Bifurcation values of functions of class C^∞

Michał Klepczarek

SUMMARY OF THE DOCTORAL DISSERTATION

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, i.e. a function of class C^∞ . The smallest set $B \subset \mathbb{R}$, relative to the inclusion relation, such that the function

$$f|_{\mathbb{R}^n \setminus f^{-1}(B)} : \mathbb{R}^n \setminus f^{-1}(B) \to \mathbb{R} \setminus B$$

is a locally trivial smooth fibration is called *the bifurcation set of* f and denoted by $B(f)$. In 1969 R. Thom proved that $B(f)$ is finite for polynomial functions f. In general, it is well known that $B(f) = K_0(f) \cup B_\infty(f)$, where $K_0(f)$ is *the set of critical values* of f and $B_\infty(f)$ is *the set of bifurcation values of* f *at infinity*, i.e. the set of points at which f is not locally trivial smooth fibration outside a large ball. The computation of $B_\infty(f)$ is an open problem that was the subject of research of, among others, S.A. Broughton, L.R.G. Dias, H.V.Há, Z. Jelonek, K. Kurdyka, T. Krasiński, T.D. Lê, A. Némethi, A. Parusiński, D. Siersma, M. Tibăr, A. Zaharia.

In order to estimate the set $B_\infty(f)$ some conditions on the function f in neighborhoods of fibers $f^{-1}(y)$ are introduced, which implies that the points y are *typical values* of f (i.e. $y \in \mathbb{R} \setminus B(f)$). Frequently used examples of such conditions are Malgrange's condition and ρ_a-regularity.

We say that f satisfies *Malgrange's condition* at a point $y \in \mathbb{R}$ if there exist a neighborhood $U \subset \mathbb{R}$ of the point y and constants $R, \delta > 0$ such that

$$|\nabla f(x)||x| \geqslant \delta \ \text{ for } x \in f^{-1}(U),\ |x| > R.$$

By $K_\infty(f)$ we denote the set of *asymptotic critical values of* f, i.e. the set of points where f doesn't satisfy Malgrange's condition:

$$K_\infty(f) = \{y \in \mathbb{R} : \exists_{(x_k)_{k=1}^\infty \subset \mathbb{R}^n} \lim_{k\to\infty} |x_k| = +\infty,\ \lim_{k\to\infty} f(x_k) = y,$$
$$\lim_{k\to\infty} |x_k||\nabla f(x_k)| = 0\}.$$

It is well known that $B_\infty(f) \subset K_\infty(f)$ and that the set $K_\infty(f)$ is finite, provided f is a polynomial.

Let $a \in \mathbb{R}^n$ and let $M(f, \rho_a)$ be the set of critical points of (f, ρ_a), where $\rho_a(x) = |x - a|^2$. We say that f is ρ_a*-regular* at $y \in \mathbb{R}$ if there exists a neighborhood $U \subset \mathbb{R}$ of y such that $M(f, \rho_a) \cap f^{-1}(U)$ is bounded. By $S_a(f)$ we denote the set of points at which f is not ρ_a-regular, i.e.

$$S_a(f) = \{y \in \mathbb{R} : \exists_{(x_k)_{k=1}^\infty \subset M(f,\rho_a)} \lim_{k\to\infty} |x_k| = +\infty,\ \lim_{k\to\infty} f(x_k) = y\}.$$

Let $S_\infty(f) := \bigcap_{a\in\mathbb{R}^n} S_a(f)$. L.R.G. Dias and M. Tibăr (2015) proved that $B_\infty(f) \subset S_\infty(f)$.

In this paper we study the problem of determining the set $B_\infty(f)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. In this case, the set $B_\infty(f)$ might be infinite.

Chapter 1 has an auxiliary character, where we recall some notions and theorems in differential geometry, algebraic and semialgebraic geometry and differential equations. In particular, we present a result due to T. Bárta, R. Chill, E. Fašangová

(2012), which says that every ordinary differential equation with a strict Lyapunov function is a gradient system.

In Chapter 2 we define a set $\pitchfork(f, D)$ of smooth vector fields $v : D_v \to \mathbb{R}^n$, $D \subset D_v \subset \mathbb{R}^n$, such that

$$\partial_{v(x)} f(x) \neq 0 \quad \text{for } x \in D.$$

Next, we introduce the set $[f, y]_v^D$ consisting of functions f^* which, roughly speaking, measures how far the points $x \in D$ are from the level set $f^{-1}(y)$. In order to control how far the points $x \in D$ are from infinity, we introduce the set of functions h satisfying the condition

$$(*) \qquad \forall_{(x_k)_{k=1}^{\infty} \subset D} \lim_{k\to\infty} |x_k| = +\infty \Rightarrow \lim_{k\to\infty} |h(x_k)| = +\infty.$$

Using the above notions we define the sets $\overline{O}_y(f), O_y(f)$ consisting of triples (v, h, f^*) such that: for some neighborhood U of y, we have $v \in \pitchfork(f, f^{-1}(U))$, h satisfies the condition (*) with $D = f^{-1}(U)$, and $f^* \in [f, y]_v^{f^{-1}(U)\setminus K}$ for some compact set $K \subset \mathbb{R}^n$. Moreover, we assume that the function

$$H(x) = \frac{\partial_{v(x)} h(x)}{\partial_{v(x)} f^*(x)}$$

is bounded on solutions of the equation $x' = v(x)$ in $f^{-1}(U) \setminus (K \cup f^{-1}(y))$ for elements of the set $\overline{O}_y(f)$. We assume that the function H is bounded on the set $f^{-1}(U) \setminus (K \cup f^{-1}(y))$ for elements of $O_y(f)$.

The main result of the doctoral thesis is Theorem 2.4.7. In this theorem we give a description of typical values of f in terms of the sets $O_y(f)$. More precisely, we have the following

Theorem 2.4.7. *A point y is a typical value of f if and only if $O_y(f) \neq \emptyset$.*

The above theorem follows directly from Theorems 2.4.3 and 2.4.6.

In Theorem 2.4.3 we prove that if $\overline{O}_y(f) \neq \emptyset$ then y is a typical value of f. This result allows the construction of new conditions characterizing certain supersets of $B_\infty(f)$ (see i.e. Chapter 3).

In Theorem 2.4.6 we show that for every typical value y of a function f there exist a neighborhood U of y, a vector field $v \in \pitchfork(f, f^{-1}(U))$ and a smooth function h satisfying condition (*) with $D = f^{-1}(U)$ such that $H = \partial_v h / \partial_v f = 0$. In particular, $O_y(f) \neq \emptyset$. As a corollary we get

Corollary 2.4.8. *For every typical value y of f there exists a metric tensor g defined in a neighborhood of $f^{-1}(y)$ such that we can trivialize function f near $f^{-1}(y)$ by integrating the gradient of f (with respect to the metric tensor g).*

Next we give some special conditions implying that y is a typical value of f. Let h satisfy the condition (*) with $D = f^{-1}(U_0) \setminus K_0$, where U_0 is a neighborhood of y and $K_0 \subset \mathbb{R}^n$ is a compact set and let $M(f, h, D')$ be the set of critical values of $(f, h) : D' \to \mathbb{R}^2$, $D' \subset f^{-1}(U_0) \setminus K_0$. We have

Theorem 2.7.2. *Under the above assumptions:*

a) *if there exists a neighborhood $U \subset U_0$ of y such that the set $M(f, h, f^{-1}(U) \setminus K_0)$ is bounded then y is a typical value of f,*

b) *if there exists a neighborhood* $U \subset U_0$ *of* y *such that the set* $M(f, h, f^{-1}(U) \setminus K_0)$ *is unbounded but the function* $|\nabla h|/|\nabla f|$ *is bounded on* $M(f, h, f^{-1}(U) \setminus K)$ *then* y *is a typical value of* f,

c) *if for every neighborhood* $U \subset U_0$ *of* y *and every compact set* $K \supset K_0$ *the set* $M(f, h, f^{-1}(U) \setminus K)$ *is unbounded and the function* $|\nabla h|/|\nabla f|$ *is unbounded on* $M(f, h, f^{-1}(U) \setminus K)$ *then for every* $v \in \pitchfork (f, f^{-1}(U))$ *we have* $(v, h, f) \notin O_y(f)$.

We define a set $S^h(f)$ by

$$S^h(f) = \{y \in \mathbb{R} : \exists_{(x_k)_{k=1}^{\infty} \subset M(f,h,\mathbb{R}^n \setminus K_0)} \lim_{k\to\infty} |x_k| = +\infty, \lim_{k\to\infty} f(x_k) = y\}.$$

Under the assumptions of Theorem 2.7.2, we get

Corollary 2.7.5. $B(f) \subset S^h(f) \cup K_0(f)$.

In Chapter 3 we show how one can use the above theorems to get well known conditions for trivializing a function: Ehresmann's fibration theorem, Fedoryuk's condition, Malgrange's condition, ρ_a-regularity and others. We also introduce some new conditions similar to Malgrange's condition. For example, we say that f satisfies *improved pre-Malgrange's condition* at y if there exists a neighborhood U of y such that $\nabla f(x) \neq 0$ for $x \in f^{-1}(U)$ and there exist constants $R, C > 0$ and $\theta < 1$ such that

$$|f(x) - y|^{\theta} |\langle \nabla f(x), x\rangle| \leqslant C|x|^2 |\nabla f(x)|^2, \text{ for } x \in f^{-1}(U) \setminus f^{-1}(y), |x| > R.$$

We prove

Theorem 3.6.2. *Let* y *be a regular value of* $f \in C^{\infty}(\mathbb{R}^n)$. *If* f *satisfies improved pre-Malgrange's condition at* y *then* y *is a typical value of* f.

In Chapter 4 we show how one can use the sets $\overline{O}_y(f)$ for more precise study of the set $B(f)$. More precisely, given a foliation $\mathbb{G} = \{G_s \subset \mathbb{R}^n : s \in S\}$ of $D \subset \mathbb{R}^n$ we consider a set $\pitchfork^{\mathbb{G}} (f, D)$ of a vector fields $v \in \pitchfork (f, D)$ tangent to the leaves of the foliation $\mathbb{G}$. By $O_y^{\mathbb{G}}(f)$ we denote a set of of triples (v, h, f^*) such that for some neighborhood U of y, we have $v \in \pitchfork^{\mathbb{G}} (f, f^{-1}(U))$, h satisfies the condition (*) with $D = f^{-1}(U)$, and $f^* \in [f, y]_v^{f^{-1}(U)\setminus K}$ for some compact set $K \subset \mathbb{R}^n$. Moreover, we assume that the function $H = \partial_v h / \partial_v f^*$ is bounded on $G_s \cap f^{-1}(U) \setminus (K \cup f^{-1}(y))$ for $s \in S$. We prove that if $O_y^{\mathbb{G}}(f) \neq \emptyset$ then y is a typical value of f (see Corollary 4.1.2). In particular, for a given submersion $g : D_g \to \mathbb{R}^r$ we introduce g-pre-Malgrange's condition at y as follows. For $x \in D_g$ by $\nabla^g f(x)$ we denote the gradient of $f|_{g^{-1}(g(x))}$. We say that f satisfies *g-pre-Malgrange's condition* at y if there exist a neighborhood U of y and a compact set $K \subset \mathbb{R}^n$ such that:

($1M_g$) $f^{-1}(U) \setminus K \subset D_g$,
($2M_g$) $\nabla^g f(x) \neq 0$ for $x \in f^{-1}(U) \setminus K$,
($3M_g$) $\forall_{s\in S} \exists_{C_s > 0} |\langle x, \nabla^g f(x)\rangle| \leqslant C_s |x|^2 |\nabla^g f(x)|^2$ for $x \in f^{-1}(U) \setminus K \cap g^{-1}(s)$.

We have the following

Theorem 4.2.4. *Let* y *be a regular value of* $f \in C^{\infty}(\mathbb{R}^n)$. *If* f *satisfies* g*-pre-Malgrange's condition at* y *then* y *is a typical value of* f.

In Chapter 5 we give two algorithms for computing some supersets of $B(f)$ when f is a polynomial. The first algorithm is based on the result due to Z. Jelonek and K. Kurdyka (2014) and uses a finite dimensional space of rational arcs. More precisely, let h satisfy the condition (*) with $D = \mathbb{R}^n \setminus K$ for some compact set $K \subset \mathbb{R}^n$. Let v be a smooth vector field on $\mathbb{R}^n$ and suppose that the function $\partial_{v(x)}h(x)/\partial_{v(x)}f(x)$ is rational and let $\Sigma_v = \{x \in \mathbb{R}^n : \partial_v f(x) = 0\}$. We define a set $K_\infty^{v,h}(f)$ such that $B_\infty(f) \subset K_\infty^{v,h}(f)$ (see Corollary 5.2.3). Moreover, $K_\infty^{v,h}(f)$ can be described in terms of the set of points at which $f|_{\Sigma_v}$ is not proper and the set of limits of f on some rational arcs (see Theorem 5.2.4). Under the above assumptions, we provide an algorithm that allows to decide whether the set $K_\infty^{v,h}(f)$ is finite and compute $K_\infty^{v,h}(f)$ in this case (see Paragraph 5.2).

The second algorithm is based on the result due to L.R.G. Dias, M. Tibăr (2014). This algorithm computes the sets $b_0(AV_i), i \in I = \{k \in \{1, ..., n\} : \frac{\partial f}{\partial x_k} \neq 0\}$ of limits at infinity of f on some rational arcs, such that $B(f) \setminus K_0(f) \subset S^h(f) \setminus K_0(f) \subset \bigcup_{i \in I} b_0(AV_i)$ (see Corollary 2.7.5 Theorem 5.3.1).