifunction — the medial axis in the open set W_t. In (3) this set is extended to a set 'eating out' the singularities of class $\mathscr{C}^p$ of the set X_t; this extension is defined by the class $\mathscr{C}^{p-1}$ of the function $m(t,\cdot)$ (univalued in the open set being the complement of $\overline{M_t}$). What is more, everything here depends in a definable way on the multidimensional parameter t.

This good dependence on the parameter is all the more an important feature of the result as it is no longer true when we turn to the subanalytic case. The reason for this is the fact that the function $(t,x) \mapsto d(x, X_t)$ is not in general subanalytic when X is such, although the distance itself $x \mapsto d(x, X)$, as is known, is subanalytic in $\mathbb{R}^n$ (see Raby's Theorem 4.3 and Example 4.4 in [14]); it is one of the most important results in subanalytic geometry. We will illustrate this using an example from the survey [14] (this is a modified version of the example from [16] Remark 3.3).

Example 3.5. Consider
$$X = \{(x, 1/x) \mid x > 0\} \cup \bigcup_{n=1}^{+\infty} \{(1/n, -n)\} \subset \mathbb{R} \times \mathbb{R}.$$
Although this set is subanalytic, the set $M = \bigcup \{(1/n, 0)\}$ is not.

Nevertheless, there is a subanalytic analogue of the last theorem once we get rid of the parameters.

Theorem 3.6 ([16] Theorem 3.2)**.** *Let* $X \subset \mathbb{R}^n$ *be subanalytic, nonempty and locally closed. Then there exists a subanalytic neighbourhood* $W \supset X$ *in which* X *is closed and*

(1) *the multifunction* $m(x) = \{y \in X \colon ||x - y|| = \mathrm{dist}(x, X)\} \neq \varnothing$, *for* $x \in W$, *is subanalytic;*

(2) *the set* $M_X = \{x \in W : \#m(x) > 1\}$ *is subanalytic and nowheredense (in particular* $m : W \setminus M_X \to \mathbb{R}^n$ *is a globally subanalytic function);*
(3) *there is a nowheredense, subanalytic set* $F \subset W$ *closed in* W *and such that* $M_X \subset F$, $F \cap X = \mathrm{Sng}X$ *and* $x \in W \setminus \overline{M_X}$ *is a point of analycity of* m *if and only if* $x \in W \setminus F$.

Let us note that in (3) we obtain the analycity of m, which is a consequence of the well-known Tamm Lemma (its geometric proof not requiring the use of Hironaka's desingularization was given by K. Kurdyka in the eighties).

We should stress the fact that the proof of the theorem above cannot be obtained by a simple cutting off of X using an increasing sequence of cubes in order to apply the preceding result to the globally definable sets $X_\nu = X \cap [-\nu, \nu]^n \subset \mathbb{R}^n$ obtained in this way. Indeed, in general there is no equality $M_X = \bigcup M_\nu$, where M_ν is the medial axis defined for X_ν.

Example 3.7. Take X to be the union of half-circles $\{x^2 + (y-\nu)^2 = (3/4)^2, y \leq \nu\}$; then $(0,\nu) \in M_\nu \setminus M_{\nu+1}$ and in particular $(0,\nu) \notin M_X$.

3.2. **Reaching of singularities.** The Nash Lemma 3.1 implies that

$$\overline{M_X} \cap X \subset \mathrm{Sng}_2 X. \tag{‡}$$

Already in [16] we observed that some singular points are reached by the medial axis.

Example 3.8. ([5] Example 3.1). Consider in $\mathbb{R}^2$ the sets

$$X_1 := \{y = x^2\},\ X_2 := \{y = |x|^{3/2}\} \text{ and } X_3 := \{y = (1 + \mathrm{sgn}x)x^2\}.$$

Then $0 \in \mathrm{Reg}_1 X_i \cap \mathrm{Sng}_2 X_i$ for $i = 2, 3$, whereas $X_1 = \mathrm{Reg}_2 X_1$.

It is easy to see that M_{X_1} is the half-line $\{0\} \times (1/2, +\infty)$ and so it does not meet X_1. On the other hand, $M_{X_2} = \{0\} \times (0, +\infty)$ reaches the $\mathscr{C}^1$-singularity of X_2. But again M_{X_3} stays away from it. This is due to the fact that although both X_2 and X_3 have the same kind of singularity, their geometric 'radii of curvature' (see below — the reaching radius) are different.

We are led to the following natural question:

Problem 2. *Characterise the points of*

$$\overline{M_X} \cap \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X \quad \textit{and} \quad \overline{M_X} \cap \mathrm{Sng}_1 X.$$

Remark 3.9. If we think of the example of a quadrant $X = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$, we see that even for $\mathscr{C}^1$-singular points the question whether the medial axis reaches them or not is not obvious at all.

Partial answers to the Problem were given in [5] where several techniques were developed that should allow to thoroughly solve the question. In particular, an important tool is the newly introduced *reaching radius* ([5] Definition 4.24):

Let $V_a = N_a(X)\cap\mathbb{S}(0,1)$ denote the intersection of the normal cone to X at a with the unit sphere ([4]).

Definition 3.10. We define the *weak reaching radius*

$$r'(a)=\inf_{v\in V_a} r_v(a)$$

where

$$r_v(a)=\sup\{t\geq 0\mid a\in m(a+tv)\}$$

is the *directional reaching radius* (or *v-reaching radius*). Next we put

$$\tilde r(a)=\liminf_{X\setminus\{a\}\ni x\to a} r'(x)$$

for the *limiting reaching radius*. Finally, we define the *reaching radius* as

$$r(a)=\begin{cases} r'(a), & a\in \mathrm{Reg}_2 X,\\ \min\{r'(a),\tilde r(a)\}, & a\in\mathrm{Sng}_2 X.\end{cases}$$

Remark 3.11. If $a\in \mathrm{int}X$, then $N_a(X)=\{0\}$ and so $V_a=\varnothing$ which gives $r'(a)=+\infty$ (as the infimum over the empty set).

Of course, if X is a hypersurface, then at $a\in\mathrm{Reg}_1 X$, we have $V_a=\{\nu(a),-\nu(a)\}$ where ν is a local unit normal vector field.

Example 3.12. The idea is that the reaching radius should vanish only at points attained by the medial axis ([5]). The reason why we consider the biggest lower bound of the radii in all possible normal directions at a is that we have to take into account the curvature and obtain a possibly finite number, e.g. for $X=\{y=x^2\}\subset\mathbb{R}^2$ we have $r'(0)=r_{(0,1)}(0)=1/2<r_{(-1,0)}(0)=+\infty$.

The need for considering also the limiting radius comes from the fact that for $X=\{y=|x|\}$ we have $r'(0)=+\infty$, while $\liminf_{X\setminus\{0\}\ni x\to 0} r'(x)=0$.

On the other hand, if $X=((-\infty,-1]\cup[1,+\infty))\times\{0\}$, then $\tilde r(-1,0)=+\infty$, while using the directions from the normal cone we see that $\inf_{v\in V_0} r_v(-1,0)=1$. This explains the final minimum in the definition.

By the Nash Lemma, $r(a)>0$ at points $a\in\mathrm{Reg}_2 X$. On the other hand, by [5] Theorem 4.28 and Lemma 4.27, $X_+ := r^{-1}(+\infty)\cap\mathrm{Reg}_1 X$ is either void, or a connected component of $\mathrm{Reg}_1 X$ and $X\subset T_aX+a$ for any $a\in X_+$. More importantly, by [5] Theorem 4.33, $M_X\neq\varnothing$ implies $X\setminus r^{-1}(+\infty)\neq\varnothing$.

Theorem 3.13. *For a definable X, the function $r\colon X\to[0,+\infty]$ is definable* ([6]) *and $a\in\overline{M_X}\cap X$ iff $r(a)=0$.*

[4] If $a\in m(x)$, then $x-a$ belongs to the normal cone to X at a, cf. (†).

[5] The notion is thus different from what is known as Federer's *reach* $\rho(X):=\inf\{d(x,M_X)\mid x\in X\}$, see [22] and [5] Subsection 4.3. It is rather awkward to use the distance $d(x,M_X)$ itself as it does not bring along enough geometric information, even though it has some interesting properties too, see [5] Corollaries 4.18 and 4.19.

[6] I.e. $r^{-1}(+\infty)$ is definable and the restriction $r|_{X\setminus r^{-1}(+\infty)}$ is a definable function.

Remark 3.14. A major role in the proof is played by the so called *proximal inequality*. We say that $v \in V_a$ is proximal for X, if for some $r > 0$, $m(a + rv) = \{a\}$. This is equivalent to the following inequality:

$$(\#) \qquad \exists r > 0\colon \forall x \in X,\ \langle x - a, v\rangle \leq \frac{1}{2r}||x - a||^2.$$

Example 3.15. It would be helpful, if $r'(a) = 0$ implied $\tilde{r}(a) = 0$. Unfortunately, the example of $X = \{(x, y, z) \mid z = 0, y \leq |x|^{3/2}\}$ shows that there may be $r'(a) = 0$ and $\tilde{r}(a) > 0$. Here $\mathrm{Sng}_1 X = \{(x, |x|^{3/2}, 0) \mid x \in \mathbb{R}\}$, so that $r' \equiv +\infty$ along $\mathrm{Reg}_1 X$.

On the other hand, if X is the graph of $f(x, y) = y|x|^{3/2}$, then $r'(0) > 0$, while $\tilde{r}(0) = 0$, in particular, $r'(0, y) = 0$ for $y \neq 0$.

Although the definition of the reaching radius seems rather technical, the Theorem above is often quite easy to apply.

Example 3.16. Consider the surface $X = \{z^3 = xy(x^4 + y^4)\}$ in $\mathbb{R}^3$ from [22]. The interesting thing here is that at each point $a \in X$ the tangent cone is flat, i.e. a plane. However, there is a discontinuity of the tangents at the origin (where the tangent is the (x, y)-plane) if we move along the x- or y-axis where the tangent planes are vertical (they contain the z-axis). Thus the origin lies in $\mathrm{Sng}_1 X$. It is easy to see that $r'(0) > 0$, but $\tilde{r}(0) = 0$ so that $0 \in \overline{M_X}$, by the last Theorem.

3.3. **Stability of the medial axis with application to the reaching of singularities.** Almost since its introduction the medial axis M_X has been known as being highly unstable under small deformations of X. F. Chazal and R. Soufflet illustrated this in [9] with a most simple example: the medial axis of a circle is its centre, but even the smallest 'protuberance' on the circle leads to the medial axis becoming a whole segment. The paper [9] is entirely devoted to showing that under some hypotheses on X there is a kind of stability of the medial axis for $\mathcal{C}^2$ deformations expressed by means of map images. However, that kind of approach consists actually in looking at the initial and the final states only — with a black box in between, where the actual deformation takes place. Even from the point of view of applications it seems more natural to see the deformation as a continuous process. Which is more, there is no need for it to be smooth. This is best expressed using the Kuratowski convergence of sets and indeed lets us have some insight of what is happening to the medial axis.

Let $\pi(t, x) = t$ for $(t, x) \in \mathbb{R}^k \times \mathbb{R}^n$. We have the following type of semicontinuity of the medial axis ([7]):

Theorem 3.17 ([18] Theorem 4.1)**.** *Assume that $X \subset \mathbb{R}^k \times \mathbb{R}^n$ is definable with closed t-sections, 0 is an accumulation point of $\pi(X)$ and $X_t \xrightarrow{K} X_0$. Then for*

[7] Recently, we have obtained with A. Denkowska a similar but more detailed result based partly on [13] for conflict sets and also for *Voronoi diagrams* which are medial axes of finite sets. The result is as yet unpublished.

$M = \{(t,x) \mid \#m(t,x) > 1\}$, where $m(t,x) = \{a \in \mathbb{R}^n \mid a \in X_t \colon ||x-a|| = d(x,X_t)\}$, we have

$$\liminf_{\pi(M)\ni t\to 0} M_t \supset M_0$$

where we posit $\liminf_{\pi(M)\ni t\to 0} M_t = \varnothing$ when $0 \notin \overline{\pi(M)\setminus\{0\}}$.

Remark 3.18. The Theorem implies that 0 cannot be an isolated point of $\pi(M) = \{t \mid M_t \neq \varnothing\}$, i.e. $M_0 = \varnothing$, if $0 \notin \overline{\pi(M)\setminus\{0\}}$.

The proof depends heavily on the Curve Selection Lemma for which the definability assumption is unavoidable. Whether there is a general counterpart of this result remains an open question.

Example 3.11 from [18] shows that we can hardly expect a better result even in the quite regular situation when we are dealing with a convergent definable one-parameter family of graphs:

Example 3.19. Consider the set $X = \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \mid y = t|x|\}$. It is definable, we have $X_t \xrightarrow{K} X_0$, but

$$M_t = \begin{cases} \{(x,y) \mid x = 0, y > 0\}, & t > 0,\\ \varnothing, & t = 0,\\ \{(x,y) \mid x = 0, y < 0\}, & t < 0,\end{cases}$$

so that there is no convergence.

The Theorem above combined with the following recent observation of A. Białożyt [3] enables us to prove a refined version of Theorem 4.6 from [5] on reaching a certain type of singular points.

Proposition 3.20 (A. Białożyt)**.** *Let $V \subset \mathbb{R}^n$ be a real cone* ([8])*. Then $M_V \neq \varnothing$ if and only if V is not a convex set.*

Proof. See [3]. If V is convex, then $m_V(x)$ is obviously univalued and $M_V = \varnothing$. On the other hand, it is easy to see that by homothety $M_V \cup \{0\}$ is a real cone, too. Assume that V is non-convex and take $x \neq y$ in V such that $[x,y] \cap V = \{x,y\}$. If the midpoint z of the segment $[x,y]$ is not in M_V take $a = m_V(z)$ and write $B_t := \mathbb{B}(a + t(z-a), td(z,V))$. Then by the choice of z, we conclude that there must be

$$\sup\{t \geq 1 \mid B_t \subset \mathbb{R}^n \setminus V\} < +\infty,$$

which implies that the central set $C_V \neq \varnothing$ and we are done due to Theorem 2.6. □

Theorem 3.21. *Assume that $X \subset \mathbb{R}^n$ is a definable with a non-convex tangent cone $V := C_0(X)$. Then $0 \in \overline{M_X}$ and $C_0(M_X) \supset \overline{M_V}$.*

[8]A real cone $V \subset \mathbb{R}^n$ is a union of half-lines starting from the origin, i.e. for any $t \geq 0$, $tV \subset V$ and we assume that $V \neq \varnothing$ by definition.

Proof. As noted in Remark 2.12, in the definable case we know that V is the Kuratowski limit when $t \to 0^+$ of the dilatations $(1/t)X$, $t > 0$. Hence by Theorem 3.17,

$$M_V \subset \liminf_{t\to 0^+} M_{(1/t)X}.$$

By homothety, we have $M_{(1/t)X} = (1/t)M_X$ and so the limit inferior is actually a limit and coincides with $C_0(M_X)$. Finally, as observed [20], for a definable set we have $C_0(E) = \lim_{t\to 0^+}(1/t)E$ also in the case when $0 \notin \overline{E}$ in which case the limit is empty. Therefore, since we know by Proposition 3.20 that $0 \in \overline{M_V}$ (as $M_V \cup \{0\}$ is a cone,by homothety), we obtain the result sought for. □

In general we can hardly expect equality between $C_0(M_X)$ and $\overline{M_V}$ in the last theorem:

Example 3.22. Consider a calyx-shaped X, i.e. the union of the horn $x^2 + y^2 + z^3 = 0$ together with $z = ||(x,y)||$ (Euclidean norm). Then $V = C_0(X)$ consists of $\{x = y = 0, z \leq 0\}$ together with $z = ||(x,y)||$ and so it is non-convex and the last Theorem applies. However, M_X contains the z-axis without the origin, so that $C_0(M_X)$ contains the whole z-axis, whereas the cone $M_V \cup \{0\}$ intersected with the z-axis is just the half-line $\{x = y = 0, z \geq 0\}$ which means that $C_0(M_V) = \overline{M_V} = M_V \cup \{0\}$ does not contain the half-line $\{x = y = 0, z < 0\}$.

3.4. **The plane case.** The plane case is far from being plain, if we may indulge in a little pun. Let us recall the following classical fact.

Lemma 3.23. *If $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected, i.e. X has a single branch ending at the origin, then the tangent cone $C_0(X)$ is a half-line that we can identify with $\mathbb{R}_+ \times \{0\}^{n-1} \subset \mathbb{R}^n$ in properly chosen coordinates and X is near zero the graph of a definable $\mathscr{C}^1$ function $f\colon [0,\varepsilon) \to \mathbb{R}$ with $f(0) = 0$ and $f'(0) = 0$.*

In the situation from this Lemma, for $0 < t \ll 1$, we can write $f(t) = at^\alpha + o(t^\alpha)$ with $a \neq 0, \alpha \geq 1$, provided $f \not\equiv 0$.

Definition 3.24. We say that X as in the Lemma above is *superquadratic* at zero iff $f \not\equiv 0$ and $\alpha < 2$ (cf. [5, Section 3.3]).

Remark 3.25. The definability of f allows us also to assume that f has constant convexity on $[0,\varepsilon)$ and is $\mathscr{C}^2$ on $(0,\varepsilon)$.

The choice of the adjective *superquadratic* in view of the fact that we require $\alpha < 2$ may seem a little strange. Its geometrical origin is shown in the following easy lemma.

Lemma 3.26 ([5] Lemma 3.17). *If $\gamma\colon [0,\varepsilon) \to [0,+\infty)$ is superquadratic with $\gamma(0) = \gamma'(0) = 0$, then for any $r > 0$ the disc $D_r := \mathbb{B}((0,r),r) \subset \{y > 0\}$ tangent to the x-axis at zero contains points of γ inside.*

Proof. It follows from the obvious observation that if $g\colon [0,r) \to \mathbb{R}_+$ denotes the usual parametrization of the lower part of the circle ∂D_r through zero, then $g(x) = \frac{1}{2r}x^2 + o(x^2)$ near zero. At the same time $\gamma(x) = ax^\alpha + o(x^\alpha)$ with $a > 0$ and $\alpha \in (0,2)$ and so there must be $g(x) < \gamma(x)$ for small x. $\square$

Following [6] we will give here the correct version of [5] Proposition 3.24. It reads:

Proposition 3.27 ([5] Proposition 3.24 – correct version)**.** *Assume that $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected. Then $0 \in \overline{M_X}$ if and only if X is superquadratic at zero.*

Proof. If X is superquadratic at zero, then by Lemma 3.26, the weak reaching radius $r'(0,0)$ is zero and so the reaching radius $r(0,0)$ is zero, too. By Theorem 3.13, it means that $0 \in \overline{M_X}$.

If X is not superquadratic at zero, then either $f \equiv 0$, or $\alpha \geq 2$, where f is the function from Lemma 3.23. In both cases f has a $\mathscr{C}^2$ extension by 0 through zero and the Nash Lemma leads to the conclusion that $0 \notin \overline{M_X}$. $\square$

Lemma 3.28. *If $X \subset \mathbb{R}^2$ is definable with* $\dim_0 X = 1$ *and $0 \in \overline{M_X} \cap X$, then* $\dim_0 M_X = 1$.

Proof. Since the assumptions imply that $0 \in \overline{M_X} \setminus M_X$, then by the Curve Selection Lemma, $\dim_0 M_X \geq 1$. On the other hand, M_X has empty interior, whence $\dim_0 M_X < 2$. $\square$

We may complete now the previous Proposition with a metric statement.

Proposition 3.29 ([6] Proposition 2.3)**.** *Assume that X is as in the previous Proposition and $0 \in \overline{M_X} \cap X$. Then the tangent cone $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero. To be more precise, if X near zero is the graph of $f\colon [0,\varepsilon) \to \mathbb{R}$ and f is, say, convex, then $C_0(M_X) = \{0\} \times [0, +\infty)$.*

Proof. From the previous Lemma we know that $\dim_0 M_X = 1$. By Lemma 3.23, we assume that X is the graph of a convex definable function $f\colon [0,+\infty) \to \mathbb{R}$ of class $\mathscr{C}^1$ that is $\mathscr{C}^2$ on $(0,\varepsilon)$, $f(0) = f'(0) = 0$ and f is superquadratic at the origin (by Proposition 3.27). Thanks to the convexity, for some neighbourhood U of the origin, we have $M_X \cap U \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$.

Take any sequence $M_X \ni a_\nu \to 0$ such that $a_\nu/||a_\nu|| \to v$. For each index we pick a point $b_\nu \in m(a_\nu) \setminus \{0\}$. Then $b_\nu \to 0$ (cf. [20] Lemma 8.5). Moreover, $v_\nu := (a_\nu - b_\nu)/d(a_\nu, X)$ is a unit normal vector to X at b_ν and for each $\theta \in [0, d(a_\nu, X))$, b_ν is the unique closest point in X to $b_\nu + \theta v_\nu$ and so the unit vector v_ν is proximal, which implies, as in the proof of Theorem 4.35 in [5], the proximal inequality (#):

$$\forall c \in X,\ \langle c - b_\nu, v_\nu \rangle \leq \frac{1}{2d(a_\nu, X)}||c - b_\nu||^2.$$

From this, after multiplying both sides by $d(a_\nu, X)$ and taking $c = 0$, we obtain

$$\frac{1}{2}||b_\nu||^2 \leq \langle a_\nu, b_\nu \rangle,$$

whence $||b_\nu||/||a_\nu|| \leq 2\cos\alpha_\nu$, where $\alpha_\nu = \angle(b_\nu, a_\nu)$. In particular all the angles α_ν are acute.

Since $||b_\nu|| \to 0$, we obtain $b_\nu/||b_\nu|| \to (1,0)$, for $C_0(X) = [0,+\infty) \times \{0\}$. Our proof will be accomplished, if we show that $\alpha_\nu \to \pi/2$, since $\alpha_\nu = \angle(b_\nu/||b_\nu||, a_\nu/||a_\nu||) \to \angle((1,0), v)$. As the angles are acute, we immediately get $\angle((1,0), v) \in [0, \pi/2]$.

We know that X is superquadratic at zero, which implies that for any $y > 0$, the origin does not belong to $m((0,y))$, by Lemma 3.26. If $b \in m((0,y))$, then b is the unique closest point for any point from the segment $[(0,y), b] \setminus \{(0,y)\}$. As earlier, by [20] Lemma 8.5, $b \to 0$ when $y \to 0^+$. Then the set $Y := \{b \in X \mid \exists y > 0 \colon b \in m((0,y))\}$ is definable and $0 \in \overline{Y} \setminus Y$. Therefore, by the Curve Selection Lemma, Y coincides with X in a neighbourhood of zero that we may take to be a ball $\mathbb{B}(0,R)$.

In particular, we can find $r, \rho > 0$ such that there is a continuous definable surjection $[0,r) \ni y \mapsto F(y) \in X \cap \mathbb{B}(0,\rho)$ satisfying $F(y) \in m((0,y))$. Then, for any $(x,y) \in \mathbb{B}(0,\rho/2)$ such that $x > 0, y > f(x)$, the distance $d((x,y),X)$ is realized in $\mathbb{B}(0,\rho) \cap X$. If b is a closest point to (x,y), then the vector $(x,y) - b$ is normal to X at b, but as $b = F(y')$ for some $y' \in [0,r)$, we conclude that $(x,y) \in [(0,y'), b]$ and so $m((x,y)) = \{b\}$. Therefore,

$$M_X \cap \{(x,y) \in \mathbb{B}(0,\rho/2) \mid x > 0, y > 0\} = \varnothing.$$

This means that $M_X \cap \mathbb{B}(0,\rho/2) \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0, x \leq 0\}$, whence $\angle((1,0), v) \in [\pi/2, \pi]$. Summing up, we obtain $\angle((1,0), v) = \pi/2$ as required. □

If we are dealing with a $\mathscr{C}^1$-smooth curve, a so called 'rolling disc' argument yields:

Theorem 3.30. *Assume that* $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$. *Then* $0 \in \overline{M_X}$ *iff* X *is superquadratic at the origin.*

Proof. [5] Theorem 3.19. □

In the presence of at least two branches, we have the following result for a $\mathscr{C}^1$-singularity:

Theorem 3.31. *Let* $X \subset \mathbb{R}^2$ *be a definable curve with* $0 \in \mathrm{Sng}_1 X$ *and assume that the germ* $(X \setminus \{0\}, 0)$ *has at least two connected components. Then* $0 \in \overline{M_X}$.

Proof. [5] Theorem 3.21. □

Remark 3.32. Due to the Nash Lemma, Proposition 3.27 together with Theorems 3.30 and 3.31 completely solve Problem 2 in the plane.

Using the main result of Birbrair and Siersma from [4], which is the following Theorem, we are able to compute in the case of plane curves the tangent cone to M_X at a point $a \in X$ reached by the medial axis.

Theorem 3.33 (Birbrair-Siersma [4]). *Let $X_1, \dots, X_k \subset \mathbb{R}^n$ be closed, definable, pairwise disjoint, nonempty sets such that 0 belongs to $K := \mathrm{Conf}(X_1, \dots, X_k)$ and let $S := \mathbb{S}(0, d(0, X_1))$ be the supporting sphere at 0. Then $C_0(K)$ is the cone spanned over the conflict set $\mathrm{Conf}_S(\tilde{X}_1, \dots, \tilde{X}_k)$ in the sphere, where $\tilde{X}_i := X_i \cap S$.*

Here, what we mean by a *cone spanned over* a subset E of the sphere S centred at zero is the set $\bigcup\{\mathbb{R}_+ v \mid v \in E\}$. The conflict set *in the sphere* is computed with respect to the geodesic metric in S (cf. Remark 2.9).

If $(X, 0) \subset \mathbb{R}^2$ is a definable pure one-dimensional closed germ, then $X \setminus \{0\}$ consist of finitely many branches $\Gamma_0, \dots, \Gamma_{k-1}$ ending at zero and dividing a small ball $\mathbb{B}(0, r)$ into k regions. For $k > 1$, if we enumerate the branches in a consecutive way, we can call these open regions $D(\Gamma_i, \Gamma_{i+1})$, $i \in \mathbb{Z}_k$. Assuming that $0 \in \overline{M_X}$, we say that a pair of *consecutive* branches Γ_i, Γ_{i+1} *contributes* to M_X at zero, if $0 \in \overline{M_X \cap D(\Gamma_i, \Gamma_{i+1})}$.

Let $1 \leq c \leq k$ be the number of contributing regions. For each such region $D(\Gamma_i, \Gamma_{i+1})$ we have two half-lines ℓ_i, ℓ_{i+1} tangent to Γ_i, Γ_{i+1} at zero, respectively. These half-lines define an oriented angle $\alpha(i, i+1) \in [0, 2\pi]$, consistent with the region ([9]).

As we know that M_X is one-dimensional, the germ $(\overline{M_X}, 0)$ consists of finitely many branches ending at zero. For a definable curve germ $(E, 0)$, we will denote by $b_0(E)$ the number of its branches at the origin.

Combining [5] Theorem 3.27 with Propositions 3.27 and 3.29, we obtain the following Tangent Cone Theorem.

Theorem 3.34 ([5] Theorem 3.27 – extended version, [6] Theorem 2.4). *Assume that $0 \in \overline{M_X} \cap X$ where X is a pure one-dimensional closed definable set in the plane. Then,*

1. *either $b_0(X) = 1$, in which case $b_0(M_X) = 1$ and $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero,*
2. *or $b_0(X) = k > 1$, in which case $b_0(M_X) \leq c + 1$ where c is the number of contributing regions, and $C_0(M_X)$ is the union of the bisectors of all the pairs of half-lines forming up $C_0(X)$ given by pairs of consecutive branches delimiting regions that contribute to M_X at zero with possibly one exception: there is at most one contributing region $D(\Gamma_i, \Gamma_{i+1})$ with angle $\alpha(i, i+1) > \pi$ in which case at least one of the curves Γ_i, Γ_{i+1} is superquadratic at zero and $M_{i,i+1} = M_X \cap D(\Gamma_i, \Gamma_{i+1})$ has at most two branches at zero and*

[9] Note that it may happen that $\alpha(i, i+1) = 2\pi$; indeed, if X consists of the two branches $\Gamma_0 = [0, +\infty) \times \{0\}$ and the superquadratic $\Gamma_1 = \{y = x^{3/2}, x \geq 0\}$, then both regions $D(\Gamma_0, \Gamma_1)$ and $D(\Gamma_1, \Gamma_0)$ are contributing. The angles are 0 and 2π, respectively.

$C_0(M_{i,i+1})$ consists of one or two half-lines orthogonal to the corresponding tangent ℓ_i or ℓ_{i+1}.

Proof. (1) is the statement of Proposition 3.29. To see that M_X near zero consists of one branch we consider the situation from the proof of Proposition 3.29. In particular $M_X \cap \mathbb{B}(0,r) \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0, x \leq 0\}$. Suppose that there are (at least) two different branches M_1, M_2 ending at zero. Then one of them, say M_1, lies in the region delimited by the other one, i.e. M_2, and $\{0\} \times [0,+\infty)$. Take a point $a \in M_2$. Then $m(a)$ contains a non-zero point b. Then, if a is sufficiently near zero, the segment $[a,b]$ intersects M_1. If c belongs to the intersection, then $m(c) = \{b\}$, contrary to $c \in M_X$.

As for (2), we can repeat the argument from the proof in [5] Theorem 3.27 with only one additional case to consider. Let $D(\Gamma_0, \Gamma_1)$ be a contributing region. The same type of argument as above shows that M_X has only one branch in $D(\Gamma_0, \Gamma_1)$ ending at zero ([10]). Let $\alpha = \alpha(0,1) \in [0, 2\pi]$ be the oriented angle consistent with $D(\Gamma_0, \Gamma_1)$.

If $\alpha \in [0, \pi)$, we proceed as in [5] Theorem 3.27: for $a \in M_X$ near zero, $m(a)$ cannot contain zero and has points both from Γ_0 and Γ_1 — these tend to zero when $a \to 0$. The set $M_X \cap D(\Gamma_0, \Gamma_1)$ coincides with the conflict set of Γ_0, Γ_1 (compare the proof of Theorem 3.21 in [5]) and the Birbrair-Siersma Theorem quoted above gives the result as in the original proof in [5].

If $\alpha = \pi$, then $\Gamma = \Gamma_0 \cup \Gamma_1$ is a $\mathscr{C}^1$ curve and $M_X \cap D(\Gamma_0, \Gamma_1)$ reaches the origin iff Γ is superquadratic at zero ([11]). But then no point from the normal cone at zero can have its distance realized at the origin (cf. Lemma 3.26) and so we are in a position that allows us to repeat the argument based on the Birbrair-Siersma Theorem just as in [5].

If $\alpha > \pi$ (clearly, there can be only one such contributing region), then the only possibility that the region $D(\Gamma_0, \Gamma_1)$ be contributing is that at least one of the two delimiting curves be superquadratic at zero and $\mathbb{B}(0,r) \setminus D(\Gamma_0, \Gamma_1)$ be non-convex. In this case we are exactly in the situation from Proposition 3.29 and the result follows. Of course, $M_X \cap D(\Gamma_0, \Gamma_1)$ may have two branches at zero which explains why we have $b_0(M_X) \leq c + 1$. □

The need for taking $c+1$ in (2) is illustrated by the following example from [6].

Example 3.35. Rotate the superquadratic curve $y = x^{3/2}$, $x \geq 0$ by $\pi/6$ anticlockwise and the curve $y = -x^{3/2}$, $x \geq 0$ by the same angle clockwise, obtaining two curves Γ_0, Γ_1 with tangent half-lines at zero $y = (1/\sqrt{3})x, x \geq 0$ and

[10] If there were only two branches of M_X in $D(\Gamma_0, \Gamma_1)$ ending at zero, it could happen that along each of them the segments joining the points to the points realizing their distance would not intersect the other branch. In that case we pick a point a in between the two branches of M_X and the segment $[a, m(a)]$ must intersect one of the branches in a point c. Then $m(a) \in m(c)$ but there is a point $b \in m(c) \setminus m(a)$ and the triangle inequality shows that $||a-b|| < ||a - m(a)||$, which is a contradiction.

[11] I.e. $D(\Gamma_0, \Gamma_1)$ is near zero the epigraph of a superquadratic function.

$y = -(1/\sqrt{3})x, x \geq 0$, respectively. Let $X = \Gamma_0 \cup \Gamma_1$. Then we have two contributing regions: $D(\Gamma_1, \Gamma_0)$ with $\alpha(1,0) = \pi/3$ and $D(\Gamma_0, \Gamma_1)$ with $\alpha(1,2) = 5\pi/3$. The medial axes has three branches at zero: the half-line $[0,+\infty) \times \{0\}$ and two curves symmetric with respect to $(-\infty, 0] \times \{0\}$, living in the quadrants $\{x \leq 0, y \geq 0\}$ and $\{x \leq 0, y \geq 0\}$, respectively. Then

$$C_0(M_X) = ([0,+\infty) \times \{0\}) \cup \{y = -\sqrt{3}x, x \leq 0\} \cup \{y = \sqrt{3}x, x \leq 0\}.$$

In the non-definable setting the tangent cone $C_0(M_X)$ when M_X reaches the set X at zero may be quite big.

Example 3.36. ([5] Example 3.28). Consider $X = \{0\} \cup \bigcup_{\nu=1}^{+\infty} \{(x_\nu, 0)\} \subset \mathbb{R}^2$ where $x_\nu = 1/\nu$. Then

$$M_X = \bigcup_{\nu=1}^{+\infty} \left\{ \frac{x_\nu + x_{\nu+1}}{2} \right\} \times \mathbb{R}$$

and so $0 \in \overline{M_X}$, but $C_0(M_X) = \{(x,y) \mid x \geq 0\}$, while $C_0(X) = [0,+\infty) \times \{0\}$.

Example 3.37. Let $X = \{(x, x^2) \mid x \in [0,1)\} \cup \{(x, x^3) \mid x \in [0,1)\}$. Then M_X near zero is clearly a curve lying between the two branches of $X \setminus \{0\}$. Since these branches have a common tangent $[0,+\infty) \times \{0\}$ at zero, this line is also the tangent cone of M_X at the origin.

From these results, we obtain a symmetry property of plane analytic curves.

Proposition 3.38 ([5] Corollary 3.30)**.** *Let $X \subset \mathbb{R}^2$ be a real-analytic curve germ at zero and such that $X \setminus \{0\}$ consists of only two branches and $0 \in \overline{M_X}$. Then in a neighbourhood U of zero, the medial axis M_X is a half-line that is a symmetry axis of $X \cap U$.*

Proof. In view of the preceding results, there are two possibilities ([12]): either $0 \in \mathrm{Sng}_1 X$, or $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$ with X superquadratic at the origin. In the first case, by [22] Corollary 5.6 we know that $C_0(X)$ is a half-line ℓ, that we may assume to be $\{0\} \times [0,+\infty)$, whereas in the second one it is a line L that we assume to be the x-axis. Using the definition of the tangent cone, we may assume in both cases that in a neighbourhood of the origin X is a graph over an interval $(-\varepsilon, \varepsilon)$ in the x-axis. Consider $F = 0$ to be an analytic equation of X in the same neighbourhood.

Let h be the branch over $(-\varepsilon, 0]$ and g the branch over $[0, \varepsilon)$. They both are $\mathcal{C}^1$ at zero and due to the Puiseux Theorem, for some integer $p > 0$, $g(t^p)$ has an analytic extension through zero onto $(-\delta, \delta)$ for some $\delta \in (0, \varepsilon)$. Then, we obtain

$$F(s, g(s)) \equiv 0, \quad s \in [0, \sqrt[p]{\delta}).$$

Therefore, by the identity principle, this holds true for $|s| < \sqrt[p]{\delta}$. But we may repeat the same argument with h and so we conclude that $g(-s) = h(s)$ for $s \in (0, \delta)$ (if

[12]Note that both may occur: $y^3 = x^4$ is $\mathscr{C}^1$ regular at zero but superquadratic at this point, cf. Example 2.13.

δ was chosen < 1) which gives the symmetry sought after (since the germ of M_X at zero depends only on the germ of X at this point) and proves that M_X is a half-line near zero, as well. $\square$

Remark 3.39. This result implies that for instance the superquadratic curve $y = \mathrm{sgn}(x)|x|^{3/2}$ is not analytic at the origin.

4. Superquadratic points

Motivated by the situation in the plane, we may introduce a notion of superquadraticity in higher dimensions. The first natural step would be the following.

Definition 4.1 ([5] Definition 3.9)**.** If X is the graph of a non-negative continuous function f at $x_0 \in X$, then we call X *superquadratic* at this point, if the function $g_f(r) := \max_{x \in \mathbb{S}(x_0,r)} f(x)$ is superquadratic, i.e. it can be written near zero as $g(r) = ar^\alpha + o(r^\alpha)$ with $\alpha < 2$.

On the other hand, a geometric interpretation as in e Lemma 3.26 suggests that it might be a good idea to consider a notion of order of vanishing.

Definition 4.2 ([5] Definition 3.10)**.** We define the *order at zero* of a continuous definable function germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ as

$$\mathrm{ord}_0 f = \sup\{\eta > 0 \mid |f(x)| \leq \mathrm{const.}||x||^\eta, ||x|| \ll 1\},$$

if $f \not\equiv 0$, and $\mathrm{ord}_0 f := +\infty$ otherwise.

Remark 4.3. Since we are in a polynomially bounded o-minimal structure, the Łojasiewicz inequality ensures the well-posedness of the definition. It is a mere exercise to prove that in one variable $g(t) = at^\alpha + o(t^\alpha)$ is written precisely with $\alpha = \mathrm{ord}_0 g$ and $|g(t)| \leq \mathrm{const.}|t|^\alpha$.

By the methods used by Bochnak and Risler in [8] Theorem 1, it is easy to show that the least upper bound in the definition is in fact attained.

The inequality defining the order is satisfied with any exponent $\alpha \leq \mathrm{ord}_0 f$ and it makes sense also for a vector-valued f; then it is written as $||f(x)|| \leq \mathrm{const.}||x||^\eta$. In the latter case, $\mathrm{ord}_0 f$ coincides with the minimal order of the components f_i of $f = (f_1, \dots, f_k)$ ([13]).

Remark 4.4. For a given function germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ the definition of differentiability at zero gives readily the following two implications:

$$f \text{ is differentiable at } 0 \text{ and } \nabla f(0) = 0 \Rightarrow \mathrm{ord}_0 f \geq 1,$$

and

$$\mathrm{ord}_0 f \geq 2 \Rightarrow \ f \text{ is differentiable at } 0 \text{ and } \nabla f(0) = 0.$$

The example of $f(x) = |x|^{3/2}$ shows that there may be $f'(0) = 0$ and $\mathrm{ord}_0 f \in (1, 2)$.

[13] Also in this case the upper bound is attained. If $|f_i(x)| \leq c_i||x||_i^\theta$ for $||x|| \ll 1$ where $c_i > 0$ and $\theta_i = \mathrm{ord}_0 f_i$, then $\max|f_i(x)| \leq (\max c_i)||x||^{\min\theta_i}$ whence $\mathrm{ord}_0 f \geq \min\theta_i$. On the other hand, for the Euclidean norm we have $||f(x)|| \leq |f_i(x)|$ for any i, whence $\mathrm{ord}_0 f \leq \theta_i$.

From a practical point of view it is natural to consider also the following notion.

Definition 4.5 ([5] Definition 3.12)**.** We call *sectional order at zero* for a definable function $f\colon (\mathbb{R}^n,0)\to(\mathbb{R},0)$, $f\not\equiv 0$, the number

$$s_0(f)=\inf\{\alpha>0 \mid f(tv)=at^\alpha+o(t^\alpha), 0\leq t\ll 1, v\in\mathbb{S}^{n-1}\colon f|_{\mathbb{R}_+v}\not\equiv 0\}.$$

The relations between these three notions are given in Proposition 3.13 from [5]:

Proposition 4.6. *Consider a non-constant, continuous, definable germ* $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$*. Then for the following three conditions:*

(1) $s_0(f)<2$*;*
(2) $\mathrm{ord}_0 f<2$*;*
(3) $|f|$ *is superquadratic at* 0*;*

we have $(1)\Rightarrow(2)\Leftrightarrow(3)$*.*

Proof. The implications $(1)\Rightarrow(2)\Rightarrow(3)$ are immediate. Indeed, if (2) does not hold, then in a neighbourhood of zero, $|f(x)|\leq C||x||^2$ for some $C>0$. Thus for $f(tv)$ we have for all $t\geq 0$ small enough, $|at^\alpha+o(t^\alpha)|\leq Ct^2$ which implies $\alpha\geq 2$ (divide both sides by t^α and take $t\to 0^+$) and so $s_0(f)\geq 2$. If (3) does not hold, then $|f(x)|\leq g_{|f|}(||x||)=a||x||^\alpha+o(||x||^\alpha)$ for some $\alpha\geq 2$. But as $\mathrm{ord}_0 g_{|f|}=\alpha$, we obtain $|f(x)|\leq \mathrm{const.}||x||^\alpha$ and so $\mathrm{ord}_0 f\geq 2$.

Furthermore, to see that $(3)\Rightarrow(2)$ suppose that $\mathrm{ord}_0 f\geq 2$ and consider the definable set $A=\{(r,x)\in[0,\varepsilon]\times\mathbb{R}^n \mid ||x||=r, g_{|f|}(||x||)=|f(x)|\}$. Then 0 is an accumulation point of A and so there is a continuous definable selection $r\mapsto(r,\gamma(r))\in A$.

Then $g_{|f|}(r)=|f(\gamma(r))|$ and it follows from the definition of the order of vanishing (note that for small r, the values $\gamma(r)$ are near zero) that $\mathrm{ord}_0 g_{|f|}\geq \mathrm{ord}_0 f$ and so $\mathrm{ord}_0 g_{|f|}\geq 2$ as required. □

Example 4.7 ([5] Example 3.15)**.** The implication $(2)\Rightarrow(1)$ does not hold in general. To see this consider the semi-algebraic function

$$f(x,y)=\begin{cases}0, & x\leq 0 \text{ or } y\leq 0,\\ \frac{y}{x}, & 0<y\leq x \text{ and } x^2+y^2>\frac{y^2}{x^2},\\ (x^2+y^2)\frac{x}{y}, & 0<y\leq x \text{ and } x^2+y^2\leq\frac{y^2}{x^2},\\ f(y,x), & 0<x<y.\end{cases}$$

It is easy to check that f is continuous. Clearly, $f|_{\mathbb{R}_+v}\not\equiv 0$ iff $v\in\mathbb{S}^2\cap\{x,y>0\}=:S$ in which case $f(tv)=t^2(v_1/v_2)$ for $0\leq t\leq v_2/v_1$ (for greater t's we get $f(tv)=v_2/v_1$), where $v=(v_1,v_2)$. Hence $s_0(f)=2$.

But if there were $\mathrm{ord}_0 f\geq 2$, then we would have in a neighbourhood of zero, $f(x,y)\leq C||(x,y)||^2$ for some constant $C>0$. In particular, this would hold for $(x,y)=tv$ for any $v\in S$ and all $t\in(0,\varepsilon)$ with an appropriate $\varepsilon>0$. However, this would lead to $v_1/v_2\leq C$ which yields a contradiction when we make $(v_1,v_2)\in S$ tend to $(1,0)$.

Remark 4.8. The equivalence (2) $\Leftrightarrow$ (3) in the last Proposition allows us to extend the Definition 4.1 to any hypersurface being the graph of a definable function.

Definition 4.9. A set $X \subset \mathbb{R}^{n+1}$ is said to be *superquadratic at* a point $a \in X$, if in some coordinates it can be written in a neighbourhood of a as the graph of a superquadratic function of n variables.

The reason why we confine ourselves — at least for the moment — to hypersurfaces is that the superquadraticity introduced above has a further geometric characterisation similar to Lemma 3.26. First, let us introduce the (open) *bi-ball* (or *bidisc* when we are in the plane) *in the direction* $v \in \mathbb{S}^{n-1}$ as the open set

$$b_v(a,r) := \mathbb{B}(a - rv, r) \cup \mathbb{B}(a + rv, r)$$

where $r > 0$.

Proposition 4.10 ([5] Proposition 3.18)**.** *Let $X \subset \mathbb{R}^n$ be a closed definable set such that the tangent cone $C_0(X)$ is a linear hyperplane and $X \cap U$ is a graph over it, for some neighbourhood U of $0 \in X$. Then the following assertions are equivalent:*

(1) *X is superquadratic at the origin.*
(2) *For any $r > 0$, $b_{\nu(0)}(0,r) \cap X \neq \varnothing$ where $\nu(0)$ is a unit normal to X at 0.*

Proof. Choose coordinates in $\mathbb{R}^n = \mathbb{R}^{n-1}_x \times \mathbb{R}_t$ so that $C_0(X) = \{t = 0\}$ and write $X = \Gamma_f$ in a neighbourhood of zero. Fix $\nu(0) = (0, 1)$.

We start with (1) $\Rightarrow$ (2). Suppose that for some $r > 0$, $b_{\nu(0)}(0,r) \cap X = \varnothing$. This implies that for all $x \in \mathbb{R}^{n-1}$ sufficiently close to zero, we have $(x, |f(x)|) \notin \mathbb{B}(r\nu(0), r)$. On the other hand, observe that for $0 < ||x|| < r$ we have $(x, (1/r)||x||^2) \in \mathbb{B}(r\nu(0), r)$. Summing up, in a neighbourhood of zero, $|f(x)| \leq (1/r)||x||^2$ which means by Proposition 4.6 that X is not superquadratic.

In order to prove (2) $\Rightarrow$ (1) assume that X is not superquadratic at zero. Then by Proposition 4.6 we conclude that $\mathrm{ord}_0 f \geq 2$, i.e. $|f(x)| \leq c||x||^2$ for $||x|| < \varepsilon$ where $c, \varepsilon > 0$ are constants. Observe that for any $0 < r < 1/(2c)$, the graph of $t = c||x||^2$ does not enter the ball $\mathbb{B}((0,r),r)$. This readily implies that $X \cap b_{\nu(0)}(0,r) = \varnothing$, provided we have taken $r < \min\{1/(2c), \varepsilon\}$. □

Now, thanks to this result and in view of Theorem 3.13 (we need only to use the directional reaching radius in this case) we easily obtain the following Proposition:

Proposition 4.11. *If X satisifes the assumptions of the previous Proposition, then*

$$X \textit{ is superquadratic at the origin } \Rightarrow\ 0 \in \overline{M_X}.$$

The converse to the implication above does not hold.

Example 4.12 ([5] Remark 4.18)**.** Consider $X = \{z = y|x|^{3/2}\}$ which is the graph of a $\mathscr{C}^1$ function $z = f(x,y)$ in $\mathbb{R}^3$. We easily check that $\mathrm{ord}_0 f \geq 2$ so that X is not superquadratic at the origin, but as it is such along all the other points of the y-axis, we have $0 \in \overline{M_X}$ by the preceding Proposition.

Remark 4.13. Clearly, in view of the last Proposition, if a point $a \in X$ belongs to the closure of superquadratic points in X, then it belongs to $\overline{M_X}$.

The converse, unfortunately, is not true and the question of the relation between superquadraticity of $\mathscr{C}^1$-smooth hypersurfaces in at least three dimensions and the reaching of singularities by the medial axis is settled by the following clever Example of A. Białożyt [3]:

Example 4.14 (A. Białożyt). Let X be the graph of the function

$$f(x,y) = \begin{cases} \frac{y^2}{x}, & |y| < x^3, x > 0; \\ 2x^2|y| - x^5, & |y| \geq x^3, x > 0; \\ 0, & x \leq 0. \end{cases}$$

Then we can check that f is of class $\mathscr{C}^1$, it is not superquadratic at any point and yet $0 \in \overline{M_X} \cap X$ as X contains a suitable part of a rotated cone.

More results about superquadraticity, its generalization for sets of codimension greater than 1 and how can that be exploited in the context of Problem 2 will be published in [3] where the following theorem is shown:

Theorem 4.15 (A. Białożyt). *If $X \subset \mathbb{R}^k_x \times \mathbb{R}^n_y$ is definable with $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$ and $C_0(X) = \mathbb{R}^k \times \{0\}^n$, then $0 \in \overline{M_X}$ provided X is superquadratic at 0 in the sense that $g_X(\varepsilon) = a\varepsilon^\alpha + o(\varepsilon^\alpha)$ with $a \neq 0$ and $\alpha < 2$ where $g_X(\varepsilon) := \max\{||y||\colon (x,y) \in X, ||x|| = \varepsilon\}$. Moreover, if $\dim_0 X = 1$, then the converse holds: $0 \in \overline{M_X}$ implies X is superquadratic at the origin.*

Remark 4.16. Although the result above, Theorem 3.21 and Remark 4.13 give an answer to Problem 2 for a quite large family of singularities, still much work has to be done in order to definitely settle the question. It seems that Theorem 3.13 should lead to some advances.

5. On the multifunction of closest points

Another question related to the medial axis in the setting of singularity theory is what can be said about the the metric properties of the multifunction $m(x)$. It appears that $m(x)$ satisfies some Łojasiewicz-type inequalities ([5] Proposition 2.16 — see below; here our recent results [20] prove useful). Note that the distance of X along M_X encodes some metric information about the singularities. This, together with the study of the link of M_X, $\mathrm{lk}(M_X, a) = M_X \cap \partial\mathbb{B}(a,\varepsilon)$ (by the Local Conical Structure Theorem it does not depend on $\varepsilon > 0$ sufficiently small), provides some information about the tangent cone of M_X at $a \in \overline{M_X} \cap X$.

Let us begin with a general semicontinuity result that holds regardless of the definability of X.

Proposition 5.1 ([5] Proposition 2.17). *The multifunction $m(x)$ is upper semi-continuous: $\limsup_{D \ni x \to x_0} m(x) = m(x_0)$ at any point $x_0 \in \mathbb{R}^n$ and for* *any dense subset D of $\mathbb{R}^n$. Along M_X we usually only have an inclusion:* $\limsup_{M_X \ni x \to x_0} m(x) \subset m(x_0)$.

Proof. See [5] and [16] for the second part of the statement. □

Another useful fact is the following observation that also holds in general.

Proposition 5.2. *Let $U \subset \mathbb{R}^n$ be open and nonempty. Assume that there is a continuous selection $\mu\colon U \to X$ for $m(x)$, i.e. for any $x \in U$, $\mu(x) \in m(x)$. Then $\mu = m|_U$, i.e. $m(x)$ is univalent on U.*

Proof. [5] Proposition 2.19. □

Following [20] we will recall the different possible notions of a fibre of a multifunction $F\colon \mathbb{R}^m \to \mathscr{P}(\mathbb{R}^n)$. For $a \in \mathrm{dom}F$, we consider

- $F^{-1}(F(a)) = \{x \in \mathbb{R}^m \mid F(x) = F(a)\}$ *the (strong) pre-image*;
- $F^*(F(a)) = \{x \in \mathrm{dom}F \mid F(x) \subset F(a)\}$ *the lower pre-image*;
- $F_*(F(a)) = \{x \in \mathbb{R}^m \mid F(x) \supset F(a)\}$ *the upper pre-image*;
- $F^{\#}(F(a)) = \{x \in \mathbb{R}^m \mid F(x) \cap F(a) \neq \varnothing\}$ *the weak pre-image.*

Finally, we may consider a *point pre-image* defined for a point $y \in F(a)$ as the section $(\Gamma_F)_y := \{x \in \mathbb{R}^m \mid y \in F(x)\}$. Obviously,

$$F^{\#}(F(a)) = \bigcup_{y \in F(a)} (\Gamma_F)_y.$$

Apart from $m(x)$ we introduced in [5] two more multifunctions of interest, namely, *the normal set* multifunction

$$N(a) = \{x \in \mathbb{R}^n \mid a \in m(x)\} = \{x \in \mathbb{R}^n \mid ||x - a|| = d(x, X)\},\ a \in X$$

and the *univalued normal set* multifunction

$$N'(a) = \{x \in \mathbb{R}^n \mid m(x) = \{a\}\}, \quad a \in X.$$

Proposition 5.3 ([5] Propositions 2.2, 2.6)**.** *In the introduced setting*

1. *$a \in N'(a) \subset N(a)$;*
2. *$N(a)$ is closed, convex and definable (respectively, subanalytic), actually $X \ni a \mapsto N(a)$ is a definable (resp. subanalytic) multifunction, when X is definable (resp. subanalytic);*
3. *$N(a) \subset N_a(X) + a$;*
4. *$x \in N'(a) \ \Rightarrow\ [a, x] \subset N'(a)$*
   and $x \in N(a) \setminus \{a\} \ \Rightarrow\ [a, x) \subset N'(a)$;
5. *For any non-isolated $a \in X$, $\limsup_{X \ni b \to a} N(b) \subset N(a)$;*
6. *$N'(a)$ is convex and definable/subanalytic (as a set and as a multifunction of $a \in X$) when X is definable/subanalytic;*
7. *$N(a) = \overline{N'(a)}$.*

Clearly, we have

$$M_X = \bigcup_{a\in X} N(a)\setminus N'(a) = \bigcup_{a\in X} N(a)\setminus \bigcup_{a\in X} N'(a) = \mathbb{R}^n \setminus \bigcup_{a\in X} N'(a)$$

cf. [5] Theorem 27.

The different types of pre-images of $m(x)$, $N(a)$ or $N'(a)$ can be explicitly computed, see [5] Subsection 2.2 and the Proposition below.

The Kuratowski convergence of closed subsets of $\mathbb{R}^n$ is metrizable, and thus by the results of [20] Section 6 we have the following Łojasiewicz-type inequalities in the definable setting:

Proposition 5.4 ([5] Proposition 2.16)**.** *Let F denote either the closed multifunction $N(x)$, $x\in X$, or the compact one $m(x)$, $x\in\mathbb{R}^n$. Then for any point x_0 in the domain of F, there are constants $C,\ell>0$ such that in a neighbourhood of x_0,*

$$\mathrm{dist}_{HK}(F(x),F(x_0)) \geq Cd(x,F^\bullet(F(x_0)))^\ell$$

where dist_{HK} *denotes the Hausdorff-Kuratowski distance* ([14]) *and* $F^\bullet(F(x_0))$ *stands for any of the pre-images introduced above. In particular,*

1. $\mathrm{dist}_K(N(x),N(x_0)) \geq C||x-x_0||^\ell$ *for all* $x\in X$ *near* $x_0\in X$*;*
2. $\mathrm{dist}_H(m(x),m(x_0)) \geq Cd(x,N'(x_0))^\ell$ *for all* $x\in\mathbb{R}^n$ *near* $x_0\in X$*;*
3. $\mathrm{dist}_H(m(x),m(x_0)) \geq Cd(x,N(x_0))^\ell$ *for all* $x\in\mathbb{R}^n$ *near* $x_0\in X$*;*
4. $\mathrm{dist}_H(m(x),m(x_0)) \geq Cd(x,\bigcup_{y\in m(x_0)} N(y))^\ell$ *for all* $x\in\mathbb{R}^n$ *near* $x_0\in\mathbb{R}^n$*;*
5. $\mathrm{dist}_H(m(x),m(x_0)) \geq Cd(x,M_X)^\ell$ *for all* $x\in\mathbb{R}^n$ *near* $x_0\in M_X$*.*

Fix a definable or subanalytic closed, nonempty proper subset X of $\mathbb{R}^n$ and put

$$M_X(k) = \{x\in M_X \mid \dim m(x) = k\}.$$

These sets are obviously definable or subanalytic, respectively.

Theorem 5.5 ([16] Theorem 4.10, Theorem 4.13)**.** *In the setting considered,*

1. *If $k=n-1$ (which is the maximal dimension possible), then* $\dim M_X(n-1)=0$*, i.e. $M_X(n-1)$ is isolated;*
2. *In general $k+\dim M_X(k)\leq n-1$ and the inequality may be strict already for $k=1$, $n=3$.*

Remark 5.6. Point (1) in the theorem above was obtained earlier by Albano and Cannarsa in [2] in a slightly different form and for a general closed set X using the Hausdorff dimension (it coincides with the analytic dimension for a subanalytic set). Point (2), on the contrary, proved in [16] using methods typically from tame geometry, is still waiting for a general counterpart.

Example 5.7. ([16] Example 4.15). Consider $X=\{x^2+y^2+z^2=1, yz=0\}$ in $\mathbb{R}^3$. Then $m(0)=X$ and so $0\in M_X(1)$ and clearly it is the only point in this set. Therefore $1+\dim M_X(1) < 3-1=2$.

[14]For compact sets it is the usual Hausdorff distance, for closed ones, it is the metric giving the Kuratowski convergence.

Using the last Theorem, A. Białożyt shows in [3]:

Theorem 5.8 (A.Białożyt)**.** *In the definable setting, for any $a \in M_X$, there is a neighbourhood U of a such that*

$$\dim_a M_X = n - 1 - \min\{\dim m(b) \mid b \in M_X \cap U\}.$$

We have to move around a due to the Białożyt wristwatch example:

Example 5.9 (A. Białożyt)**.** Let X be the boundary of $\mathbb{B}_2(0,3) \cup ((-1,1) \times \mathbb{R})$ in $\mathbb{R}^2$. Then for $a = 0$, we have $\dim m(a) = 1$ and $\dim_a M_X = 1$, as $M_X = \{0\} \times \mathbb{R}$. Only taking any other point $b \in M_X \setminus \{a\}$ gives the equality from the Theorem.

Further studies on the subject are presented in [3].

6. Closing remarks

Several results concerning the topological structure of the medial axis are known. For instance in [21] Theorem 1.B it is shown that for a domain $D \subset \mathbb{R}^n$ that does not contain any half-space (cf. Remark 2.2) the set $M_X \cap D$ is connected. This intuitive result has an astonishingly intricate proof (see [17] for a self-contained one; see also the inspiring paper [23]). Moreover, Fremlin proves also an interesting fact [21] Proposition 1.F which hints at the fact that the medial axis should not have 'bad' singularities itself, i.e. no cusps are allowed (this is partly confirmed by the results of [4]). Also Yomdin in his very nice paper [28] presented a general structural result concerning the medial axis, however the proof is fallacious as it is based on a non-existing (and most probably false) version of a Lipschitz Implicit Function Theorem (LIFT). We discuss this problem in [19] obtaining a correct LIFT which, nonetheless, allows us to reprove Yomdin's stability result only in a generic case in $\mathbb{R}^3$.

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