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Chapter 22
On ψ-density topologies on the real line and on the plane

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In this chapter we discuss topologies called ψ-density topologies. The definition of them is based on Taylor's strengthening the Lebesgue Density Theorem. All of ψ-density topologies are essentially weaker than the density topology $\mathcal{T}_d$ but still essentially stronger than $\mathcal{T}_{nat}$. The notion of ψ-density topology was involved in the research work of many mathematicians. They concentrated mostly on the differences between density topology and ψ-density topologies on the real line. We would like to present the main results of that research but we will focus on ordinary and strong ψ-density topologies on the plane.

22.1 The density topology on the real line

The classic Lebesgue Density Theorem [19] claims that for any Lebesgue measurable set $A \subset \mathbb{R}$ the equality

$$\lim_{h\to 0+} \frac{\lambda\left(A \cap [x-h, x+h]\right)}{2h} = 1 \tag{22.1}$$

holds for all points $x \in A$ except for the set of Lebesgue measure zero. Denoting

$$\Phi_d(A) = \left\{ x \in \mathbb{R} \colon \lim_{h \to 0+} \frac{\lambda\,(A \cap [x-h, x+h])}{2h} = 1 \right\}$$

we can equivalently say that $\lambda\,(A \Delta \Phi_d\,(A)) = 0$ for any $A \in \mathcal{L}$. The operator Φ_d is a lower density operator i.e. for any $A, B \in \mathcal{L}$ it has the following properties:

(1) $\Phi_d(\emptyset) = \emptyset$, $\Phi_d(\mathbb{R}) = \mathbb{R}$;
(2) $\Phi_d(A \cap B) = \Phi_d(A) \cap \Phi_d(B)$;
(3) $\lambda(A \Delta B) = 0 \Longrightarrow \Phi_d(A) = \Phi_d(B)$;
(4) $\lambda(A \Delta \Phi_d(A)) = 0$.

It is well known that a family

$$\mathcal{T}_d = \{A \in \mathcal{L} : A \subset \Phi_d\,(A)\}$$

forms a topology called the density topology and denoted by $\mathcal{T}_d$. Let us recall its several properties.

Theorem 22.1. *The density topology has the properties:*

(a) *If* $\lambda\,(N) = 0$ *then* N *is* $\mathcal{T}_d-$*closed;*
(b) $\mathcal{T}_{nat} \subsetneq \mathcal{T}_d$*;*
(c) $(\mathbb{R}, \mathcal{T}_d)$ *is neither first countable, nor Lindelöf, nor separable;*
(d) A *is* $\mathcal{T}_d-$*compact* $\Longleftrightarrow A$ *is finite;*
(e) $\lambda\,(N) = 0 \Longleftrightarrow N$ *is* $\mathcal{T}_d-$*nowhere dense* $\Longleftrightarrow N$ *is* $\mathcal{T}_d-$*meager;*
(f) $(\mathbb{R}, \mathcal{T}_d)$ *is a Baire space;*
(g) $\mathrm{int}_{\mathcal{T}_d}(A) = A \cap \Phi_d\,(B)$, *where* B *is a measurable kernel of* A*;*
(h) $(\mathbb{R}, \mathcal{T}_d)$ *is completely regular but not normal;*
(i) A *is connected in* $(\mathbb{R}, \mathcal{T}_d) \Longleftrightarrow A$ *is connected in* $(\mathbb{R}, \mathcal{T}_{nat})$*;*
(j) $\mathcal{T}_d$ *is invariant under translations and multiplications by nonzero numbers.*

The proofs of these properties will be presented in the next chapter. Notice that:

- properties (a)-(d) follow from properties (1)-(3) of the operator Φ_d;
- properties (e)-(g) are true by the Lebesgue Density Theorem (compare [19]);
- a proof of completely regularity is much more complicated and connected with the Lusin-Menchoff Theorem (compare [11]); $(\mathbb{R}, \mathcal{T}_d)$ is not normal, because there is no possibility to separate $\mathbb{Q}$ from $\mathbb{Q} + \sqrt{2}$ by $\mathcal{T}_d$-open sets (compare [11]);
- (i) was proved by Goffman and Waterman via properties of approximately continuous functions (see [12]);
- (j) again follows straightforward from the properties of Lebesgue measure.

22.2 Taylor's strengthening the Lebesgue Density Theorem

In The Scottish Book one can find a problem formulated by Stanisław Ulam (1936) as follows: "It is known that in sets of positive measure there exist points of density 1. Can one determine the speed of convergence of this ratio for almost all points of the set?" (Problem 146, [20]). In other words it is the question about possibility of strengthening the Lebesgue Density Theorem. The answer to Ulam's question was given by S. J. Taylor in 1959, [21]. Taylor modified the condition (22.1) by introducing in denominator of the fraction a new factor ψ, which is a nondecreasing continuous function from $(0,\infty)$ to $(0,\infty)$ such that $\lim_{x\to 0+}\psi(x)=0$ (the family of such functions will be denoted by $\widehat{\mathcal{C}}$). He proved two important theorems ([21], Th.3 and 4).

The First Taylor's Theorem *For any Lebesgue measurable set $A\subset\mathbb{R}$ there exists a function $\psi\in\widehat{\mathcal{C}}$ such that*

$$\lim_{d(I)\to 0}\frac{\lambda(A'\cap I)}{\lambda(I)\psi(\lambda(I))}=0$$

for almost all $x\in A$, where I is any interval containing x.

The Second Taylor's Theorem *For any function $\psi\in\widehat{\mathcal{C}}$ and a real number α, $0<\alpha<1$, there exists a perfect set $E\subset[0,1]$ such that $\lambda(E)=\alpha$ and*

$$\limsup_{d(I)\to 0}\frac{\lambda(E'\cap I)}{\lambda(I)\psi(\lambda(I))}=\infty$$

for all $x\in E$.

In [22] S.J. Taylor formulated an alternative form of Egoroff's theorem and, by a consequence, he obtained the following theorem.

The Third Taylor's Theorem *Given any Lebesgue measurable set E in m-dimensional Euclidean space, there exist $\psi\in\widehat{\mathcal{C}}$ and $S\subset E$ such that $\lambda_m(E\setminus S)=0$ and for $x\in S$*

$$\lim_{d(I)\to 0}\frac{1}{\psi(d(I))}\left(\frac{\lambda_m(I\cap E)}{\lambda_m(I)}-1\right)=0 \tag{22.2}$$

where I is any rectangle containing x with slides parallel to the coordinate axes of $\mathbb{R}^m$ and $d(I)$ stands for a diameter of I.

22.3 ψ-density topology on the real line

Following Taylor, in [23] there was introduced a notion of ψ-density point which involved only intervals I with center at x. Fix $\psi \in \widehat{\mathcal{C}}$.

Definition 22.2 ([23]). We say that $x \in \mathbb{R}$ is a ψ-density point of a set $A \in \mathcal{L}$ if

$$\lim_{h \to 0+} \frac{\lambda(A' \cap [x-h, x+h])}{2h\psi(2h)} = 0.$$

In particular, if $\psi = id$ we obtain superdensity introduced by Zajicek in [15]. For any set $A \in \mathcal{L}$ we denote

$$\Phi_\psi(A) = \{x \in \mathbb{R} : x \text{ is a } \psi\text{-density point of } A\}.$$

The operator Φ_ψ is not a lower density operator, in fact the properties (1)-(3) are fulfilled but (4) fails by The Second Taylor's Theorem. However, for any measurable set A, $\Phi_\psi(A)$ is measurable (in fact it is a $F_{\sigma\delta}$ set) and a family

$$\mathcal{T}_\psi = \{A \in \mathcal{L} \colon A \subset \Phi_\psi(A)\}$$

constitutes a topology called ψ-density topology. Clearly, for any $\psi \in \widehat{\mathcal{C}}$, $\mathcal{T}_{nat} \subset \mathcal{T}_\psi \subset \mathcal{T}_d$ and null sets are $\mathcal{T}_\psi-$closed. Therefore, the space $(\mathbb{R}, \mathcal{T}_\psi)$ is neither first countable, nor second countable, nor Lindelöf, nor separable; each set compact in $\mathcal{T}_\psi$ is finite; a set is measurable if it is Borel in $\mathcal{T}_\psi$ ([7]).

Moreover, using the condition (i) from Theorem 22.1, we can easy conclude that the family of connected sets in topology $\mathcal{T}_\psi$ coincides with the family of connected sets in $\mathcal{T}_{nat}$ (and $\mathcal{T}_d$).

The set $\mathbb{R} \setminus \mathbb{Q} \in \mathcal{T}_\psi \setminus \mathcal{T}_{nat}$. Let E be the set constructed in The Second Taylor's Theorem. It is easy to see that $\Phi_d(E) \in \mathcal{T}_d \setminus \mathcal{T}_\psi$. Therefore,

$$\mathcal{T}_{nat} \subsetneq \mathcal{T}_\psi \subsetneq \mathcal{T}_d.$$

Now we will look at the properties which distinguish topologies $\mathcal{T}_\psi$ and $\mathcal{T}_d$. The set constructed in The Second Taylor's Theorem has positive measure and is $\mathcal{T}_\psi-$nowhere dense. In density topology nowhere dense sets must have measure zero. E. Wagner-Bojakowska proved:

Theorem 22.3 (compare [24], Th.8). *There exist $\mathcal{T}_\psi-$closed and $\mathcal{T}_\psi-$nowhere dense sets $E_n \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_{n=1}^\infty E_n$.*

Hence $\left(\mathbb{R}, \mathcal{T}_\psi\right)$ is not a Baire space while $(\mathbb{R}, \mathcal{T}_d)$ is.

To describe the interior operation in ψ-density topology let $\Phi_\psi^1(A) = \Phi_\psi(A)$. If α is an ordinal number, $1 < \alpha < \Omega$ and $\alpha = \beta + 1$, where $1 \leq \beta < \Omega$, then $\Phi_\psi^\alpha(A) = \Phi_\psi(\Phi_\psi^\beta(A))$. If α is a limit number, $\alpha < \Omega$, then $\Phi_\psi^\alpha(A) = \bigcap_{1\le\beta<\alpha} \Phi_\psi^\beta(A)$. E. Wagner-Bojakowska and W. Wilczyński proved in [25] the following theorem.

Theorem 22.4. *(a) For each $\psi \in \widehat{\mathcal{C}}$, $A \in \mathcal{L}$ and each countable ordinal $\alpha > 0$*

$$\operatorname{int}_\psi(A) \subset A \cap \Phi_\psi^\alpha(A).$$

(b) For each $\psi \in \widehat{\mathcal{C}}$ and $A \in \mathcal{L}$ there exists an ordinal β, $1 \le \beta < \Omega$ such that

$$\operatorname{int}_\psi(A) = A \cap \Phi_\psi^\beta(A).$$

(c) For each $\psi \in \widehat{\mathcal{C}}$ and each countable ordinal $\alpha > 0$ there exists $A \in \mathcal{L}$ such that

$$\operatorname{int}_\psi(A) \neq A \cap \Phi_\psi^\alpha(A).$$

The spaces $(\mathbb{R}, \mathcal{T}_d)$ and $(\mathbb{R}, \mathcal{T}_\psi)$ satisfy different separate axioms. In [4] it is proved that $\psi-$density topologies does not satisfy the Lusin-Menchoff Theorem and $(\mathbb{R}, \mathcal{T}_\psi)$ is not regular. Since $\mathcal{T}_{nat} \subset \mathcal{T}_\psi$, $(\mathbb{R}, \mathcal{T}_\psi)$ is a Hausdorff space.

Clearly, for $\psi \in \widehat{\mathcal{C}}$, any translation of a set belonging to $\mathcal{T}_\psi$ belongs to $\mathcal{T}_\psi$ either. It seems interesting that the invariance under multiplication depends on ψ. In fact, it is strictly connected with the condition which we call (Δ_2), by analogy with well known condition used in Orlicz spaces. We will say that $\psi \in \widehat{\mathcal{C}}$ fulfills (Δ_2) condition ($\psi \in \Delta_2$) if

$$\limsup_{x\to 0+} \frac{\psi(2x)}{\psi(x)} < \infty.$$

In [9] it is proved that:

Theorem 22.5. *The topology $\mathcal{T}_\psi$ is invariant under multiplication if and only if $\psi \in \Delta_2$.*

Note that $\psi(x) = \begin{cases} x^{\frac{1}{x}} & \text{for } x \in (0,1) \\ 1 & \text{for } x \geqslant 1 \end{cases}$ does not satisfies (Δ_2), but functions $\psi(x) = x^\alpha$ satisfy this condition for any $\alpha > 0$.

It is obvious that

$$\bigcup\{\mathcal{T}_\psi\colon \psi \in \widehat{\mathcal{C}}\} \subset \mathcal{T}_d.$$

In [27] it was proved that the inclusion is proper and topology generated by this union is equal to density topology. Moreover, (compare [23]),

$$\bigcap\{\mathcal{T}_\psi\colon\ \psi\in\widehat{\mathcal{C}}\}=\{U\setminus N\colon\ U\in\mathcal{T}_{nat}\wedge\lambda(N)=0\}.$$

It is evident that the same ψ-density topologies can be obtained via different functions from $\widehat{\mathcal{C}}$. If $\limsup_{x\to 0+}\frac{\psi_1(x)}{\psi_2(x)}<\infty$ and $\liminf_{x\to 0+}\frac{\psi_1(x)}{\psi_2(x)}>0$ then $\mathcal{T}_{\psi_1}=\mathcal{T}_{\psi_2}$ ([23]). However, there exist functions $\psi_1,\psi_2\in\widehat{\mathcal{C}}$ such that they fulfill the conditions $0<\limsup_{x\to 0+}\frac{\psi_1(x)}{\psi_2(x)}<\infty$ and $\liminf_{x\to 0+}\frac{\psi_1(x)}{\psi_2(x)}=0$ and $\mathcal{T}_{\psi_1}=\mathcal{T}_{\psi_2}$. The necessary and sufficient condition were given by E. Wagner-Bojakowska and W. Wilczyński in [26]. For $\psi_1,\psi_2\in\widehat{\mathcal{C}}$ and $k\in\mathbb{N}$ they put:

$$\begin{aligned}
A_k^+ &=\left\{x\in\mathbb{R}_+;\psi_1(2x)<\frac{1}{k}\psi_2(2x)\right\},\\
B_k^+ &=\left\{x\in\mathbb{R}_+;\psi_2(2x)<\frac{1}{k}\psi_1(2x)\right\},\\
A_k &=A_k^+\cup(-A_k^+),\quad B_k=B_k^+\cup(-B_k^+).
\end{aligned}$$

and proved:

Theorem 22.6 ([26], Th. 8). *Let* $\psi_1,\psi_2\in\widehat{\mathcal{C}}$*,*

$$\varepsilon_k=\limsup_{x\to 0+}\frac{m(A_k\cap[-x,x])}{2x\psi_1(2x)},\ \eta_k=\limsup_{x\to 0+}\frac{m(B_k\cap[-x,x])}{2x\psi_2(2x)}$$

for $k\in\mathbb{N}$*. The topologies* $\mathcal{T}_{\psi_1},\mathcal{T}_{\psi_2}$ *are equal if and only if* $\lim_{k\to\infty}\varepsilon_k=\lim_{k\to\infty}\eta_k=0$.

A brief survey of ψ-density topologies one can find in [2] and [3]. Some other properties of ψ-density topologies were also examined. G. Horbaczewska considered resolvability of ψ-density topologies and proved that for any function $\psi\in\widehat{\mathcal{C}}$ such topologies are maximally resolvable and, assuming Martin's Axiom, extraresolvable ([10]). A. Goździewicz-Smejda and E. Łazarow introduced the notion of ψ-sparse sets and ψ-sparse topologies ([13], [14]). E. Łazarow and A. Vizváry examined the category analogue of ψ-density topologies ([16]). In [17] E. Łazarow and K. Rychert introduced the notion of ψ-porosity and ψ-superporosity and they compared them with the classical notions of porosity and superporosity.

22.4 ψ-density topologies on the plane.

Defining density points and $\psi-$density points on the real line there are used the closed intervals with the common center. On $\mathbb{R}^m$ there are two standard differentiation basis - we can use cubes or rectangles. To simplify considerations we will present the definitions and results obtained for $\mathbb{R}^2$, but all of them can be applied in the same manner for $m > 2$.

We will denote by $\mathcal{T}_{nat}^2$ - the natural topology on $\mathbb{R}^2$. For any $x = (x_1, x_2) \in \mathbb{R}^2$ and $h, k > 0$, let $Sq(x,h)$ denote a square $[x_1 - h, x_1 + h] \times [x_2 - h, x_2 + h]$ and $R(x,h,k)$ denote a rectangle $[x_1 - h, x_1 + h] \times [x_2 - k, x_2 + k]$. Recall that $x \in \mathbb{R}^2$ is an ordinary density point of a set $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0+} \frac{\lambda_2(A' \cap Sq(x,h))}{4h^2} = 0.$$

Let $\Phi_d^o(A) = \{x \in \mathbb{R}^2 \colon x \text{ is an ordinary density point of a set } A\}$ for $A \in \mathcal{L}_2$ and $\mathcal{T}_d^o$ denotes the family of all sets $A \in \mathcal{L}_2$ such that $A \subset \Phi_d^o(A)$. The family $\mathcal{T}_d^o$ is a topology called the ordinary density topology on the plane (see [28], section 4). Analogously, $x \in \mathbb{R}^2$ is a strong density point of a set $A \in \mathcal{L}_2$ if

$$\lim_{\substack{h \to 0+ \\ k \to 0+}} \frac{\lambda_2(A' \cap R(x,h,k))}{4hk} = 0.$$

In the same way for $A \in \mathcal{L}_2$ we define the set

$$\Phi_d^s(A) = \{x \in \mathbb{R}^2 \colon x \text{ is a strong density point of a set } A\}.$$

The family $\mathcal{T}_d^s = \{A \in \mathcal{L}_2 \colon A \subset \Phi_d^s(A)\}$ is a topology called the strong density topology on the plane. Both operators Φ_d^o and Φ_d^s are lower density operators, so topologies $\mathcal{T}_d^o$ and $\mathcal{T}_d^s$ satisfy properties analogous to (a)-(g) from Theorem 22.1.

We will study properties of ordinary and strong $\psi-$density on the plane. Because the results have never been published in English we will present proofs of some theorems. More detailed information about this can be found in [5].

Suppose that $\psi \in \widehat{\mathcal{C}}$. We say that $x \in \mathbb{R}^2$ is an ordinary ψ-density point of a set $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0+} \frac{\lambda_2(A' \cap Sq(x,h))}{4h^2 \psi(4h^2)} = 0.$$

Analogously, we say that $x \in \mathbb{R}^2$ is a strong $\psi-$density point of A if

$$\lim_{\substack{h \to 0+ \\ k \to 0+}} \frac{\lambda_2(A' \cap R(x,h,k))}{4hk\psi(4hk)} = 0.$$

We say that $x \in \mathbb{R}^2$ is an ordinary (strong) ψ-dispersion point of a set A if x is an ordinary (strong) ψ-density point of A'. We denote by $\Phi^o_\psi(A)$ ($\Phi^s_\psi(A)$) the set of all ordinary (strong) ψ-density points of a set A.

Using The Third Taylor's Theorem we can easy obtain on the plane a result analogous to The First Taylor's Theorem for ordinary $\psi-$density.

Theorem 22.7. *For any $A \in \mathcal{L}_2$ there exists a function $\psi \in \widehat{\mathcal{C}}$ such that*

$$\lambda_2\left(A \setminus \Phi^o_\psi(A)\right) = 0.$$

Proof. By The Third Taylor's Theorem, there exists $\psi^* \in \widehat{\mathcal{C}}$ such that

$$\lim_{h\to 0+} \frac{1}{\psi^*\left(\sqrt{2}h\right)} \left(\frac{\lambda_2\left(Sq\left(x,h\right)\cap A\right)}{4h^2} - 1\right) = 0$$

for almost all $x \in A$. Let $\psi\left(t\right) = \psi^*\left(\sqrt{\frac{t}{2}}\right)$ for $t > 0$. Then $\psi \in \widehat{\mathcal{C}}$ and , for any $h > 0$

$$\frac{\lambda_2\left(A'\cap Sq\left(x,h\right)\right)}{\psi\left(4h^2\right)4h^2} = \frac{1}{\psi\left(4h^2\right)}\left(\frac{\lambda_2\left(Sq\left(x,h\right)\right) - \lambda_2\left(Sq\left(x,h\right)\cap A\right)}{4h^2}\right) =$$
$$= \frac{1}{\psi^*\left(\sqrt{2}h\right)}\left(1 - \frac{\lambda_2\left(Sq\left(x,h\right)\cap A\right)}{4h^2}\right).$$

□

We will prove that the analogues result for strong $\psi-$density is not valid. Observe, that The Third Taylor's Theorem refers to rectangles, but in the denominator of the formula (22.2) we have a diameter of I. We will show that - roughly speaking - we can not put $\lambda_2\left(I\right)$ instead of $d\left(I\right)$.

Lemma 22.8. *If $B \in \mathcal{L}_1$ satisfies the property*

$$\lambda\left(B \cap [-h,h]\right) > 0$$

for any $h > 0$ then $(0,0)$ is not a strong $\psi-$density point of a set $A = B' \times \mathbb{R}$ for any $\psi \in \widehat{\mathcal{C}}$.

Proof. Let $\psi \in \widehat{\mathcal{C}}$. We will show that for any $\varepsilon > 0$, $\delta > 0$ and $h \in (0,\delta)$ there is a number $k \in (0,\delta)$ such that

$$\frac{\lambda_2\left(A'\cap R\left(x,h,k\right)\right)}{4hk\cdot\psi\left(4hk\right)} > \varepsilon.$$

Fix $\varepsilon > 0$, $\delta > 0$ and $h \in (0,\delta)$. Since $\lambda(B \cap [-h,h]) > 0$, there exists a positive number

$$\alpha = \frac{\lambda(B \cap [-h,h])}{2h}.$$

A number h is fixed and $\lim_{t\to 0+}\psi(t) = 0$, so there is $k \in (0,\delta)$ such that $\psi(4hk) < \frac{\alpha}{\varepsilon}$. Therefore,

$$\frac{\lambda_2(A' \cap R(x,h,k))}{4hk \cdot \psi(4hk)} = \frac{\lambda(B \cap [-h,h]) \cdot 2k}{2h \cdot 2k \cdot \psi(4hk)} = \frac{\alpha}{\psi(4hk)} > \varepsilon.$$

□

Moreover, if $\lambda(B \cap [x_1 - h, x_1 + h]) > 0$ for any $h > 0$, then (x_1,x_2) is not a strong $\psi-$density point of a set $A = B' \times \mathbb{R}$ for any $x_2 \in \mathbb{R}$ and any $\psi \in \widehat{\mathcal{C}}$. Observe that this property of a strong $\psi-$density is rather unusual. It is easy to check that

Proposition 22.9. *If x_1 is a density point of $B \in \mathcal{L}_1$ then for any $x_2 \in \mathbb{R}$, (x_1,x_2) is an ordinary and strong density point of $B \times \mathbb{R}$.*

Proposition 22.10. *If x_1 is a $\psi-$density point of $B \in \mathcal{L}_1$, for some $\psi \in \widehat{\mathcal{C}}$ then for any $x_2 \in \mathbb{R}$, the point (x_1,x_2) is a ψ^*-ordinary density point of $B \times \mathbb{R}$, where $\psi^*(t) = \psi(t^2)$ for $t > 0$. On the other hand, if for some $\psi \in \widehat{\mathcal{C}}$ and $B \in \mathcal{L}_1$, (x_1,x_2) is a $\psi-$ordinary point of $B \times \mathbb{R}$, then x_1 is a $\widehat{\psi}-$density point of B, for $\widehat{\psi}(t) = \psi(\sqrt{t})$.*

For strong $\psi-$density we obtain a "strong strengthening" of the Second Taylor's Theorem.

Theorem 22.11. *For any $\alpha \in (0,1)$ there is a set $E \subset [0,1] \times [0,1]$ such that $\lambda_2(E) = \alpha$ and for any $\psi \in \widehat{\mathcal{C}}$ no point of E is a strong $\psi-$density point of a set E.*

Indeed, we can take a nowhere dense set $C \subset [0,1]$ of measure α and put $A = C \times [0,1]$ and use Lemma 22.8.

On the other hand, using Proposition 22.10, we can prove a theorem analogous to The Second Taylor's Theorem for ordinary $\psi-$density.

Theorem 22.12. *For each function $\psi \in \widehat{\mathcal{C}}$ and number $\alpha \in (0,1)$ there is a set $E \subset [0,1] \times [0,1]$ such that $\lambda_2(E) = \alpha$ and no point of E is its ordinary $\psi-$density point.*

Straightforward from the definitions of $\Phi^o_\psi(A)$ and $\Phi^s_\psi(A)$ we obtain (compare [23], Th.1.3):

Theorem 22.13. *For any $A, B \in \mathcal{L}_2$*

(1) if $A \subset B$ then $\Phi^o_\psi(A) \subset \Phi^o_\psi(B)$;
(2) if $A \sim B$ then $\Phi^o_\psi(A) = \Phi^o_\psi(B)$;
(3) $\Phi^o_\psi(\emptyset) = \emptyset$ and $\Phi^o_\psi(\mathbb{R}^2) = \mathbb{R}^2$;
(4) $\Phi^o_\psi(A \cap B) = \Phi^o_\psi(A) \cap \Phi^o_\psi(B)$;
(5) $\Phi^o_\psi(A) \subset \Phi^o_d(A)$.

The same properties are satisfied for the operator Φ^s_ψ.

Put
$$\mathcal{T}^o_\psi = \{A \in \mathcal{L}_2 : A \subset \Phi^o_\psi(A)\}$$
and
$$\mathcal{T}^s_\psi = \{A \in \mathcal{L}_2 : A \subset \Phi^s_\psi(A)\}.$$

Theorem 22.14. *Let $\psi \in \widehat{\mathcal{C}}$. The families $\mathcal{T}^o_\psi$ and $\mathcal{T}^s_\psi$ form topologies on the plane, stronger than the Euclidean topology $\mathcal{T}^2_{nat}$ and weaker then the ordinary density topology $\mathcal{T}^o_d$. Moreover $\mathcal{T}^s_\psi \subsetneq \mathcal{T}^o_\psi \cap \mathcal{T}^s_d$.*

Proof. Inclusions $\mathcal{T}^2_{nat} \subset \mathcal{T}^s_\psi \subset \mathcal{T}^o_\psi \subset \mathcal{T}^o_d$ follow immediately from the definitions. To prove that $\mathcal{T}^o_\psi$ ($\mathcal{T}^s_\psi$) is a topology, it is enough to observe that the union of an arbitrary subfamily of $\mathcal{T}^o_\psi$ ($\mathcal{T}^s_\psi$) belongs to $\mathcal{T}^o_\psi$ ($\mathcal{T}^s_\psi$). The only difficulty is to show that it is a measurable set. It is true because $\mathcal{T}^s_\psi \subset \mathcal{T}^o_\psi \subset \mathcal{T}^o_d$ and $\mathcal{T}^o_d$ is closed under arbitrary unions, and $\mathcal{T}^o_d \subset \mathcal{L}_2$.

Let $\widehat{\psi}(t) = \psi\left(\sqrt{t}\right)$ and $A = \bigcup\limits_{n=1}^{\infty} (a_n, b_n)$, $0 < b_{n+1} < a_n < b_n$ for $n = 1, 2, \ldots$ be an interval set such that 0 is a right $\widehat{\psi}-$density point of A (for example $a_n = \frac{1}{2^{n+1}} + \frac{1}{4^{n+1}} \cdot \widehat{\psi}(\frac{1}{2^n})$ and $b_n = \frac{1}{2^n})$. Clearly, the set $B = -A \cup \{0\} \cup A$ belongs to $\mathcal{T}_{\widehat{\psi}}$ and $\mathcal{T}_d$. Therefore, the set $B \times \mathbb{R}$, by Proposition 22.9, belongs to $\mathcal{T}^s_d$, and - by Proposition 22.10 - belongs to $\mathcal{T}^o_\psi$. However, from Lemma 22.8 it follows that $(0,0)$ is not a strong $\psi-$density of $B \times \mathbb{R}$, so $B \times \mathbb{R} \notin \mathcal{T}^s_\psi$.

The set $\mathbb{R}^2 \setminus (\mathbb{Q} \times \mathbb{Q}) \in \mathcal{T}^s_\psi \setminus \mathcal{T}^2_{nat}$. Finally, we will define a set D such that $D' \in \mathcal{T}^o_d \setminus \mathcal{T}^o_\psi$. There is $n_0 \in \mathbb{N}$ such that $\psi(\frac{1}{2^{n_0}}) \leqslant 1$. Let

$$D = \bigcup_{n=n_0}^{\infty} \left[\frac{1}{2^n} - \frac{1}{2^n}\psi(\frac{1}{4^{n-1}}), \frac{1}{2^n}\right] \times \mathbb{R}.$$

For simplicity we write $Sq(h)$ instead of $Sq((0,0),h)$. For each $h \in \left[\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$

$$\frac{\lambda_2(D \cap Sq(h))}{4h^2} \leqslant \frac{\lambda_2\left(D \cap Sq(\frac{1}{2^n})\right)}{4\frac{1}{(2^{n+1})^2}} \leqslant \frac{\frac{2}{2^n}\psi\left(\frac{1}{4^{n-1}}\right)\frac{2}{2^n}}{\frac{1}{4^n}} = 4\psi\left(\frac{1}{4^{n-1}}\right).$$

Since $n \to \infty$ when $h \to 0+$, the point $(0,0)$ is an ordinary density point of D'. On the other hand, for any $n \geqslant n_0$,

$$\frac{\lambda_2(D \cap Sq(\frac{1}{2^n}))}{4(\frac{1}{2^n})^2 \psi(4(\frac{1}{2^n})^2)} \geqslant \frac{\lambda_2([\frac{1}{2^n} - \frac{1}{2^n}\psi(\frac{1}{4^{n-1}}), \frac{1}{2^n}] \times [-\frac{1}{2^n} \cdot \frac{1}{2^n}])}{4\frac{1}{4^n}\psi(\frac{1}{4^{n-1}})} = \frac{1}{2}$$

and $(0,0)$ is not an ordinary ψ-density point of D'. The set $D' \setminus \{(0,0)\} \in \mathcal{T}^2_{nat}$, so $D' \in \mathcal{T}^o_d \setminus \mathcal{T}^o_\psi$. □

For any function $\psi \in \widehat{\mathcal{C}}$ the spaces $(\mathbb{R}, \mathcal{T}^o_\psi)$ and $(\mathbb{R}, \mathcal{T}^s_\psi)$ are Hausdorff and not separable. Any set of two-dimensional measure zero is closed. Each compact subspace of $(\mathbb{R}, \mathcal{T}^o_\psi)$ or $(\mathbb{R}, \mathcal{T}^s_\psi)$ is finite. Moreover

Theorem 22.15. *Let $\psi \in \widehat{\mathcal{C}}$. Then*

$$\bigcap\{\mathcal{T}^s_\psi \colon \psi \in \widehat{\mathcal{C}}\} = \bigcap\{\mathcal{T}^o_\psi \colon \psi \in \widehat{\mathcal{C}}\} = \{U \setminus P \colon U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\}.$$

Proof. It is not difficult to check that a measurable set A belongs to the family $\left\{U \setminus P : U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\right\}$ if and only if

$$\forall (x \in A) \exists (\delta_x > 0) \forall \left(h, k \in (0, \delta_x) \quad \lambda_2\left(A' \cap R(x,h,k\right) = 0\right).$$

Therefore, if $A \in \left\{U \setminus P : U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\right\}$, then $A \in \mathcal{T}^s_\psi$, for any $\psi \in \widehat{\mathcal{C}}$. Hence $\left\{U \setminus P : U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\right\} \subset \bigcap\{\mathcal{T}^s_\psi \colon \psi \in \widehat{\mathcal{C}}\}$.

Suppose, that $A \notin \left\{U \setminus P : U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\right\}$. Therefore, there is a point $x \in A$ such that

$$\lambda_2\left(A' \cap Sq\left(x, \frac{1}{n}\right)\right) > 0$$

for any $n \in \mathbb{N}$. The sequence $\left\{\lambda_2\left(A' \cap Sq\left(x, \frac{1}{n}\right)\right)\right\}$ is decreasing and tends to 0. There exists a function $\psi \in \widehat{\mathcal{C}}$, such that

$$\psi\left(\frac{4}{n^2}\right) = \lambda_2\left(A' \cap Sq\left(x, \frac{1}{n}\right)\right).$$

Since

$$\frac{\lambda_2\left(A' \cap Sq\left(x, \frac{1}{n}\right)\right)}{\frac{4}{n^2}\psi\left(\frac{4}{n^2}\right)} = \frac{n^2}{4},$$

$A \notin \mathcal{T}^o_\psi \supset \mathcal{T}^s_\psi$. Therefore,

$$\bigcap\{\mathcal{T}^s_\psi \colon \psi \in \widehat{\mathcal{C}}\} \subset \bigcap\{\mathcal{T}^o_\psi \colon \psi \in \widehat{\mathcal{C}}\} \subset \left\{U \setminus P : U \in \mathcal{T}^2_{nat} \wedge \lambda_2(P) = 0\right\}.$$

□

Recall that operators Φ_d^o and Φ_d^s are lower density operators, so $\left(\mathbb{R}^2, \mathcal{T}_d^o\right)$ and $\left(\mathbb{R}^2, \mathcal{T}_d^s\right)$ are Baire spaces.

Theorem 22.16. *The plane is a first category set in* $\left(\mathbb{R}^2, \mathcal{T}_\psi^s\right)$ *and in* $\left(\mathbb{R}^2, \mathcal{T}_\psi^o\right)$ *for any* $\psi \in \widehat{\mathcal{C}}$.

Proof. Let $(C_n)_{n\in\mathbb{N}}$ be a sequence of Cantor-type sets on the real line of positive one-dimensional Lebesgue measure, such that $\lambda\left(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} C_n\right) = 0$. Fix a function $\psi \in \widehat{\mathcal{C}}$. All sets $C_n \times \mathbb{R}$ are $\mathcal{T}_\psi^s-$closed sets and, by Lemma 22.8, $\Phi_\psi^s (C_n \times \mathbb{R}) = \emptyset$. Therefore, there are $\mathcal{T}_\psi^s-$nowhere dense sets. The set $(\mathbb{R} \setminus \bigcup_{n=1}^{\infty} C_n) \times \mathbb{R}$ is $\mathcal{T}_\psi^s-$nowhere dense because it is a set of measure zero.

Note that, by The First Taylor's Theorem, there are functions $\psi \in \widehat{\mathcal{C}}$ such that sets $C_n \times \mathbb{R}$ have a nonempty $\mathcal{T}_\psi^o-$interior. In the next part of the proof we will divide the plane differently for different ψ.

Let $\psi \in \widehat{\mathcal{C}}$ and $\psi^*(t) = \psi\left(t^2\right)$. There exist $\mathcal{T}_{\psi^*}-$closed and $\mathcal{T}_{\psi^*}-$nowhere dense sets $E_n \subset \mathbb{R}$ such that $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$ (Theorem 22.3). Therefore, $\mathrm{int}_{\mathcal{T}_{\psi^*}}(E_n) = \emptyset$. The sets $E_n \times \mathbb{R}$ are $\mathcal{T}_\psi^o-$closed and $\mathrm{int}_{\mathcal{T}_\psi^o}(E_n \times \mathbb{R}) = \emptyset$. □

Theorem 22.17. *The space* $\left(\mathbb{R}^2, \mathcal{T}_\psi^s\right)$ *is not regular for any* $\psi \in \widehat{\mathcal{C}}$.

Proof. As in the first part of the proof of Theorem 22.16 we can show an "universal" closed set and a point which can not be separated by $\mathcal{T}_\psi^s-$open sets for any $\psi \in \widehat{\mathcal{C}}$. Fix a point $(x_0, y_0) \in \mathbb{R}^2$. In [11], Th. 5 it is proved that the set $L = \{(x_0, y) : y \in \mathbb{R}\} \setminus \{(x_0, y_0)\}$ can not be separated from this point by $\mathcal{T}_d^s-$open sets.

For every $\psi \in \widehat{\mathcal{C}}$, the set L is $\mathcal{T}_\psi^s-$closed, as a null set. Suppose that there are disjoint sets U and V, open in $\left(\mathbb{R}^2, \mathcal{T}_\psi^s\right)$ such that $(x_0, y_0) \in U$ and $L \subset V$. Then U and V belong to $\mathcal{T}_d^s$ and separate (x_0, y_0) from L, which gives a contradiction. □

A proof of the analogous property for ordinary $\psi-$density is more complicated.

Theorem 22.18. *The space* $\left(\mathbb{R}^2, \mathcal{T}_\psi^o\right)$ *is not regular for any* $\psi \in \widehat{\mathcal{C}}$.

Proof. Let $\psi \in \widehat{\mathcal{C}}$ and $\psi^*(t) = \psi\left(t^2\right)$. There exists an increasing sequence $(E_n)_{n\in\mathbb{N}}$ of closed, $\mathcal{T}_{\psi^*}-$nowhere dense subsets of $[0,1]$ such that

$$\lambda\left([0,1] \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0$$

(compare [4]). Let $E = \bigcup_{n=1}^{\infty} E_n$ and $A = E \times \mathbb{R}$. Observe that, for any $n \in \mathbb{N}$, $E_n \times \mathbb{R}$ is $\mathcal{T}_\psi^o-$closed and $\mathcal{T}_\psi^o-$boundary set, and $A \cap ((0,1) \times \mathbb{R}) \in \mathcal{T}_\psi^o$.

Suppose that $\left(\mathbb{R}^2, \mathcal{T}_\psi^o\right)$ is regular. Fix $z = (z_1, z_2) \in A \cap ((0,1) \times \mathbb{R})$. Then there exists a set $U \in \mathcal{T}_\psi^o$ such that $z \in U$ and $Cl_{\mathcal{T}_\psi^o}(U) \subset A \cap ((0,1) \times \mathbb{R})$. We will find a sequence $\left(x^{(n)}\right)_{n \in \mathbb{N}}$ of elements of the set U such that $x = \lim_{n \to \infty} x^{(n)} \notin A$ and

$$\limsup_{n \to \infty} \frac{\lambda_2 (U \cap Sq(x,h))}{4h^2} > 0.$$

Since $\operatorname{int}_{\mathcal{T}_\psi^o}(E_1 \times \mathbb{R}) = \emptyset$ and $\operatorname{int}_{\mathcal{T}_\psi^o}(U) \neq \emptyset$, the set $U \setminus (E_1 \times \mathbb{R})$ has positive measure, so we can choose a point $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}\right) \in U \setminus (E_1 \times \mathbb{R})$. The sequence $(E_n)_{n \in \mathbb{N}}$ is increasing and $x_1^{(1)} \in E \setminus E_1$, so there is $k_1 > 1$ such that $x_1^{(1)} \in E_{k_1} \setminus E_{k_1 - 1}$. Therefore, $x_1^{(1)}$ belongs to some component J of $\mathbb{R} \setminus E_{k_1}$ and J is open. Denote it by (a_1, b_1). Let

$$\varepsilon_1 = \min\left(x_1^{(1)} - a_1, b_1 - x_1^{(1)}\right).$$

The set U belongs to $\mathcal{T}_\psi^o$ and $x^{(1)} \in U$. Therefore, $x^{(1)} \in \Phi_\psi^o(U) \subset \Phi_d^o(U)$ and there exists a number $r_1 \in (0, \varepsilon_1)$ such that

$$\frac{\lambda_2\left(U \cap Sq\left(x^{(1)}, r_1\right)\right)}{4r_1^2} \geq \frac{3}{4}.$$

It is not difficult to check, that for any $y \in Sq\left(x^{(1)}, \frac{1}{4}r_1\right)$

$$\frac{\lambda_2\left(U \cap Sq\left(y, \frac{3}{4}r_1\right)\right)}{4\left(\frac{3}{4}r_1\right)^2} \geq \frac{1}{4}.$$

Moreover, for any $y \in Sq\left(x^{(1)}, \frac{1}{4}r_1\right)$ the distance between y and $E_{k_1-1} \times \mathbb{R}$ (and between y and $E_i \times \mathbb{R}$ for $i < k_1 - 1$) is greater then $\frac{3}{4}r_1$.

The set

$$U_1 = U \cap \left(\left(x_1^{(1)} - \frac{1}{4}r_1, x_1^{(1)} + \frac{1}{4}r_1\right) \times \left(x_2^{(1)} - \frac{1}{4}r_1, x_2^{(1)} + \frac{1}{4}r_1\right)\right)$$

belongs to $\mathcal{T}_\psi^o$. Since $\operatorname{int}_{\mathcal{T}_\psi^o}(E_{k_1} \times \mathbb{R}) = \emptyset$, there is a point $x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}\right) \in U \setminus (E_1 \times \mathbb{R})$. Therefore, there is $k_2 > k_1$ such that $x_1^{(2)} \in E_{k_2} \setminus E_{k_2 - 1}$. Denote by (a_2, b_2) the component of $\mathbb{R} \setminus E_{k_2}$ such that $x_1^{(2)} \in (a_2, b_2)$ and by ε_2 the minimum of numbers $x_1^{(2)} - a_1$, $b_1 - x_1^{(2)}$ and $\frac{1}{4}r_1$. Let $r_2 \in (0, \varepsilon_2)$ be such a number that

$$\frac{\lambda_2\left(U \cap Sq\left(x^{(2)}, r_2\right)\right)}{4r_2^2} \geq \frac{3}{4}.$$

We now proceed by induction and find a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}$ of elements of U, a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers and a decreasing sequence $(r_n)_{n\in\mathbb{N}}$ tending to zero such that

$$x^{(n)} \in U \cap E_{k_n} \cap Sq\left(x^{(n-1)}, \frac{1}{4}r_n\right), \tag{22.3}$$

$$\frac{\lambda_2\left(U \cap Sq\left(x^{(n)}, r_n\right)\right)}{4r_n^2} \geq \frac{3}{4}$$

for any $n > 1$ and

$$dist\left(x^{(n)}, E_i \times \mathbb{R}\right) \geq \frac{3}{4}r_n \tag{22.4}$$

for any $i < k_n$.

From (22.3) we know that the sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}}$ is convergent. By (22.4), $x = \lim_{n\to\infty} x^{(n)} \notin A$. Finally, for any $n \in \mathbb{N}$, $x \in Sq\left(x^{(n)}, \frac{1}{4}r_n\right)$ and consequently

$$\frac{\lambda_2\left(U \cap Sq\left(x, \frac{3}{4}r_n\right)\right)}{4\left(\frac{3}{4}r_n\right)^2} \geq \frac{1}{4}.$$

It follows that

$$\limsup_{n\to\infty} \frac{\lambda_2\left(U \cap Sq\left(x, h\right)\right)}{4h^2} \geq \frac{1}{4}.$$

and $x \in Cl_{\mathcal{T}_d^o}(U)$. Since $Cl_{\mathcal{T}_d^o}(U) \subset Cl_{\mathcal{T}_\psi^o}(U)$, the set $Cl_{\mathcal{T}_\psi^o}(U)$ is not a subset of $A \cap ((0,1) \times \mathbb{R})$. This contradiction proves that the space $\left(\mathbb{R}^2, \mathcal{T}_\psi^o\right)$ is not regular. $\square$

From the definitions of operators and topologies $\mathcal{T}_\psi^o$ and $\mathcal{T}_\psi^s$ it follows that, for any function $\psi \in \widehat{\mathcal{C}}$, if $A \in \mathcal{T}_\psi^o$ and $x \in \mathbb{R}$ then $x + A = \{x + a : a \in A\} \in \mathcal{T}_\psi^o$ (if $A \in \mathcal{T}_\psi^s$, then $x + A \in \mathcal{T}_\psi^s$).

As it would be expected, invariance under multiplications is connected with (Δ_2) condition. Observe first that:

Lemma 22.19. *Let $\psi \in \widehat{\mathcal{C}}$. If there is $\alpha_0 > 1$ such that* $\limsup_{t\to 0+} \frac{\psi(\alpha_0 t)}{\psi(t)} = \infty$ *then* $\limsup_{t\to 0+} \frac{\psi(\alpha t)}{\psi(t)} = \infty$ *for any $\alpha > 1$.*

Theorem 22.20. *Let ψ be a function from the family $\widehat{\mathcal{C}}$.*

1. *If* $A \in \mathcal{T}_\psi^o$, $\alpha > 0$ *and* $\limsup_{t\to 0+} \frac{\psi(t)}{\psi(\alpha^2 t)} < \infty$ *then* $\alpha A = \{\alpha a : a \in A\} \in \mathcal{T}_\psi^o$.
2. *If* $\alpha > 0$ *and* $\limsup_{t\to 0+} \frac{\psi(t)}{\psi(\alpha^2 t)} = \infty$ *then there is such a set* B *that* $\alpha B \notin \mathcal{T}_\psi^o$.

Proof. Suppose that $A \in \mathcal{T}_\psi^o$. To prove the first condition of the theorem it is enough to show that for any $x \in A$, αx is an ordinary ψ-density point of αA. Let $x \in A$. The point $(0,0)$ is an ordinary ψ-density point of a set $A - x = \{a - x : a \in A\}$. For any $h > 0$

$$(\alpha(A-x))' \cap Sq(h) = \left(\alpha(A-x)'\right) \cap Sq(h) = \alpha\left((A-x)' \cap Sq\left(\frac{h}{\alpha}\right)\right).$$

Therefore,

$$\begin{aligned}&\limsup_{h\to 0+} \frac{\lambda_2((\alpha(A-x))' \cap Sq(h))}{4h^2\psi(4h^2)} \leqslant \\ &\leqslant \frac{1}{\alpha}\lim_{h\to 0+} \frac{\lambda_2((A-x)' \cap Sq(\frac{h}{\alpha}))}{4(\frac{h}{\alpha})^2\psi(4(\frac{h}{\alpha})^2)} \cdot \limsup_{t\to 0+} \frac{\psi(t)}{\psi(\alpha^2 t)} = 0\end{aligned}$$

what means that $(0,0)$ is an ordinary ψ-density point of a set $\alpha A - \alpha x$.

In the second part of the proof we will use a construction from the real line. Let $\psi^*(t) = \psi(t^2)$ for $t > 0$. Hence

$$\limsup_{t\to 0+} \frac{\psi^*(t)}{\psi^*(\alpha t)} = \limsup_{t\to 0+} \frac{\psi(t)}{\psi(\alpha^2 t)} = \infty.$$

Repeating the proof of Theorem 2.8 from [23] we can construct an interval set $A = \bigcup_{n=1}^\infty [a_n, b_n]$ with $0 < b_{n+1} < a_n < b_n$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = 0$ such that 0 is a ψ^*-dispersion point of A and is not a ψ^*-dispersion point of a set αA. It is easy to check that $A' \in \mathcal{T}_{\psi^*}$ and $(\alpha A)' \notin \mathcal{T}_{\psi^*}$. Therefore, the set $A' \times \mathbb{R}$ is open in topology $\mathcal{T}_\psi^o$ but the set $(\alpha A)' \times \mathbb{R} = \alpha \cdot (A' \times \mathbb{R})$ is not. □

Corollary 22.21. *An ordinary* ψ*−density topology is invariant under multiplication by positive numbers if and only if* $\psi \in \Delta_2$.

One of the clearest differences between $\mathcal{T}_d^o$ and $\mathcal{T}_d^s$ is connected with rotations. It is well known that the ordinary density topology on the plane is invariant under rotations and the strong density topology is not. It can be surprising, that invariance $\mathcal{T}_\psi^o$ under rotations again depends on the (Δ_2) condition.

Theorem 22.22. *Suppose that* $\psi \in \widehat{\mathcal{C}}$. *If* $\psi \in \Delta_2$ *then, for any set* $A \in \mathcal{T}_\psi^o$, *the set* B *received from* A *by turning around a fixed point, belongs to topology* $\mathcal{T}_\psi^o$.

Proof. We will show that if the point (0,0) is an ordinary ψ-dispersion point of a set A then (0,0) is an ordinary ψ-dispersion point of the set A^θ received from A by turning around (0,0) of an angle $\theta \in (0, 2\pi)$.

Let $Sq^\theta(h)$ denotes a square $Sq(h)$ turned around (0,0) of θ. Because $\lambda_2(A^\theta \cap Sq(h)) = \lambda_2(A \cap Sq^{-\theta}(h)) \leqslant \lambda_2(A \cap Sq(h\sqrt{2})$, we obtain

$$\frac{\lambda_2(A^\theta \cap Sq(h))}{4h^2\psi(4h^2)} \leqslant \frac{\lambda_2(A \cap Sq(h\sqrt{2})}{4h^2\psi(4h^2)} = \frac{\lambda_2(A \cap Sq(h\sqrt{2})}{\frac{1}{2}4(h\sqrt{2})^2\psi(4(h\sqrt{2})^2)} \cdot \frac{\psi(2\cdot 4h^2)}{\psi(4h^2)}.$$

From the assumption we know that $\limsup_{t\to 0+} \frac{\psi(2\cdot 4h^2)}{\psi(4h^2)} < \infty$. Therefore,

$$\limsup_{h\to 0+} \frac{\lambda_2(A^\theta \cap Sq(h))}{4h^2\psi(4h^2)} = 0.$$

Fix a point $s = (s_1, s_2) \in \mathbb{R}^2$ and an angle $\theta \in (0, 2\pi)$. We will show that if x is an ordinary ψ-dispersion point of a set A then the point y, received from x by rotating x around the point s of θ, is an ordinary ψ-dispersion point of the set B received from A by the same rotate. Suppose that

$$\lim_{h\to 0+} \frac{\lambda_2(A' \cap Sq(x,h))}{4h^2\psi(4h^2)} = 0.$$

Denote by $Sq^*(y,h)$ a square received from $Sq(y,h)$ by rotating of an angle $-\theta$ around s. Thus, like in previous case, $\lambda_2(B' \cap Sq(y,h)) = \lambda_2(A' \cap Sq^*(y,h)) \leq \lambda_2(A' \cap Sq(x, \sqrt{2}))$ and

$$\lim_{h\to 0+} \frac{\lambda_2(B' \cap Sq(y,h))}{4h^2\psi(4h^2)} = 0.$$

□

Theorem 22.23. *Suppose that $\psi \in \widehat{\mathcal{C}}$. If $\psi \notin \Delta_2$ then, for any angle $\theta \in (0, \frac{\pi}{4}]$, there exists a set $A \in \mathcal{L}_2$ such that $(0,0)$ is not an ordinary ψ-dispersion point of a set A, but is an ordinary ψ-dispersion point of a set A rotated of an angle $-\theta$ around the point $(0,0)$.*

Proof. Fix an angle $\theta \in (0, \frac{\pi}{4}]$. Since $\cos(\frac{\pi}{4} - \theta) > \frac{\sqrt{2}}{2}$, a number $2\cos^2(\frac{\pi}{4} - \theta)$ is greater then one. Let $\alpha \in (1, 2\cos^2(\frac{\pi}{4} - \theta))$. We know that $\limsup_{t\to 0+} \frac{\psi(\alpha t)}{\psi(t)} = \infty$. Therefore, there is a sequence $(t_n)_{n\in\mathbb{N}} \searrow 0$ such that

$$\lim_{n\to\infty} \frac{\psi(\alpha t_n)}{\psi(t_n)} = \infty. \tag{22.5}$$

We will construct a sequence $(d_n)_{n\in\mathbb{N}} \searrow 0$ such that $\psi(4d_1) < \frac{1}{4}$ and

$$\psi(\alpha \cdot 4d_n^2) > n \cdot \psi(4d_n^2), \tag{22.6}$$

$$\sqrt{\psi(4d_n^2)} < \sqrt{2} \cdot \cos(\frac{\pi}{4} - \theta) - \sqrt{\alpha}, \tag{22.7}$$

$$d_{n+1} \leqslant d_n \cdot \sqrt{\psi(4d_n^2)} \tag{22.8}$$

for any $n \in \mathbb{N}$. Let $b_n = \frac{\sqrt{t_n}}{2}$. From (22.5) it follows that

$$\lim_{n\to\infty} \frac{\psi(\alpha \cdot 4b_n^2)}{\psi(4b_n^2)} = \infty.$$

Thus, we can choose a subsequence $(c_n)_{n\in\mathbb{N}}$ of a sequence $(b_n)_{n\in\mathbb{N}}$, such that $(c_n)_{n\in\mathbb{N}}$ satisfies (22.6). Since $\lim_{n\to\infty} c_n = 0$, almost all c_n satisfy (22.7). We can choose a subsequence $(d_n)_{n\in\mathbb{N}}$ such that (22.8) is true and $\psi(4d_1) < \frac{1}{4}$. Let

$$h_i = d_i \cdot \sqrt{\psi(4d_i^2)} \tag{22.9}$$

and denote by A_i a triangle with vertices (d_i, d_i), $(d_i, d_i - h_i)$ and $(d_i - h_i, d_i)$. We define

$$A = \bigcup_{i=1}^{\infty} A_i.$$

Observe that from (22.8) and (22.9) it follows

$$\lambda_2(\bigcup_{k=i}^{\infty} A_k) = \lambda_2(A_i) + \lambda_2(\bigcup_{k=i+1}^{\infty} A_k) \leqslant \lambda_2(A_i) + \lambda_2([0, d_{i+1}]^2) = \tag{22.10}$$

$$= \lambda_2(A_i) + d_{i+1}^2 \leqslant \lambda_2(A_i) + h_i^2 = 3\lambda_2(A_i),$$

for any $i \in \mathbb{N}$. It is obvious that $(0,0)$ is not an ordinary ψ-dispersion point of a set A because, for each i,

$$\frac{\lambda_2(A \cap Sq(d_i))}{4d_i^2 \psi(4d_i^2)} \geqslant \frac{\frac{1}{2}h_i^2}{4d_i^2 \psi(4d_i^2)} = \frac{1}{8}.$$

We will prove that $(0,0)$ is an ordinary ψ-dispersion point of a set B, received from A by turning by $-\theta$ around $(0,0)$. Notice, that $\lambda_2(B \cap Sq(\sqrt{2}d_i)) = \lambda_2(A \cap Sq(d_i))$ for any $i \in \mathbb{N}$ and $\theta \in (0, \frac{\pi}{4}]$. Let t be an arbitrary point of $(0, d_1]$. There is $i \in \mathbb{N}$ such that $t \in (\sqrt{2}d_{i+1}, \sqrt{2}d_i]$. We will consider two cases:

1^0. If $t \in [\sqrt{\alpha}d_i, \sqrt{2}d_i]$ then from (22.10) and (22.9) it follows

$$\frac{\lambda_2(B\cap Sq(t))}{4t^2\psi(4t^2)} \leqslant \frac{\lambda_2(B\cap Sq(\sqrt{2}d_i))}{4\alpha d_i^2\psi(4\alpha d_i^2)} = \frac{\lambda_2(A\cap Sq(d_i))}{4\alpha d_i^2\psi(4\alpha d_i^2)} \leqslant \frac{3\lambda_2(A_i)}{4\alpha d_i^2\psi(4\alpha d_i^2)} \leqslant$$
$$\leqslant \frac{3\cdot\frac{1}{2}d_i^2\psi(4d_i^2)}{4\alpha d_i^2 i\psi(4d_i^2)} = \frac{3}{8\alpha i};$$

2^0. Let $t\in(\sqrt{2}d_{i+1},\sqrt{\alpha}d_i)$. Note that

$$\{(d_i,d_i)\}^\theta = \left\{\left(\sqrt{2}d_i\cos\left(\frac{\pi}{4}+\theta\right),\sqrt{2}d_i\cos\left(\frac{\pi}{4}-\theta\right)\right)\right\}.$$

By (22.7) we have

$$\sqrt{2}d_i\cdot\cos(\frac{\pi}{4}-\theta)-h_i = d_i\left(\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)-\sqrt{\psi(4d_i^2)}\right) > d_i\sqrt{\alpha} > d_i,$$

so $A_i^\theta\cap([0,d_i]\times[0,d_i])=\emptyset$. Therefore, $B\cap Sq(t)=B\cap Sq(\sqrt{2}d_{i+1})$. Reminding that $\alpha<2$, we obtain from (22.9) and (22.6)

$$\frac{\lambda_2(B\cap Sq(t))}{4t^2\psi(4t^2)} \leqslant \frac{\lambda_2(B\cap Sq(\sqrt{2}d_{i+1}))}{4\cdot 2d_{i+1}^2\psi(4\cdot 2d_{i+1}^2)} = \frac{\lambda_2(A\cap Sq(d_{i+1}))}{8\cdot d_{i+1}^2\psi(4\cdot 2d_{i+1}^2)} \leqslant$$
$$\leqslant \frac{3\lambda_2(A_{i+1})}{8\cdot d_{i+1}^2\psi(\alpha\cdot 4d_{i+1}^2)} \leqslant \frac{3}{16(i+1)}.$$

Since $i\to\infty$ when $t\to 0+$, it follows that $(0,0)$ is an ordinary ψ-dispersion point of a set B.

□

As we can expect, a strong ψ-density topology is not invariant under rotation, either. Firstly, we construct a set $C\in\mathcal{L}_2$ such that $(0,0)$ is a strong ψ-dispersion point of this set and $\lambda_2(C\cap Sq(r))>0$ for any $r>0$.

Example 22.24. Suppose that $\psi\in\widehat{\mathcal{C}}$, $(b_n)_{n\in\mathbb{N}}$ is a decreasing sequence tending to 0 and $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive numbers with

$$a_{n+1}\leq\frac{1}{\sqrt{2}}a_n$$

for $n\in\mathbb{N}$. Let, for any $n\in\mathbb{N}$,

$$c_n = a_{n+1}\cdot b_n\cdot\sqrt{\psi\left(4a_{n+1}^2\right)},$$

$$C_n = [a_n-c_n,a_n]\times[a_n-c_n,a_n]$$

and

$$C = \bigcup_{n=1}^{\infty} C_n.$$

Obviously, $\lambda_2(C \cap Sq(r)) > 0$ for any $r > 0$ and $\lambda_2(C_{n+1}) \leq \frac{1}{2}\lambda_2(C_n)$ for any $n \in \mathbb{N}$. Moreover, there is $n_0 \in \mathbb{N}$ such that $\psi\left(4a_{n+1}^2\right) < 1$ and $b_n < \sqrt{2} - 1$ for $n > n_0$. Therefore, for $n > n_0$,

$$a_n - c_n \geq \sqrt{2}a_{n+1} - a_{n+1}b_n\sqrt{\psi\left(4a_{n+1}^2\right)} > \sqrt{2}a_{n+1} - a_{n+1}b_n > a_{n+1},$$

so squares C_n and C_{n+1} are disjoint.

We will check that $(0,0)$ is a strong ψ-dispersion point of C. Fix arbitrary $k, h \in (0, a_{n_0}]$. There is $n \geq n_0$ such that $\min(k,h) \in (a_{n+1}, a_n]$. From the construction of the set C it follows that

$$C \cap R((0,0),h,k) \subset C \cap ([0,a_n] \times [0,a_n]).$$

Hence

$$\frac{\lambda_2(C \cap R((0,0),h,k))}{4hk \cdot \psi(4hk)} \leq \frac{c_n^2}{4a_{n+1}^2 \cdot \psi\left(4a_{n+1}^2\right)} = \frac{b_n^2}{2}.$$

Now we will use a lemma which is, in fact, the strengthening of Lemma 22.8.

Lemma 22.25. *Suppose that $A \in \mathcal{L}_2$. If there exists a sequence $(a_n)_{n\in\mathbb{N}} \searrow 0$ such that all points $(a_n, 0)$ belong to an interior of A (in the natural topology on the plane) then $(0,0)$ is not a strong ψ-dispersion point of A for any $\psi \in \widehat{\mathcal{C}}$.*

Proof. Fix a function $\psi \in \widehat{\mathcal{C}}$ and $n \in \mathbb{N}$. There exists a positive number $\delta_n < \min\{a_n, 1\}$, such that $Sq((a_n,0),\delta_n) \subset A$. Hence, for any $k \in (0,\delta_n)$,

$$\lambda_2(A \cap R((0,0),a_n,k)) \geq 2k\delta_n.$$

Since $\lim_{t\to 0+} \psi(t) = 0$, there is $\varepsilon_n \in (0,\delta_n)$ such that $\psi(4a_nk) < \frac{\delta_n}{a_n}$ for any $k \in (0,\varepsilon_n]$. Therefore,

$$\frac{\lambda_2(A \cap R((0,0),a_n,k))}{4a_nk \cdot \psi(4a_nk)} \geq \frac{2k\delta_n}{4a_nk \cdot \psi(4a_nk)} > \frac{1}{2}.$$

Let $k_1 = \varepsilon_1$ and $k_n = \min\left\{\varepsilon_n, \frac{1}{2}k_{n-1}\right\}$ for $n \geq 2$. Then

$$\limsup_{n\to\infty} \frac{\lambda_2(A \cap R((0,0),a_n,k_n))}{4a_nk_n \cdot \psi(4a_nk_n)} \geq \frac{1}{2}$$

and, consequently, $(0,0)$ is not a strong ψ-dispersion point of A. $\square$

From the latter example and lemma it follows that $(0,0)$ it is not a strong ψ-dispersion point of the set C rotated of $\frac{\pi}{4}$. Therefore, $\mathbb{R}^2 \setminus C \in \mathcal{T}_\psi^s$ and $\mathbb{R}^2 \setminus C^{\frac{\pi}{4}} \notin \mathcal{T}_\psi^s$.

Modifying a bit the construction in Example 22.24 we can construct, for any angle $\theta \in \left(0, \frac{\pi}{4}\right)$, the set $D \in \mathcal{L}_2$ such that $(0,0)$ it is a strong ψ-dispersion point of D and $(0,0)$ it is not a strong ψ-dispersion point of the set D rotated of $-\theta$ (compare [5], Example 2.12).

Corollary 22.26. *For any $\psi \in \widehat{\mathcal{C}}$ the strong ψ-density topology on the plane is not invariant under rotation.*

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