http://dx.doi.org/10.18778/7525-971-1.19

Chapter 19
Stability Aspects of the Jensen-Hosszú Equation

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2010 Mathematics Subject Classification: 39B22, 39B52.
Key words and phrases: general solutions of functional equations, Jensen and Hosszú functional equations, Hyers-Ulam stability.

19.1 Introduction

The main motivation for the study of the subject of the stability is due to Ulam (cf. [13]). In 1940, Ulam presented some unsolved problems and among them posed the following question. Let G_1 be a group and G_2 be a group with a metric d. Given a real number $\delta > 0$, does there exist $\varepsilon > 0$ such that if a map $\varphi : G_1 \to G_2$ satisfies

$$d(\varphi(xy), \varphi(x)\varphi(y)) \leq \varepsilon$$

for all $x, y \in G_1$, then there exists a homomorphism $\Phi : G_1 \to G_2$ such that

$$d(\varphi(x), \Phi(x)) \leq \delta$$

for all $x \in G_1$? If the answer to this question is "yes" then we say that the equation

$$d(\varphi(xy), \varphi(x)\varphi(y)) = 0, \qquad x, y \in G_1$$

is stable. Since then several, not necessarily equivalent, definitions of the stability have appeared. An extensive survey concerning this topic may be found in an excellent paper [10] by Z. Moszner. In this article our consideration are devoted to the following functional equation

$$g(x+y-xy)+h(xy)=2f\left(\frac{x+y}{2}\right), \qquad x,y\in E \tag{19.1}$$

in the class of functions $f,g,h: E\to X$, where E is the set of all reals or the closed unit real interval and X is a real Banach space. We choose the following definition of the stability: given $\varepsilon\geq 0$ and let $F,G,H: E\to X$ be functions satisfying the inequality

$$\left\|2F\left(\frac{x+y}{2}\right)-G(x+y-xy)-H(xy)\right\|\leq\varepsilon, \qquad x,y\in E;$$

we say that functional equation 19.1 is stable if and only if there exist functions $f,g,h: E\to X$ fulfilling equation 19.1 and $\delta_1,\ \delta_2,\ \delta_3: [0,\infty)\to[0,\infty)$, vanishing at zero and continuous, such that

$$\|F(x)-f(x)\|\leq\delta_1(\varepsilon);$$
$$\|G(x)-g(x)\|\leq\delta_2(\varepsilon);$$
$$\|H(x)-h(x)\|\leq\delta_3(\varepsilon)$$

for every $x\in E$.

19.2 Stability of Cauchy, Jensen and Hosszú functional equations

We start with a result of D. Hyers [2] which contains a positive answer to the Ulam's problem.

Theorem 19.1. *Given $\varepsilon\geq 0$. Let X,Y be real Banach spaces and let $f: X\to Y$ satisfy the inequality*

$$\|f(x+y)-f(x)-f(y)\|\leq\varepsilon \tag{19.2}$$

for all $x,y\in X$. Then there exists a unique additive function $L: X\to Y$ (i.e. L satisfies Cauchy functional equation)

$$L(x+y)=L(x)+L(y), \qquad x,y\in X,$$

and the following estimation

$$\|f(x)-L(x)\| \le \varepsilon, \qquad x \in X.$$

Proof. *(Sketch)* Setting in (19.2) $y = x$ we obtain

$$\|f(2x)-2f(x)\| \le \varepsilon, \qquad x \in X.$$

By induction we show that for every positive integer n we have

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \le \left(1-\frac{1}{2^n}\right)\varepsilon.$$

Defining

$$L_n(x) := \frac{f(2^n x)}{2^n}, \qquad x \in X,$$

one can check that $(L_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence for each $x \in X$ and its limit $L(x)$ satisfies required conditions. □

Assume, as previously, that X, Y are Banach spaces, $\varepsilon \ge 0$ is a given constant and $h : X \to Y$ satisfies the following inequality

$$\left\|h\left(\frac{x+y}{2}\right) - \frac{h(x)+h(y)}{2}\right\| \le \varepsilon, \tag{19.3}$$

for all $x, y \in X$. Then function H given by the formula $H(x) := h(x) - h(0)$, $x \in X$, also satisfies this inequality, i.e.,

$$\left\|H\left(\frac{x+y}{2}\right) - \frac{H(x)+H(y)}{2}\right\| \le \varepsilon, \qquad x, y \in X.$$

Moreover, observe that $H(0) = 0$ and

$$\|H(x+y) - H(x) - H(y)\| \le 4\varepsilon, \qquad x, y \in X.$$

On account of Theorem 19.1 there exists an additive function $L : X \to Y$ such that

$$\|H(x) - L(x)\| \le 4\varepsilon, \qquad x \in X.$$

Note that function $J : X \to Y$ defined as $J(x) := L(x) + h(0)$, $x \in X$, satisfies the following Jensen functional equation of the form

$$J(x) + J(y) = 2\,J\left(\frac{x+y}{2}\right), \tag{19.4}$$

$x, y \in X$, and estimation

$$\|h(x) - J(x)\| \le 4\varepsilon, \tag{19.5}$$

for every $x \in X$. Thus we have proved the following corollaries.

Corollary 19.2. *Given $\varepsilon \geq 0$, let X, Y be Banach spaces and let $h : X \to Y$ satisfy (19.3) for all $x, y \in X$. Then there exists a function $J : X \to Y$ fulfilling Jensen equation (19.4) for all $x, y \in X$ and estimation (19.5) for every $x \in X$. In other words, Jensen functional equation is stable in the class of functions transforming X into Y.*

Corollary 19.3. *Let X, Y be Banach spaces. Then $J : X \to Y$ is a Jensen function if and only if $J(x) = L(x) + c$, $x \in X$, where $c \in Y$ is a constant, and $L : X \to Y$ is additive.*

Observe that Jensen functional equation may be considered in the class of functions defined on a convex subset of X. But if the domain of J is a bounded subset of X then the method used above cannot be applied. The first step to solve the problem of the stability of Cauchy functional equation in the class of functions defined on a real interval was set out by F. Skof in [11]. She has proven that if $f : [0, a) \to Y, a > 0$, (Y - a Banach space) satisfies inequality (19.2) for all $x, y \in [0, a)$ such that $x + y \in [0, a)$, then there exists an additive function $L : \mathbb{R} \to Y$ such that

$$\|f(x) - L(x)\| \leq 3\varepsilon$$

for every $x \in [0, a)$. In [3], using this theorem we showed that if $E \subset \mathbb{R}^n$ is a bounded set and x_0 is a point of its interior, $\frac{1}{2}(E - x_0) \subset (E - x_0)$, and $h : E \to Y$ satisfies inequality (19.3) for all $x, y \in E$ then there exist a Jensen function $J : E \to Y$ and a constant K (depending on the set E) such that

$$\|h(x) - J(x)\| \leq K\varepsilon.$$

for every $x \in E$. Applying this theorem M. Laczkovich has proved that in the case where $X = \mathbb{R}$ the assumption of the boundedness of E is superfluous.

Theorem 19.4. *[7] Given $\varepsilon \geq 0$. If $E \subset \mathbb{R}^n$ is convex and $h : E \to \mathbb{R}$ satisfies inequality (19.3), i.e.,*

$$\left| h\left(\frac{x+y}{2}\right) - \frac{h(x) + h(y)}{2} \right| \leq \varepsilon,$$

for all $x, y \in E$, then there exists a Jensen function $J : E \to \mathbb{R}$ such that

$$|h(x) - J(x)| \leq C\varepsilon,$$

for every $x \in E$, where C is a constant only depending on E.

We omit sketch of the proof of this theorem because it is based also on some other results concerning local stability of convexity.

In the class of real functions defined on $\mathbb{R}$ (or on the interval $[0,1]$) the Jensen equation (19.4) is equivalent to the equation of the form

$$\psi(x+y-xy)+\psi(xy)=\psi(x)+\psi(y),$$

for all $x,y\in\mathbb{R}$ (or $x,y\in[0,1]$) which is referred to as the Hosszú functional equation. This equation was mentioned for the first time by M. Hosszú [1] at the International Symposium on Functional Equations held in Zakopane (Poland). The proof of the equivalence of Jensen and Hosszú functional equations may be found in [6] in the case of functions transforming the set of all reals into itself. In the case of real functions defined on a unit interval it was proven by K. Lajkó [8]. In 1996 L. Losonczi [9] proved the stability of the Hosszú equation in the class of real functions defined on the set of all reals and posed the problem of the stability of this equation in the class of real functions defined on the unit interval $[0,1]$. Surprisingly, in this case the Hosszú equation is not stable. J. Tabor (Jr.) [12] proved that for every $\varepsilon>0$ one can find a function $f_\varepsilon:[0,1]\to\mathbb{R}$ such that

$$|f_\varepsilon(x+y-xy)+f_\varepsilon(xy)-f_\varepsilon(x)-f_\varepsilon(y)|\le\varepsilon,\qquad \text{for all } x,y\in[0,1], \quad (19.6)$$

and, simultaneously, for every Jensen function $J:[0,1]\to\mathbb{R}$ (or function satisfying the Hosszú functional equation)

$$\sup\{|f_\varepsilon(x)-J(x)|;x\in[0,1]\}=\infty. \quad (19.7)$$

In fact, J. Tabor (Jr.) gave a very general method of constructions of such type examples. Using his ideas we give here an example of a function $f_\varepsilon:[0.1]\to\mathbb{R}$ with $\varepsilon=ln\,4$ fulfilling conditions (19.6) and (19.7).

We define function $f:[0.1]\to\mathbb{R}$ in the following way:

$$f(x):=\begin{cases}0, & \text{if} \quad x\in\{0,1\};\\ ln\,n, & \text{if} \quad x\in\left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right]\cup\left[1-\frac{1}{2^n},1-\frac{1}{2^{n+1}}\right),\end{cases}$$

for arbitrary positive integer n. For $x=0$ or $y=0$ condition (19.6) is trivially fulfilled. Let $x\in\left(\frac{1}{2^{n+1}},\frac{1}{2^n}\right]$, $y\in\left(\frac{1}{2^{n+k+1}},\frac{1}{2^{n+k}}\right]$ where n is an arbitrary positive and k is a non-negative integer. Then $f(x)=ln\ n,\ f(y)=ln\ (n+k), f(xy)\le ln\ (2n+k+1)$ and $f(x+y-xy)\le ln\ n$. It is not hard to check that condition (19.6) is fulfilled for all $x,y\in\left[0,\frac{1}{2}\right]$ with $\varepsilon=ln\ 3$. Assume that $x,y\in[\frac{1}{2},1)$. Then $1-x,\ 1-y\in\left(0,\frac{1}{2}\right]$ and hence

$$|f(1-x+1-y-(1-x)(1-y))+f((1-x)(1-y))-f(1-x)-f(1-y)| \le ln\, 3.$$

Consequently,

$$|f(1-xy)+f(1-(x+y-xy))-f(1-x)-f(1-y)| \le ln\, 3,$$

from which (19.6) easy follows. Assume now that $x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]$ and $y \in \left[1-\frac{1}{2^m}, 1-\frac{1}{2^{m+1}}\right)$ for arbitrary positive integers n, m. Then $f(x) = ln\, n,\ f(y) = ln\, m,\ f(xy) \le ln\,(n+1)$ and $f(x+y-xy) \le ln\,(m+1)$. Thus in this case condition (19.6) is fulfilled with $\varepsilon = ln\, 4$. Assume that there exists a Jensen function $J : [0,1] \to \mathbb{R}$ such that $\sup\{|f(x)-J(x)|; x \in [0,1]\} < \infty$. Then J has to be a continuous function because it is locally bounded at $\frac{1}{2}$ (for example [6]). Thus J is of the form $J(x) = kx + c,\ x \in [0,1]$, where k and c are real constants and hence it satisfies condition (19.7), because f is unbounded.

19.3 Stability of the Jensen-Hosszú functional equation

In the previous part of the paper we gave an example of two equivalent functional equations (the Jensen and the Hosszú equations) one of which was stable, but one non-stable. The problem of the stability of the following Jensen-Hosszú equation

$$f(x+y-xy)+f(xy) = 2f\left(\frac{x+y}{2}\right)$$

now appears in a natural way. In this equation left-hand side coincides with the left-hand side of the Hosszú equation and the right-hand side coincides with the right-hand side of the Jensen equation. We will show that in the class of real functions defined on the whole set $\mathbb{R}$, as well as on the unit interval, the Jensen-Hosszú equation is stable. In fact, we shall prove much more. Namely, in the case when the domain of functions f, g, h coincides with the set of all reals then equation (19.1) is stable which is not the case when the domain of f, g and h is the unit interval $[0,1]$. In the former case, we obtain stability of our equation only if two among three functions f, g, h are equal. We start with the case when the domain of f, g, h is the set of all real numbers. The following theorem was proven in [4].

Theorem 19.5. *[4] Let $\varepsilon \ge 0$ be a fixed real number and let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions satisfying the following condition*

$$\left|2f\left(\frac{x+y}{2}\right)-g(x+y-xy)-h(xy)\right|\le\varepsilon,\qquad x,y\in\mathbb{R}. \tag{19.8}$$

Then there exist functions $f_1,g_1,h_1:\mathbb{R}\to\mathbb{R}$ fulfilling equation (19.1) and the following estimations

$$|f(x)-f_1(x)|\le 7\varepsilon,\ \ |g(x)-g_1(x)|\le 11\varepsilon,\ \ \textit{and}\ \ |h(x)-h_1(x)|\le 24\varepsilon,\ \ x\in\mathbb{R}.$$

Proof. Putting in (19.8) $x=y=0$ we get

$$|2f(0)-g(0)-h(0)|\le\varepsilon.$$

If $F(x)=f(x)-f(0),\ G(x)=g(x)-g(0),\ H(x)=h(x)-h(0),\ x\in\mathbb{R}$, then the triple $\{F,G,H\}$ satisfies the analogue condition, i.e.,

$$\left|2F\left(\frac{x+y}{2}\right)-G(x+y-xy)-H(xy)\right|\le 2\varepsilon,\qquad x,y\in\mathbb{R}, \tag{19.9}$$

and, moreover,

$$F(0)=G(0)=H(0)=0.$$

Setting $y=0$ in (19.9) we obtain

$$\left|2F\left(\frac{x}{2}\right)-G(x)\right|\le 2\varepsilon,\qquad x\in\mathbb{R}. \tag{19.10}$$

For arbitrary $u\in\mathbb{R}$ and $v\le 0$ the equation

$$z^2-(u+v)z+v=0$$

has two solutions x and y fulfilling the following equalities

$$u+v=x+y\qquad\text{and}\qquad v=xy.$$

Consequently,

$$\left|2F\left(\frac{u+v}{2}\right)-G(u)-H(v)\right|\le 2\varepsilon,\qquad u\in\mathbb{R},\ v\le 0. \tag{19.11}$$

Setting $u=0$ in (19.11) we obtain

$$\left|2F\left(\frac{v}{2}\right)-H(v)\right|\le 2\varepsilon,\qquad v\le 0. \tag{19.12}$$

By virtue of (19.10), (19.11) and (19.12), for all $u\in\mathbb{R}$ and each $v\le 0$, we have

$$\left|2F\left(\frac{u+v}{2}\right)-2F\left(\frac{u}{2}\right)-2F\left(\frac{v}{2}\right)\right| \le \left|2F\left(\frac{u+v}{2}\right)-G(u)-H(v)\right|+$$
$$+\left|2F\left(\frac{u}{2}\right)-G(u)\right|+\left|2F\left(\frac{v}{2}\right)-H(v)\right| \le 6\varepsilon,$$

which can be rewritten in the following equivalent form

$$|F(u+v)-F(u)-F(v)| \le 3\varepsilon, \qquad u\in\mathbb{R},\ v\le 0.$$

According to a well-known theorem ([11], see also remarks below Corollary 19.3 there exists a uniquely determined additive function $A:\mathbb{R}\to\mathbb{R}$ such that

$$|F(v)-A(v)| \le 3\varepsilon, \qquad v\le 0.$$

Using also (19.10) we obtain for $v\le 0$

$$|G(v)-A(v)| \le \left|G(v)-2F\left(\frac{v}{2}\right)\right|+\left|2F\left(\frac{v}{2}\right)-2A\left(\frac{v}{2}\right)\right| \le 8\varepsilon,, \qquad (19.13)$$

and, similarly, using (19.12) instead of (19.10)

$$|H(v)-A(v)| \le 8\varepsilon, \qquad v\le 0.$$

It follows from (19.11) (by putting $u=-v$) that

$$|G(-v)+H(v)| \le 2\varepsilon, \qquad v\le 0.$$

For arbitrary $v>0$ we have

$$|G(v)-A(v)| \le |G(v)+H(-v)|+|A(-v)-H(-v)| \le 2\varepsilon+8\varepsilon=10\varepsilon,$$

which together with (19.13) imply that

$$|G(u)-A(u)| \le 10\varepsilon, \qquad u\in\mathbb{R}. \qquad (19.14)$$

According to (19.14) and (19.10)

$$|F(u)-A(u)| \le \frac{1}{2}|2F(u)-G(2u)|+\frac{1}{2}|G(2u)-A(2u)| \le 6\varepsilon, \qquad u\in\mathbb{R}.$$

Putting $x=v>0$ and $y=1$ in (19.9) we get

$$\left|2F\left(\frac{v+1}{2}\right)-G(1)-H(v)\right| \le 2\varepsilon,$$

and, consequently,

$$|H(v)-A(v)| \le \left|H(v)+G(1)-2F\left(\frac{v+1}{2}\right)\right| + 2\left|F\left(\frac{v+1}{2}\right)-A\left(\frac{v+1}{2}\right)\right| + \\ + |G(1)-A(1)| \le 24\varepsilon.$$

Therefore,

$$|F(x)-A(x)| \le 6\varepsilon,\ |G(x)-A(x)| \le 10\varepsilon \quad \text{and} \quad |H(x)-A(x)| \le 24\varepsilon, \quad x \in \mathbb{R}.$$

Let us put $d := 2f(0)-g(0)-h(0)$, $f_1(x) := A(x)+f(0)-d$, $g_1(x) := A(x)+g(0)-d$, $h(x) := A(x)+h(0)$, $x \in \mathbb{R}$. We observe that $|d| \le \varepsilon$ and

$$2f_1\left(\frac{x+y}{2}\right) - g_1(x+y-xy) - h_1(xy) = 0, \qquad x,y \in \mathbb{R}.$$

Moreover,

$$|f(x)-f_1(x)| \le 7\varepsilon,\ |g(x)-g_1(x)| \le 11\varepsilon,\ \text{and}\ |h(x)-h_1(x)| \le 24\varepsilon, \quad x \in \mathbb{R}.$$

This completes the proof. □

In the next step we will show that the general solution of the equation (19.1) are Jensen functions in $(0,1)$. Assume (like in (19.8)) that

$$\left|2f\left(\frac{x+y}{2}\right) - g(x+y-xy) - h(xy)\right| \le \varepsilon, \qquad x,y \in [0,1].$$

Setting $F(x) = f(x)-f(0), G(x) = g(x)-g(0), H(x) = h(x)-h(0)$, $x \in [0,1]$, we obtain

$$\left|2F\left(\frac{x+y}{2}\right) - G(x+y-xy) - H(xy)\right| \le 2\varepsilon, \qquad x,y \in [0,1], \tag{19.15}$$

and, moreover,

$$F(0) = G(0) = H(0) = 0.$$

As an easy consequence ($y = 0$) we get

$$\left|2F\left(\frac{x}{2}\right) - G(x)\right| \le 2\varepsilon, \qquad x \in [0,1]. \tag{19.16}$$

We define a subset D of $[0,1]^2$ in the following way

$$D = \{(u,v) \in [0,1]^2; (u+v)^2 - 4v \ge 0\}.$$

For every $(u,v) \in D$ the equation $x^2 - (u+v)x + v = 0$ has solutions x_1, x_2 fulfilling conditions

$$x_1 + x_2 = u + v, \qquad x_1 x_2 = v.$$

It is not hard to check that $x_1, x_2 \in [0,1]$. Hence and by virtue of (19.15) we infer that

$$\left|2F\left(\frac{u+v}{2}\right) - G(u) - H(v)\right| \le 2\varepsilon, \qquad (u,v) \in D. \tag{19.17}$$

Observe that if $(u,v) \in D$ and $u+v \le 1$ then $(u+v,v) \in D$ whence

$$\left|2F\left(\frac{u+2v}{2}\right) - G(u+v) - H(v)\right| \le 2\varepsilon, \quad (u,v) \in D_0 := \{(u,v) \in D; u+v \le 1\}.$$

This together with (19.17) yields

$$\left|2F\left(\frac{u+2v}{2}\right) - G(u+v) - 2F\left(\frac{u+v}{2}\right) + G(u)\right| \le 4\varepsilon, \qquad (u,v) \in D_0$$

and using (24) we get

$$\left|F\left(\frac{u+2v}{2}\right) - 2F\left(\frac{u+v}{2}\right) + F\left(\frac{u}{2}\right)\right| \le 4\varepsilon, \qquad (u,v) \in D_0.$$

Putting

$$s = \frac{u+2v}{2}, \quad t = \frac{u}{2}, \qquad (u,v) \in D_0,$$

after simple calculation we have

$$\left|2F\left(\frac{s+t}{2}\right) - F(s) - F(t)\right| \le 4\varepsilon, \qquad (s,t) \in \Delta, \tag{19.18}$$

where Δ is the region bounded by the curves

$$s+t = 1, \quad s \in \left[\frac{3}{8}, \frac{5}{8}\right], \quad (s+t)^2 - 4(s-t) = 0, \quad s \in \left[0, \frac{5}{8}\right]$$

and

$$(s+t)^2 - 4(t-s) = 0, \quad s \in \left[0, \frac{3}{8}\right].$$

More precisely, $(s,t) \in \Delta$ if and only if $s+t \le 1,\ (s+t)^2 \ge 4(s-t)$.

Evidently, $[2\sqrt{2} - \frac{5}{2}, \frac{1}{2}]^2 \subset \Delta$. On account of a theorem of Laczkovich [7] (see Theorem 19.4) there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ and a real constant c such that

$$|F(u)-A(u)-c| \le \mu_1\varepsilon, \qquad u \in \left[2\sqrt{2}-\frac{5}{2},\frac{1}{2}\right],$$

where μ_1 does not depend on F. Let us put $s_0=\frac{1}{2}$, $s_1=2\sqrt{2}-\frac{5}{2}$ and $s_{n+1}=s_n(1-\frac{1}{2}s_n)$ for arbitrary positive integer n. Note that $(s_n)_{n\in\mathbb{N}}$ is a decreasing sequence converging to zero and, moreover, $s_n-s_{n+1}\le s_{n-1}-s_n$, $n\in\mathbb{N}_0$. We will show that

$$|F(u)-A(u)-c| \le \mu_n\varepsilon, \qquad u \in \left[s_n,\frac{1}{2}\right] \tag{19.19}$$

where μ_n does not depend on F. It is true for $n=1$. Observe that $[s_n,s_{n-1}]^2\subset\Delta$ and if $u\in[s_{n+1},s_n]$ and $v=2s_n-u$, then $\frac{u+v}{2},v\in[s_n,s_{n-1}],n\in\mathbb{N}$. For induction method assume (19.19) for a positive integer n. Take an arbitrary $u\in[s_{n+1},s_n]$. According to (19.18) and (19.19) we obtain

$$|F(u)-A(u)-c| \le \left|F(u)+F(v)-2F\left(\tfrac{u+v}{2}\right)\right|+2\,\left|F\left(\tfrac{u+v}{2}\right)-A\left(\tfrac{u+v}{2}\right)-c\right|$$
$$+|A(v)+c-F(v)| \le 4\varepsilon+2\mu_n\varepsilon+\mu_n\varepsilon =: \mu_{n+1}\varepsilon,$$

which ends the proof of (19.19). The sequence (s_n) tends to zero if n tends to infinity thus there exists a positive integer N such that

$$|F(u)-A(u)-c| \le \mu_N\varepsilon, \qquad u \in \left[s_N,\frac{1}{2}\right]\supset\left[\frac{1}{4},\frac{1}{2}\right]. \tag{19.20}$$

It follows from (19.20) and (19.16) that

$$\begin{aligned}|G(u)-A(u)-2c| &\le \left|G(u)-2F\left(\frac{u}{2}\right)\right|+2\left|F\left(\frac{u}{2}\right)-A\left(\frac{u}{2}\right)-c\right|\\ &\le 2\varepsilon+2\mu_N\varepsilon=2(\mu_N+1)\varepsilon, \qquad u\in[2s_N,1]\supset\left[\frac{1}{2},1\right].\end{aligned} \tag{19.21}$$

Setting $y=\frac{1}{2}$ in (19.15) we have

$$\left|2F\left(\frac{1}{4}+\frac{1}{2}x\right)-G\left(\frac{1}{2}+\frac{1}{2}x\right)-H\left(\frac{1}{2}x\right)\right| \le 2\varepsilon, \qquad x\in[0,1].$$

Therefore, according to (19.21) and (19.20), for every $x\in\left[0,\frac{1}{2}\right]$, we get

$$\left|H\left(\frac{1}{2}x\right)-A\left(\frac{1}{2}x\right)\right|\le\left|H\left(\frac{1}{2}x\right)+G\left(\frac{1}{2}+\frac{1}{2}x\right)-2F\left(\frac{1}{4}+\frac{1}{2}x\right)\right|+$$
$$\left|-G\left(\frac{1}{2}+\frac{1}{2}x\right)+A\left(\frac{1}{2}+\frac{1}{2}x\right)+2c\right|+\left|2F\left(\frac{1}{4}+\frac{1}{2}x\right)-2A\left(\frac{1}{4}+\frac{1}{2}x\right)-2c\right|\le$$
$$\le 2\varepsilon+2(\mu_N+1)\varepsilon+2\mu_N\varepsilon=4(\mu_N+1)\varepsilon,$$

whence

$$|H(x)-A(x)|\le\mu\varepsilon,\qquad x\in\left[0,\frac{1}{4}\right],$$

where $\mu=4(\mu_N+1)$. Consequently,

$$\begin{cases}|F(u)-A(u)-c|\le\mu_n\varepsilon, & u\in\left[s_n,\frac{1}{2}\right];\\ |G(u)-A(u)-2c|\le\mu_n'\varepsilon, & u\in[2s_n,1];\\ |H(u)-A(u)|\le\mu\varepsilon, & u\in\left[0,\frac{1}{4}\right].\end{cases}\tag{19.22}$$

Thus we are in a position to prove the following theorem.

Theorem 19.6. *[5] Let $f,g,h:[0,1]\to\mathbb{R}$ be functions satisfying functional equation (19.1). Then there exists an additive function $A:\mathbb{R}\to\mathbb{R}$ and a constant $c\in\mathbb{R}$ such that*

$$\begin{aligned} f(x)&=f(0)+A(x)+c, && x\in(0,1);\\ g(x)&=g(0)+A(x)+2c, && x\in(0,1];\\ h(x)&=h(0)+A(x), && x\in[0,1).\end{aligned}\tag{19.23}$$

Conversely, if functions f,g,h are defined by (19.23) and $2f(w)=g(w)+h(w)$ for $w\in\{0,1\}$, then functional equation (19.1) is fulfilled.

Proof. Putting $F(x)=f(x)-f(0)$, $G(x)=g(x)-g(0)$, $H(x)=h(x)-h(0)$, $x\in[0,1]$, we observe that the triple (F,G,H) satisfies the following functional equation

$$2F\left(\frac{x+y}{2}\right)=G(x+y-xy)+H(xy),\qquad x,y\in[0,1].$$

It follows from (19.22) (with $\varepsilon=0$) that

$$\begin{cases}F(x)=A(x)+c, & x\in\left[s_n,\frac{1}{2}\right];\\ G(x)=A(x)+2c, & x\in[2s_n,1];\\ H(x)=A(x), & x\in\left[0,\frac{1}{4}\right].\end{cases}$$

Taking $n\to\infty$ we obtain

$$\begin{cases} F(x) = A(x) + c, & x \in \left(0, \frac{1}{2}\right]; \\ G(x) = A(x) + 2c, & x \in (0,1]; \\ H(x) = A(x), & x \in \left[0, \frac{1}{4}\right]. \end{cases} \tag{19.24}$$

Let us put $\rho_0 = \frac{1}{2},\ \rho_{n+1} = \frac{1}{2}(1+\rho_n^2),\ n \in \mathbb{N}_0$. Evidently, sequence $(\rho_n)_{n\in\mathbb{N}}$ is increasing and converging to 1. Using induction method we will prove that

$$\begin{cases} F(u) = A(u) + c, & u \in (0, \rho_n]; \\ H(u) = A(u) & u \in [0, \rho_n^2]. \end{cases} \tag{19.25}$$

By virtue of (19.24) it is true for $n = 0$. Assume (19.25) for an $n \in \mathbb{N}_0$. For each $u \in [\rho_n, \rho_{n+1}]$ there exists an $x \in [0,1]$ such that $u = \frac{x+\rho_n^2}{2}$. Setting $y = \rho_n^2$ in (32) and applying the induction assumption we have

$$\begin{aligned} 2F\left(\frac{x+\rho_n^2}{2}\right) &= G(x+\rho_n^2-\rho_n^2 x) + H(\rho_n^2 x) \\ &= A(x+\rho_n^2-\rho_n^2 x) + 2c + A(\rho_n^2 x) = 2\left[A(\frac{x+\rho_n^2}{2}) + c\right], \end{aligned}$$

which proves the first equality of (19.25) for $n+1$. If $u \in [\rho_n^2, \rho_{n+1}^2]$ then there exists an $x \in (0, 2\rho_{n+1}]$ such that $x(2\rho_{n+1} - x) = u$. Therefore

$$\begin{aligned} H(u) = H(x(2\rho_{n+1}-x)) &= 2F(\rho_{n+1}) - G(x+2\rho_{n+1}-x-u)) \\ &= 2[A(\rho_{n+1})+c] - A(x+2\rho_{n+1}-x-u) - 2c = A(u), \end{aligned}$$

which proves the second equality of (19.25) for $n+1$ and ends the proof of (19.25). Since the part "conversely" is obvious, our assertion follows now from (19.24) and (19.25), because (34) hold for every non-negative integer. □

19.3.1 The case $h = f$

We have the following theorem.

Theorem 19.7. *[5] Let ε be a nonnegative number and let $f, g : [0,1] \to \mathbb{R}$ be functions satisfying the following condition*

$$\left|2f\left(\frac{x+y}{2}\right) - g(x+y-xy) - f(xy)\right| \le \varepsilon, \qquad x, y \in [0,1].$$

Then there exists an additive function $A:\mathbb{R}\to\mathbb{R}$ and constants $c,\rho_1,\rho_2\in\mathbb{R}$ (ρ_1,ρ_2 - not depending on f,g) such that

$$\begin{aligned}&|f(x)-f(0)-A(x)-c|\le\rho_1\varepsilon, && x\in[0,1];\\ &|g(x)-g(0)-A(x)-2c|\le\rho_2\varepsilon, && x\in[0,1].\end{aligned}\tag{19.26}$$

Proof. Let $F(x)=f(x)-f(0),\ G(x)=g(x)-g(0),\ x\in[0,1]$. Then also

$$\left|2F\left(\frac{x+y}{2}\right)-G(x+y-xy)-F(xy)\right|\le 2\varepsilon,\qquad x,y\in[0,1].\tag{19.27}$$

Setting here $y=\frac{1}{2}$ we have

$$\left|2F\left(\frac{1}{4}+\frac{1}{2}x\right)-G\left(\frac{1}{2}+\frac{1}{2}x\right)-F\left(\frac{1}{2}x\right)\right|\le 2\varepsilon,\qquad x\in[0,1].$$

We may use conditions (19.16) and (19.22). Taking N such that $\left[s_N,\frac{1}{2}\right]\supset\left[\frac{1}{4},\frac{1}{2}\right]$ we infer that

$$|c|\le(\mu+\mu_N)\varepsilon.$$

Now, it follows from (19.22) and (19.16) that

$$\begin{aligned}&|F(u)-A(u)|\le 2(\mu+\mu_N)\varepsilon=k\,\varepsilon,\quad u\in\left[0,\frac{1}{2}\right];\\ &|G(u)-A(u)|\le\left|G(u)-2F\left(\frac{u}{2}\right)\right|+\left|2F\left(\frac{u}{2}\right)-2A\left(\frac{u}{2}\right)\right|\\ &\quad\le 2(1+k)\,\varepsilon=\rho_2\,\varepsilon,\quad u\in[0,1].\end{aligned}$$

Hence

$$\begin{aligned}\left|2F\left(\frac{1}{4}+\frac{1}{2}x\right)-2A\left(\frac{1}{4}+\frac{1}{2}x\right)\right|&\le\left|2F\left(\frac{1}{4}+\frac{1}{2}x\right)-G\left(\frac{1}{2}+\frac{1}{2}x\right)-F\left(\frac{1}{2}x\right)\right|\\ +\left|G\left(\frac{1}{2}+\frac{1}{2}x\right)-A\left(\frac{1}{2}+\frac{1}{2}x\right)\right|&+\left|F\left(\frac{1}{2}x\right)-A\left(\frac{1}{2}x\right)\right|\le(2+\rho_2+k)\,\varepsilon,\end{aligned}$$

whence

$$|F(u)-A(u)|\le\left(1+\frac{k+\rho_2}{2}\right)\,\varepsilon,\qquad u\in\left[0,\frac{3}{4}\right].$$

Similarly, setting $y=\frac{3}{4}$ in (19.27), we get

$$|F(u)-A(u)|\le\left(1+\frac{k+\rho_2}{2}+\frac{k+\rho_2}{4}\right)\,\varepsilon,\qquad u\in\left[0,\frac{7}{8}\right].$$

Using induction method one can prove that

$$|F(u)-A(u)| \le \left(1+\frac{k+\rho_2}{2}+\cdots+\frac{k+\rho_2}{2^p}\right)\varepsilon, \qquad u \in \left[0,1-\frac{1}{2^{p+1}}\right],\ p\in\mathbb{N},$$

and therefore

$$|F(u)-A(u)| \le (1+k+\rho_2)\,\varepsilon = \rho_1\,\varepsilon, \qquad u\in[0,1).$$

Since $|2F(1)-G(1)-F(1)| \le 2\varepsilon$ then

$$|F(1)-A(1)| \le |F(1)-G(1)|+|G(1)-A(1)| \le (2+\rho_2)\,\varepsilon,$$

which ends the proof of estimations (19.26). □

19.3.2 The cases $f=g$ and $g=h$

The following two theorems easily follow from (19.22).

Theorem 19.8. *[5] Let $\varepsilon \ge 0$ be fixed and let $f,h:[0,1]\to\mathbb{R}$ be functions satisfying the following condition*

$$\left|2f\left(\frac{x+y}{2}\right)-f(x+y-xy)-h(xy)\right| \le \varepsilon, \qquad x,y\in[0,1].$$

Then there exist an additive function $A:\mathbb{R}\to\mathbb{R}$ and constants $\rho_1,\rho_2\in\mathbb{R}$ (not depending on f,g) such that

$$\begin{aligned} |f(x)-A(x)-f(0)| &\le \rho_1\varepsilon, \qquad x\in[0,1];\\ |h(x)-A(x)-h(0)| &\le \rho_2\varepsilon, \qquad x\in[0,1]. \end{aligned} \tag{19.28}$$

Proof. As in the proof of Theorem 19.6, we define functions F and H. It follows from (19.22) that $|c| \le (\mu_N+\mu_N')\,\varepsilon$, where $N\in\mathbb{N}$ is so chosen that $s_N \le \frac{1}{4}$, and

$$|F(u)-A(u)| \le 2(\mu_N+\mu_N')\,\varepsilon, \qquad u\in[s_N,1].$$

By (19.4) we have also $|2F(\frac{x}{2})-F(x)| \le 2\,\varepsilon,\ x\in[0,1]$. The last two inequalities imply that for every $x\in\left(0,\frac{1}{4}\right]$ we have

$$|F(x)-A(x)| \le \frac{1}{2}[|2F(x)-F(2x)|+|F(2x)-A(2x)|\,] \le \frac{1}{2}(2+2(\mu_N+\mu_N'))\,\varepsilon.$$

This ends the proof of the first estimation of (19.28). If $x\in[0,1]$ then

$$|H(x)-A(x)| \le \left|H(x)+F(1)-2F\left(\frac{1+x}{2}\right)\right| + |F(1)-A(1)|$$
$$+\left|2F\left(\frac{1+x}{2}\right)-2A\left(\frac{1+x}{2}\right)\right| \le (2+3\,\rho_1)\,\varepsilon.$$

□

In the same way one can prove the following theorem.

Theorem 19.9. *[5] Let $\varepsilon \ge 0$ be fixed and let $f,g : [0,1] \to \mathbb{R}$ be functions satisfying the following condition*

$$\left|2f\left(\frac{x+y}{2}\right)-g(x+y-xy)-g(xy)\right| \le \varepsilon, \qquad x,y \in [0,1].$$

Then there exist an additive function $A : \mathbb{R} \to \mathbb{R}$ and constants $\rho_1, \rho_2 \in \mathbb{R}$ (not depending on f,g) such that

$$|f(x)-A(x)-f(0)| \le \rho_1\varepsilon, \qquad x \in [0,1];$$

$$|g(x)-A(x)-g(0)| \le \rho_2\varepsilon, \qquad x \in [0,1].$$

Proof. Putting $F(x) = f(x)-f(0)$ and $G(x) = g(x)-g(0)$, $x \in I$, we observe that

$$\left|2F\left(\frac{x+y}{2}\right)-G(x+y-xy)-G(xy)\right| \le 2\,\varepsilon, \qquad x,y \in [0,1].$$

According to (19.22) we get $|c| \le \frac{1}{2}(\mu'_N+\mu)$, where $N \in \mathbb{N}$ is so chosen that $s_N \le \frac{1}{8}$. Therefore

$$|G(u)-A(u)| \le \frac{3}{2}(\mu'_N+\mu)\,\varepsilon = \rho_2\,\varepsilon, \qquad u \in [0,1].$$

Now, for all $x,y \in [0,1]$ we obtain

$$\left|F\left(\frac{x+y}{2}\right)-A\left(\frac{x+y}{2}\right)\right| \le \left|F\left(\frac{x+y}{2}\right)-\frac{1}{2}G(x+y-xy)-\frac{1}{2}G(xy)\right| +$$
$$+\frac{1}{2}|G(x+y-xy)-A(x+y-xy)|+\frac{1}{2}|G(xy)-A(xy)| \le (1+\rho_2)\,\varepsilon.$$

□

19.3.3 Counterexample

The following example shows that in the general case of the inequality (19.8) an analogous assertion does not hold.

Example 19.10. For arbitrary $n \in \mathbb{N}$, we put $r_0 = 0,\ r_1 = \frac{1}{2},\ r_{n+1} = \frac{1+r_n^2}{2}$, and

$$
\begin{aligned}
f(u) &= n, && \text{iff } \frac{x+y}{2} = u \in [r_{n-1}, r_n),\ f(1) = 0;\\
g(u) &= 0, && u \in [0,1];\\
h(u) &= 2n+1, && \text{iff } xy = u \in [r_{n-1}^2, r_n^2),\ h(1) = 0.
\end{aligned}
$$

Functions f,g,h are well defined on the unit interval I. Note that if $\frac{x+y}{2} \in [r_0, r_1)$ then $xy \in [r_0^2, r_1^2)$ and, moreover, if $\frac{x+y}{2} \in [r_n, r_{n+1})$ then $xy \in [r_{n-1}^2, r_{n+1}^2)$ for each positive integer n. Therefore for all $x,y \in [0,1]$ we have

$$\left| 2f\left(\frac{x+y}{2}\right) - g(x+y-xy) - h(xy) \right| \le 1,$$

which means that estimation (19.8) is fulfilled. On the other hand, every additive function $A : \mathbb{R} \to \mathbb{R}$ for which set $\{g(x) - A(x);\ x \in I\}$ is bounded is of the form $A(x) = kx$, with a real constant k. But then sets $\{f(x) - A(x);\ x \in [0,1]\}$ as well as $\{h(x) - A(x);\ x \in [0,1]\}$ are unbounded. □

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