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S. Saidi Goraghani and R. A. Borzooei

# ON INJECTIVE $MV$-MODULES

**Abstract**

In this paper, by considering the notion of $MV$-module, which is the structure that naturally correspond to $lu$-modules over $lu$-rings, we study injective $MV$-modules and we investigate some conditions for constructing injective $MV$-modules. Then we define the notions of essential $A$-homomorphisms and essential extension of $A$-homomorphisms, where $A$ is a product $MV$-algebra, and we get some of there properties. Finally, we prove that a maximal essential extension of any $A$-ideal of an injective $MV$-module is an injective $A$-module, too.

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*Keywords:* $(MV, PMV)$-algebra, $MV$-module, Injective $MV$-module, Essential extension.

## 1. Introduction

$MV$-algebras were defined by C.C. Chang [2] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: $CN$-algebras, Wajsberg algebras, bounded commutative $BCK$-algebras and bricks. It is discovered that $MV$-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional $C^*$-algebras. They are also naturally related to Ulam's searching games with lies. $MV$-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang that nontrivial

$MV$-algebras are subdirect products of $MV$-chains, that is, totally ordered $MV$-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an $MV$-algebra. The categorical equivalence between $MV$-algebras and $lu$-groups leads to the problem of defining a product operation on $MV$-algebras, in order to obtain structures corresponding to $l$-rings. A *product MV-algebra* (or $PMV$-algebra, for short) is an $MV$-algebra which has an associative binary operation ".". It satisfies an extra property which will be explained in Preliminaries. During the last years, $PMV$-algebras were considered and their equivalence with a certain class of $l$-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible $MV$-algebras and the $MV$-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of $MV$-modules was introduced as an action of a $PMV$-algebra over an $MV$-algebra by A. Di Nola [5]. Recently, some reasearchers worked on $MV$-modules (see [1, 10, 7]. For example, in 2016, R. A. Borzooei and S. Saidi Goraghani introduced free $MV$-modules. Since $MV$-modules are in their infancy, stating and opening of any subject in this field can be useful.

Now, in this paper, we present the definition of injective $MV$-modules and obtain some interesting results on them. Also, we define the notions of essential $A$-homomorphisms and essential extension of $A$-homomorphisms, where $A$ is a $PMV$-algebra. Finally, we prove that every maximal essential extension of an $A$-ideal in injective $A$-module $I$ is injective if it was included in $I$. In fact, we open new fields to anyone that is interested to studying and development of $MV$-modules.

## 2. Preliminaries

In this section, we review some definitions and related lemmas and theorems that we use in the next sections.

DEFINITION 2.1. *[3] An MV-algebra is a structure* $M = (M, \oplus, ', 0)$ *of type* $(2, 1, 0)$ *such that:*
$(MV1)$ $(M, \oplus, 0)$ *is an Abelian monoid,*
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
*If we define the constant* $1 = 0'$ *and operations* $\odot$ *and* $\ominus$ *by* $a \odot b = (a' \oplus b')'$,

*$a \ominus b = a \odot b'$, then*
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
*for every $a, b \in M$.*

*Now, let $M = (M, \oplus, ', 0)$ be an $MV$-algebra. It is clear that $(M, \odot, 1)$ is an Abelian monoid. If we define auxiliary operations $\vee$ and $\wedge$ on $M$ by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a bounded distributive lattice. An $MV$-algebra $M$ is a Boolean algebra if and only if the operation "$\oplus$" is idempotent, that is $x \oplus x = x$, for every $x \in M$.*

*A subalgebra of an $MV$-algebra $M$ is a subset $S$ of $M$ containing the zero element of $M$, closed under the operation of $M$ and equipped with the restriction to $S$ of these operations. In an $MV$-algebra $M$, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in M$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements $a, b$ of the $MV$-algebra $M$, $a \leq b$ if and only if $a, b$ satisfy the above equivalent conditions $(i) - (iv)$. An ideal of $MV$-algebra $M$ is a subset $I$ of $M$, satisfying the following conditions: $(I1)$: $0 \in I$, $(I2)$: $x \leq y$ and $y \in I$ imply $x \in I$, $(I3)$: $x \oplus y \in I$, for every $x, y \in I$.*

*In an $MV$-algebra $M$, the distance function $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies $(i)$: $d(x, y) = 0$ if and only if $x = y$, $(ii)$: $d(x, y) = d(y, x)$, $(iii)$: $d(x, z) \leq d(x, y) \oplus d(y, z)$, $(iv)$: $d(x, y) = d(x', y')$, $(v)$: $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$.*

*Let $I$ be an ideal of $MV$-algebra $M$. We denote $x \sim y$ $(x \equiv_I y)$ if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on $M$. Denote the equivalence class containing $x$ by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in M\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an $MV$-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$.*

*Let $M$ and $K$ be two $MV$-algebras. A mapping $f : M \to K$ is called an $MV$-homomorphism if $(H1)$: $f(0) = 0$, $(H2)$: $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$: $f(x') = (f(x))'$, for every $x, y \in M$. If $f$ is one to one (onto), then $f$ is called an $MV$-monomorphism ($MV$-epimorphism) and if $f$ is onto and one to one, then $f$ is called an $MV$-isomorphism.*

Lemma 2.2. [3] *In every MV-algebra $M$, the natural order "$\leq$" has the following properties:*
*(i) $x \leq y$ if and only if $y' \leq x'$,*
*(ii) if $x \leq y$, then $x \oplus z \leq y \oplus z$, for every $z \in M$.*

Lemma 2.3. [3] *Let $M$ and $N$ be two MV-algebras and $f : M \to N$ be an MV-homomorphism. Then the following properties hold:*
*(i) For each ideal $J$ of $N$, the set*

$$f^{-1}(J) = \{x \in M : f(x) \in J\}$$

*is an ideal of $A$. Hence, $Ker(f) = f^{-1}(\{0\})$ is an ideal of $M$,*
*(ii) $f(x) \leq f(y)$ if and only if $x \ominus y \in Ker(f)$,*
*(iii) $f$ is injective if and only if $Ker(f) = \{0\}$.*

Definition 2.4. *[4] A product MV-algebra (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on $A$ such that the following property is satisfied: if $x + y$ is defined, then $x.z + y.z$ and $z.x + z.y$ are defined and $(x + y).z = x.z + y.z$, $z.(x + y) = z.x + z.y$, for every $x, y, z \in A$, where "$+$" is the partial addition on $A$. A unit of PMV-algebra $A$ is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If $A$ has a unit, then $e = 1$. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation.*

Lemma 2.5. [4] *Let $A$ be a PMV-algebra. Then $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$, for every $a, b, c \in A$.*

Definition 2.6. *[5] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ be defined by $\Phi(a, x) = ax$, which satisfies the following axioms:*
*(AM1) if $x+y$ is defined in $M$, then $ax+ay$ is defined in $M$ and $a(x+y) = ax + ay$,*
*(AM2) if $a+b$ is defined in $A$, then $ax+bx$ is defined in $M$ and $(a+b)x = ax + bx$,*
*(AM3) $(a.b)x = a(bx)$, for every $a, b \in A$ and $x, y \in M$.*
*Then $M$ is called a (left) MV-module over $A$ or briefly an $A$-module. We say that $M$ is a unitary MV-module if $A$ has a unit and*
*(AM4) $1_A x = x$, for every $x \in M$.*

Corollary 2.7. [7] *Let $M$ be a unitary $A$-module. If $N \subseteq M$ is a nonempty set, then we have:*

$$(N] = \{x \in M : x \leq \alpha_1 x_1 \oplus \alpha_2 x_2 \oplus \cdots \oplus \alpha_n x_n,\ for\ some\ x_1, \cdots, x_n \in N, \alpha_1, \cdots, \alpha_n \in A\}.$$

*In particular, for* $a \in M$, $(a] = \{x \in M : x \leq n(\alpha a),\ for\ some\ integer\ n \geq 0\ and\ \alpha \in A\}$.

Lemma 2.8. [5] *Let* $A$ *be a* $PMV$*-algebra and* $M$ *be an* $A$*-module. Then*
$(a)$ $0x = 0$, $a0 = 0$
$(b)$ $(na)x = a(nx)$, *for any* $n \in N$,
$(c)$ $ax' \leq (ax)'$,
$(d)$ $a'x \leq (ax)'$,
$(e)$ $(ax)' = a'x + (1x)'$,
$(f)$ $x \leq y$ *implies* $ax \leq ay$,
$(g)$ $a \leq b$ *implies* $ax \leq bx$,
$(h)$ $a(x \oplus y) \leq ax \oplus ay$,
$(i)$ $d(ax, ay) \leq ad(x, y)$,
$(j)$ *if* $x \equiv_I y$, *then* $ax \equiv_I ay$, *where* $I$ *is an ideal of* $A$,
$(k)$ *if* $M$ *is a unitary* $MV$*-module, then* $(ax)' = a'x + x'$, *for every* $a, b \in A$ *and* $x, y \in M$.

Definition 2.9. *[5] Let* $A$ *be a* $PMV$*-algebra and* $M_1$, $M_2$ *be two* $A$*-modules. A map* $f : M_1 \to M_2$ *is called an* $A$*-module homomorphism (or* $A$*-homomorphism, for short) if* $f$ *is an* $MV$*-homomorphism and* $(H4)$*:* $f(ax) = af(x)$, *for every* $x \in M_1$ *and* $a \in A$.

Definition 2.10. *[5] Let* $A$ *be a* $PMV$*-algebra and* $M$ *be an* $A$*-module. Then an ideal* $N \subseteq M$ *is called an* $A$*-ideal of* $M$ *if* $(I4)$*:* $ax \in N$, *for every* $a \in A$ *and* $x \in N$.

Definition 2.11. *[10] Let* $M$ *be a unitary* $A$*-module and there exists* $k \in \mathbb{N}$ *such that* $\sum_{i=1}^{n} a_i' m_i \leq (\sum_{i=1}^{n} a_i m_i)'$, *for every* $1 \leq n \leq k$, $a_i \in A$ *and* $m_i \in M$. *Then* $M$ *is called an* $A_k$*-module. If* $\sum_{i=1}^{n} a_i' m_i \leq (\sum_{i=1}^{n} a_i m_i)'$, *for every* $n \in \mathbb{N}$, *then* $M$ *is called an* $A_{\mathbb{N}}$*-module.*

Lemma 2.12. [10] *In* $PMV$*-algebra* $A$, $(\alpha \oplus \beta)a \leq \alpha m \oplus \beta a$, *for every* $\alpha, \beta, a \in A$.

## 3. Injective $MV$-modules

In the follows, let $A$ be a $PMV$-algebra and $M$ be an $MV$-algebra unless otherewise specified.

In this section, we present the definition of injective $MV$-modules and we give some properties about them.

DEFINITION 3.1. *[8] Let $M$ be an $A$-module. $M$ is called an injective $A$-module if for every $m \in M$ and $0 \neq a \in A$, there exists $c \in M$ such that $ac = m$.*

EXAMPLE 3.2. *Consider the real unit interval $[0,1]$. Let $x \oplus y = min\{x+y, 1\}$ and $x' = 1 - x$, for all $x, y \in [0,1]$. Then $([0,1], \oplus, ', 0)$ is an $MV$-algebra, where "$+$" and "$-$" are the ordinary operations in $\mathbb{R}$. Also, the rational numbers in $[0,1]$ and for each integer $n \geq 2$, the $n$-element set*

$$L_n = \{0, \frac{1}{n-1}, \cdots, \frac{n-2}{n-1}, 1\}$$

*yield examples of subalgebras of $[0,1]$ (See [3]). Now, by using this example, we get some injective $MV$-modules.*
*$(i)$ Consider $ab = a.b$, for every $a, b \in L_2$, where "." is ordinary operation in $\mathbb{R}$. Then $(L_2, \oplus, ., ', 0)$ is a $PMV$-algebra and $L_2$ as $L_2$-module is an injective $L_2$-module.*
*$(ii)$ $[0,1]$ as $L_2$-module is an injective $L_2$-module.*
*$(iii)$ Consider $a.b = max\{a, b\}$, for every $a, b \in L_3$. Then it is routine to show that $(L_3, \oplus, ', ., 0)$ is a $PMV$-algebra and by cosidering $ab = a.b$, we have $L_3$ is a $L_3$-module. Moreover, $L_3$ is an injective $L_3$-module.*

DEFINITION 3.3. *Let $I$ be an ideal of $M$ and $a \in I$. If every $b \in I$ can be showed as $b = xa$, for some $x \in A$, then we say $I$ is an $MV$-principle ideal of $M$, and we write $I = \prec a \succ$.*

EXAMPLE 3.4. *Let $A = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "." be defined on $A$ as follows:*

| $\oplus$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 1 | 2 | 3 |
| 2 | 2 | 2 | 2 | 3 |
| 3 | 3 | 3 | 3 | 3 |

| . | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 | 1 |
| 2 | 0 | 1 | 2 | 2 |
| 3 | 0 | 1 | 2 | 3 |

*Consider $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a $PMV$-algebra. Also $I = \{0, 1, 2\}$ and $J = \{0, 1\}$ are ideals of $A$. Since $1 = 1.2$, $2 = 2.2$, $I = \prec 2 \succ$ is an $MV$-principle ideal of $A$. Also, $J = \prec 1 \succ$ is an $MV$-principle ideal of $A$.*

Proposition 3.5. *Let $M$ be an $A_2$-module, where $M$ is a boolean algebra. Then $I = \{xa : x \in A\}$ is an MV-principle ideal of $M$, for every $a \in M$.*

Proof: It is clear that $0 \in I$. Let $xa, ya \in I$, for any $x, y \in A$. Since $x \leq x \oplus y$ and $y \leq x \oplus y$, by Lemma 2.8($f$), we have $ax \leq a(x \oplus y)$ and $ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$. So by Lemma 2.2(ii), we have $ax \oplus ay \leq a(x \oplus y) \oplus ay$ and $a(x \oplus y) \oplus ay \leq a(x \oplus y) \oplus a(x \oplus y) = a(x \oplus y)$. Hence, $ax \oplus ay \leq a(x \oplus y)$, for every $a \in A$ and $x, y \in M$. Now, by Lemma 2.12, $ax \oplus ay = a(x \oplus y)$ and so $ax \oplus ay \in I$. Let $t \leq x.a \in I$, for $t \in M$. Then $1.t' \oplus x.a = 1$ and so $(t' \oplus a)' \oplus x'a = 0$. It results that $(t' \oplus a)' = 0$ and so $t' \oplus a = 1$. Hence we have

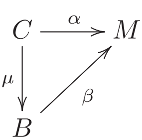
$$t = t \wedge xa = (t' \oplus (t' \oplus xa)')' = (t' \oplus (t' \oplus a)' \oplus x'a)' = (t' \oplus x'a)' = (t' \oplus a)' \oplus xa = xa.$$

It means that $t \in I$. Therefore, $I$ is an ideal of $M$. □

Note. We can consider $A$ as $A_2$-module. Then in proposition 3.5, $I = \{x.a : x \in A\}$ is an MV-principle ideal of $A$.

Definition 3.6. *[10] Let $M_1$ and $M_2$ be two $A$-modules. Then the map $f : M_1 \to M_2$ is called an $A'$-homomorphism if and only if it satisfies in $(H1)$, $(H3)$, $(H4)$ and*
*$(H'2)$ : if $x + y$ is defined in $M_1$, then $h(x + y) = h(x \oplus y) = h(x) \oplus h(y)$, for every $x, y \in M_1$, where "$+$" is the partial addition on $M_1$. If $h$ is one to one (onto), then $h$ is called an $A'$-monomorphism (epimorphism). If $h$ is onto and one to one, then $h$ is called an $A'$-isomorphism and we write $M_1 \cong' M_2$.*

Theorem 3.7. *Let all ideals of $A$ be MV-principle and $M$ be an injective $A$-module. Then for every $A$-module $C$ and every $A'$-homomorphism $\alpha : C \longrightarrow M$ and $A'$-monomorphism $\mu : C \longrightarrow B$, there is an $A$-homomorphism $\beta : B \longrightarrow M$ such that the diagram*




*is commutative, that is $\beta\mu = \alpha$.*

Proof: Let $M$ be an injective $A$-module, $\mu : D \longrightarrow B$ be an $A'$-monomorphism and $\alpha : D \longrightarrow M$ be an $A$-homomorphism, for MV-algebras $D$

and $B$. With out lost of generality, let $D$ be an $A$-ideal of $B$ (because $\mu$ is an $A$-monomorphism). Consider

$$\Omega = \{(D_j, \alpha_j) : D \subseteq D_j \subseteq B,\ \alpha_j : D_j \longrightarrow M,\ \alpha_j \mid_D = \alpha\}.$$

Then by Zorn's lemma, $\Omega$ has a maximal element $(D_m, \alpha_m)$. We claim that $D_m = B$. If $D_m \neq B$, then $D_m \subsetneqq B$ and so there is $b \in B$ such that $b \notin D$. Let $I = \{a \in A : ab \in D_m\}$. Since $0 \in I$, we have $I \neq \emptyset$. We show that $I$ is an ideal of $A$. Let $a_1, a_2 \in I$. Then $a_1b, a_2b \in D_m$. By Lemma 2.12, $(a_1 \oplus a_2)b \leq a_1b \oplus a_2b \in D_m$ and so $a_1 \oplus a_2 \in I$. Now, let $t \leq a \in I$, for some $t \in A$. Then by Lemma 2.8 $(g)$, $tb \leq ab \in D_m$ and so $tb \in D_m$. It means that $t \in I$. Hence $I$ is an ideal of $A$ and so there is $a_0 \in A$ such that $I = \prec a_0 \succ$. If $a_0 = 0$, then we consider an arbitrary element $c \in M$. If $a_0 \neq 0$, then we consider $a_0b \in D_m$ and so $m = \alpha_m(a_0b) \in M$. Since $M$ is an injective $A$-module, there is $c \in M$ such that $m = \alpha_m(a_0b) = a_0c$. Now, let $D_M = \{a_m \oplus tb : t \in A,\ a_m \in D_m\}$. Since $b \notin D_m$, we have $D_m \subset D_M$. We define $\alpha_M : D_M \longrightarrow M$ by

$$\alpha_M(a_m \oplus tb) = \begin{cases} \alpha_m(a_m) + tc, & \text{if} \quad \alpha_m(a_m) + tc, a_m + tb \text{ are defined} \\ \\ 0, & \text{otherwise} \end{cases}$$

The first, we show that $\alpha_M$ is well defined. It is sufficient that we show $\alpha_m(tb) = tc$. Since $tb \in D_m$, we have $t \in I$ and since $I = \prec a_0 \succ$, there is $z \in A$ such that $t = za_0$ and so

$$\alpha_m(tb) = \alpha_m(za_0b) = z\alpha_m(a_0b) = za_0c = tc$$

The proof of $(H1)$ is clear. If $a_{m1} + t_1b + a_{m2} + t_2b$ is defined, then

$$\begin{aligned} \alpha_M(a_{m_1} \oplus t_1b) \oplus (a_{m_2} \oplus t_2b)) &= \alpha_M(a_{m_1} \oplus a_{m_2} \oplus t_1b \oplus t_2b) \\ &= \alpha_M(a_{m1} + a_{m2} + t_1b + t_2b) \\ &= \alpha_M(a_{m1} + a_{m2} + (t_1 + t_2)b) \\ &= \alpha_m(a_{m1} + a_{m2}) + (t_1 + t_2)c \\ &= \alpha_m(a_{m1}) + t_1c \oplus \alpha_m(a_{m2}) + t_2c \\ &= \alpha_M(a_{m1}) \oplus \alpha_M(a_{m2}) \end{aligned}$$

and so $(H2)'$ is true, for any $a_{m1} \oplus t_1b,\ a_{m2} \oplus t_2b \in D_M$. By definition of $\alpha_m$, for every $a_m \oplus tb \in D_M$, we have

$$\begin{aligned}(\alpha_M(a_m \oplus tb))' &= (\alpha_m(a_m) \oplus tc)' \\ &= (\alpha_m(a_m) \oplus \alpha_m(tb))' \\ &= (\alpha_m(a_m \oplus tb))' \\ &= \alpha_m((a_m) \oplus tb)') \\ &= \alpha_m((a_m) \oplus tb)') \oplus 0 \\ &= \alpha_M((a_m) \oplus tb)' \oplus 0) = \alpha_M((a_m) \oplus tb)')\end{aligned}$$

and so $(H3)$ is true. Now, for every $a \in A$ and $a_m \oplus tb \in D_M$, we have

$$\begin{aligned}(\alpha_M(a(a_m \oplus tb)) &= \alpha_M(aa_m \oplus (a.t)b) \\ &= \alpha_m(aa_m) \oplus (a.t)c \\ &= a\alpha_m(a_m) \oplus a(tc) \\ &= a(\alpha_m(a_m) \oplus tc) \\ &= a\alpha_M(a_m \oplus tb)\end{aligned}$$

and so $(H4)$ is true. Hence $\alpha_M$ is an $A'$-homomorphism and so $(D_m, \alpha_m) \lneqq (D_M, \alpha_M)$, which is a contradiction, by maximality of $(D_m, \alpha_m)$. Therefore, $D_m = B$. □

EXAMPLE 3.8. [0,1] *as $L_2$-module satisfies in the conditions of Theorem 3.7.*

THEOREM 3.9. *Every non cyclic $L_2$-module can be embeded in an injective $L_2$-module.*

PROOF: Let $M$ be a non cyclic $L_2$-module. It is clear that $M \neq 0$ and so there is $0 \neq a \in M$. Consider $A$-ideal $(a]$ of $M$. We define $\alpha : (a] \longrightarrow [0,1]$ by $\alpha(x) = m\frac{p}{q}$, where $\frac{p}{q} \in [0,1]$ and by using of Corollary 2.7,

$$m = min\{n \mid x \leq n(\beta a),\ for\ some\ integer\ n \geq 0\ and\ \beta \in L_2\}$$

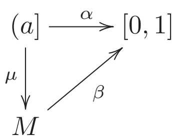
It is easy to see that $\alpha$ is well defined. We show that $\alpha$ is an $MV$-homomorphism. Since $\alpha(0) = 0$, $(H1)$ is true. Let $x_1, x_2 \in (a]$. Then $m_1 = min\{n : x_1 \leq n(\beta a),\ for\ some\ integer\ n \geq 0\ and\ \beta \in L_2\}$ and $m_2 = min\{n : x_2 \leq n(\beta a),\ for\ some\ integer\ n \geq 0\ and\ \beta \in L_2\}$. Let $m = m_1 + m_2$ and $q$ be the smallest common multiple of $m, m_1$ and $m_2$. Then

$$\alpha(x_1 \oplus x_2) = m\frac{p}{q} = (m_1 + m_2)\frac{p}{q} = m_1\frac{p}{q} + m_2\frac{p}{q} = \alpha(x_1) + \alpha(x_2) = \alpha(x_1) \oplus \alpha(x_2)$$

and so $(H2)$ is true. Now, let $\frac{s}{g} \in [0,1]$ and $x \in (a]$. Since $x \leq n(\beta a)$, for some integer $n \geq 0$ and $\beta \in L_2$, by Lemma 2.8 $(b)$ and $(f)$, we have $\frac{s}{g}x \leq \frac{s}{g}(n(\beta a)) = (n\frac{s}{g})(\beta a)$ and so $m = k\frac{s}{g}$, where

$$k = min\{n \mid \frac{s}{g}x \leq n(\frac{s}{g})(\beta a),\ for\ some\ integer\ n \geq 0\ and\ \beta \in L_2\}$$

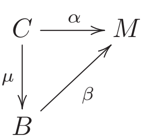
Hence $\alpha(\frac{s}{g}x) = m\frac{p_1}{q_1} = k\frac{s}{g}\frac{p_1}{q_1}$, where $q_1|k$. On the other hand, $\frac{s}{g}\alpha(x) = \frac{s}{g}k\frac{p_1}{q_1}$ and so $(H4)$ is true. Since $M$ is not cyclic, $1 \notin (a]$ and so $x' \notin (a]$, for every $x \in (a]$. It means that $(H3)$ is true. Hence $\alpha$ is an $MV$-homomorphism. If we consider the inclusion map $\mu : (a] \longrightarrow M$, then by Example 3.8 and Theorem 3.7, the following diagram




is commutative, that is $\beta\mu = \alpha$. It is routine to see that $\beta$ is an $A$-monomorphism. Hence $M$ is embeded in an injective $L_2$-module. □

Open Problem. Under what suitable an $A$-module can be embeded in an injective $A$-module?

Theorem 3.10. *Let $A$ be unital, $a.b = b$ implies that $a = 1$, for every $a, b \in A$ and for every $A$-module $C$, every $A'$-homomorphism $\alpha : C \longrightarrow M$ and $A'$-monomorphism $\mu : C \longrightarrow B$ there is an $A$-homomorphism $\beta : B \longrightarrow M$ such that the diagram*




*is commutative, that is $\beta\mu = \alpha$. Then $M$ is an injective $A$-module.*

Proof: Let for every $A$-module $C$ and every $A'$-homomorphism $\alpha : C \longrightarrow M$ and $A'$-monomorphism $\mu : C \longrightarrow B$ there is an $A$-homomorphism $\beta : B \longrightarrow M$ such that $\beta\mu = \alpha$. Also, let $m \in M$ and $0 \neq a \in A$. Consider $\alpha : A \longrightarrow M$ by $\alpha(1) = m$ (or $\alpha(t) = tm$) and $\mu : A \longrightarrow A$ by $\mu(1) = a$

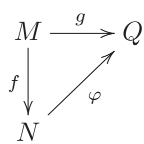
(or $\mu(t) = ta$), for every $t \in A$. It is easy to see that $\alpha$ and $\mu$ are $A'$-homomorphism. Let $x \in ker\mu$. Then $\mu(x) = xa = 0$ and so $x'a \oplus a' = 1$. It means that $a \leq x'a \leq a$ and so $x'a = a$. Hence $x' = 1$ and so $x = 0$. It results that $ker\mu = \{0\}$ and so by Lemma 2.3 $(ii)$, $\mu$ is an $A'$-monomorphism. Then by hypothesis, there is an $A$-homomorphism $\beta : A \longrightarrow M$ such that $\beta\mu = \alpha$. Since $A$ is an $A$-module, we have

$$m = \alpha(1) = \beta\mu(1) = \beta(\mu(1)) = \beta(a) = \beta(a1) = a\beta(1).$$

Now, consider $c = \beta(1)$ and so $M$ is an injective $A$-module. □

EXAMPLE 3.11. *The example 3.4 satisfies in the condition : $a.b = b$ implies that $a = 1$, for every $a, b \in A$ (note that $1_A = 3$).*

LEMMA 3.12. *Every $A'$-homomorphism $f : I \longrightarrow Q$ extends to an $A'$-homomorphism $F : A \longrightarrow Q$, for any ideal $I$ of $A$ if and only if for every $A'$-homomorphisms $f : M \longrightarrow N$ and $g : M \longrightarrow Q$, there is $A$-homomorphism $\varphi : N \longrightarrow Q$ such that the diagram*




*is commutative, that is $\varphi f = g$.*

PROOF: (⇒) Let $\Omega = \{(C, \phi) : M \subseteq C \subseteq N,\ \phi : C \longrightarrow Q,\ \phi|_M = g\}$. A routine application of Zorn's lemma shows that $\Omega$ has a maximal element $(D, \varphi)$. We show that $D = N$ and therefore $\varphi$ would be required extension of $g$. Let $n \in N$. Then by the proof of Theorem 3.7, $I_n = \{a \in A : an \in D\}$ is an ideal of $A$. Define $\alpha : I_n \longrightarrow Q$ by $\alpha(a) = \varphi(an)$. Note that

$$\alpha(a') = \varphi(a'n) = (\varphi(an+n'))' = (\varphi(an)+\varphi(n'))' = (\alpha(a)+(\alpha(1))')' = (\alpha(a))'.$$

Hence $(H')$ is true. The proof of $(H1)$, $(H3)$ and $(H4)$ are routine. Then $\alpha$ is an $A'$-homomorphism and so $\alpha$ extends to $A'$-homomorphism $\beta : I_n \longrightarrow Q$. Define $\varphi' : D \oplus An \longrightarrow Q$ by $\varphi'(d \oplus an) = \varphi(d) \oplus \beta(a)$, for every $d \in D$ and $a \in A$. Since $\beta(a) = \alpha(a) = \varphi(an)$, for every $a \in I_n$ and $\beta(a) = \phi(an)$, for every $a \in I_n$, we conclude that $\varphi'$ is well defined. It is routine to see that $\varphi'$ is an $A'$-homomorphism. Since $(D, \varphi) \leq (D \oplus An, \varphi')$, by maximality $(D, \varphi)$, we have $D = D \oplus An$ and so $D = N$.

(⇐) The proof is clear. □

Theorem 3.13. *Let $A$ be unital, all ideals of $A$ be principle and $a.b = 1$ implies that $a = 1$, for every $a, b \in A$. Then $M$ is an injective $A$-module.*

Proof: Let $I$ be an ideal of $A$ and $f : I = \prec a \succ \longrightarrow M$ be an $A'$-homomorphism. Define $F : A \longrightarrow M$ by $F(x) = f(x.a)$. It is clear that $F$ is well defined. We show that $F$ is an $A'$-homomorphism. The proofs of $(H_1)$ and $(H_2')$ are routine. We have

$$F(x') = f(x'.a) = (f(x.a + a'))' = (f(x.a) + f(a'))' = \\ = (F(x) + (f(a))')' = (F(x) + (F(1))')' = (F(x))'.$$

Therefore, $F$ is an $A'$-homomorphism and so by Lemma 3.12 and Theorem 3.10, $M$ is an injective $A$-module. □

## 4. Essential extensions

In this section, we define the notions of essential $A$-homomorphisms and essential extension of an $A$-homomorphism, where $A$ is a $PMV$-algebra and we obtain more results on them. Then by these notions, we obtain some results on injective $MV$-modules.

Definition 4.1. *Let $\mu : M \longrightarrow B$ be an $A'$-monomorphism such that $\mu(M) \cap H \neq \{0\}$, for every no zero $A$-ideal $H$ of $B$. Then $\mu$ is called an essential $A$-homomorphism. In special case, if $M$ is an $A$-ideal of $B$ ($\mu$ is inclusion map), then $B$ is called an essential extension of $\mu$.*

Proposition 4.2. *[9] Let $A$ be a $PMV$-algebra. Then $\Sigma_{i=1}^{n} A$ is a $PMV$-algebra.*

Example 4.3. *By Proposition 4.2, $A \oplus A$ is an $MV$-algebra. If operation $\bullet : A \times (A \oplus A) \longrightarrow (A \oplus A)$ is defined by $a \bullet (b, c) = (a.b, a.c)$, for every $a, b, c \in A$, then it is easy to show that $A \oplus A$ is an $A$-module. consider $A = L_2$ and $\phi : A \oplus A \longrightarrow L_4$, where $\phi(1, 0) = \frac{1}{3}$, $\phi(0, 1) = \frac{2}{3}$, $\phi(0, 0) = 0$ and $\phi(1, 1) = 1$. Then it is clear that $\phi$ is well defined. It is easy to show that $\phi$ is an $A'$-homomorphism. Since $\phi(L_2 \oplus L_2) = L_4$, $\phi$ is an essential $A$-homomorphisms.*

Theorem 4.4. *Let $M$ be an $A$-module and $B$ be an $A$-ideal of $M$. Then $M$ is an essential extension of $B$ if and only if for every $0 \neq b \in M$, there exist $a \in A$ and $c \in B$ such that $c \leq n(ab)$, for some integer $n$.*

Proof: ($\Rightarrow$) Let $M$ be an essential extension of $B$ and $0 \neq b \in M$. Then $H = (b]$ is a non zero $A$-ideal of $M$ and so $B \cap H \neq \{0\}$. It results that there

exists $0 \neq c \in M \cap H$. Since $c \in H$, there is $a \in A$ such that $c \leq n(ab)$, for some integer $n$.
($\Leftarrow$) Let for every $0 \neq b \in M$, there exists $a \in A$ and $c \in B$ such that $c \leq n(ab)$, for some integer $n$. Also, let $H$ be a non zero $A$-ideal of $M$. Then there is $0 \neq b \in H$ such that $c \leq n(ab) \in H$ and so $c \in H$. Hence $B \cap H \neq \{0\}$ and so $B$ is an essential extension of $B$. □

Proposition 4.5. *Let $M$ be an $A$-module and $B$ be a non zero $A$-ideal of $M$. Then there is a maximal essential extension $E$ of $B$ such that $B \subseteq E \subseteq M$.*
Proof: Let

$$K = \{C_i \mid C_i \textit{ is an } A-\textit{ideal of } M \textit{ that is an essential extension of } B\}$$

Since $B \in K$, $K \neq 0$. For every chain $\{C_i\}_{i\in I}$ of elements of $K$, $C = \bigcup_{i\in I} C_i$ is an $A$-ideal of $M$. Now, let $b \in B$. Since $C_i$ is an essential extension of $B$, there are $a \in A$ and $c \in C_i$ such that $c \leq n(ab)$, for every $i \in I$ and for some integer $n$. Hence, for every $b \in B$, there are $a \in A$ and $c \in C$ such that $c \leq n(ab)$ and so by Theorem 4.4, $C$ is an essential extension of $B$. Now, by Zorn's Lemma, $K$ has a maximal elements as $E$ that is essential extension of $B$ inclusion in $M$. □

In the follow, we will show that every maximal essential extension of an $A$-ideal of injective $A$-module $I$ is injective if it was included in $I$. The first we prove the following lemma that we call the short five lemma and its corollaries in $MV$-modules:
Definition 4.6. *Let $\{M_i\}_{i\in I}$ be a family of $A$-modules and $\{f_i : M_i \to M_{i+1} : i \in I\}$ be a family of $A$-module homomorphism. Then*

$$\cdots \to M_{i-1} \overset{\mathrm{f_{i-1}}}{\to} M_i \overset{\mathrm{f_i}}{\to} M_{i+1} \to \cdots$$

*is exact if $Imf_i = Kerf_{i+1}$, for every $i \in I$. In special case,*

$$0 \to M_1 \overset{\mathrm{f_1}}{\to} M_2 \overset{\mathrm{g_1}}{\to} M_3 \to 0$$

*is called a short exact sequence.*
Example 4.7. (*i*) *Let $M$ be an $A$-module and $N$ be an $A$-ideal of $M$. Then*

$$0 \to N \overset{\subseteq}{\to} M \overset{\pi}{\to} \frac{M}{N} \to 0$$

*is a short exact sequence.*
(*ii*) *Let $f : M_1 \to M_2$ be an $A$-module homomorphism. Then*

$$0 \to Kerf \overset{\subseteq}{\to} M_1 \overset{\pi}{\to} \frac{M_1}{Kerf} \to 0$$

*is a short exact sequence.*

LEMMA 4.8. $(i)$ *Let*

$$0 \to A_1 \overset{f_1}{\to} B_1 \overset{g_1}{\to} C_1 \to 0$$

*and*

$$0 \to A_2 \overset{f_2}{\to} B_2 \overset{g_2}{\to} C_2 \to 0$$

*be two exact sequences of A-modules,* $\alpha : A_1 \to A_2$ *and* $\gamma : C_1 \to C_2$ *be A-isomorphism,* $\beta : B_1 \to B_2$ *be an A-homomorphism,* $\beta \circ f_1 = f_2 \circ \alpha$ *and* $\gamma \circ g_1 = g_2 \circ \beta$. *Then* $\beta$ *is an A-isomorphism.*
$(ii)$ *For the short exact sequence*

$$0 \to A_1 \overset{f}{\to} B \overset{g}{\to} A_2 \to 0$$

*of A-modules, if there is an A-homomorphism* $k : B \to A_1$ *such that* $kf = I$ *(I is identity map), then* $B \simeq A_1 \oplus A_2$, *where* $A_1 \oplus A_2 = \{a_1 \oplus a_2 : a_1 \in A_1, a_2 \in A_2\}$ *( we say* $0 \to A_1 \to^{f} B \to^{g} A_2 \to 0$ *is split exact).*
$(iii)$ *If* $J$ *is a unitary A-module, then* $J$ *is an injective A-module if and only if every short exact sequence*

$$0 \to J \to T \to B \to 0$$

*of A-modules is split exact.*

PROOF: $(i)$ It is routine to see that $\beta$ is an $A$-monomorphism. We show that $\beta$ is an $A$-epimorphism. Consider arbitrary element $x \in B_2$. Then $g_2(x) \in C_2$ and so there is $z \in C_1$ such that $\gamma(z) = g_2(x)$. Since $g_1$ is an $A$-epimorphism, there is $b_1 \in B_1$ such that $g_1(b_1) = z$ and so $\gamma g_1(b_1) = g_2(x)$. It results that $g_2\beta(b_1) = g_2(x)$ and so by Lemma 2.3, $\beta(b_1) \ominus x \in Kerg_2 = Imgf_2$. Hence there is $a \in A_2$ such that $f_2(a) = \beta(b_1) \ominus x$. Since $a \in A_2$, there is $d \in A_1$ such that $\alpha(d) = a$ and so $f_2\alpha(d) = \beta(b_1) \ominus x$. It results that $\beta(f_1(d)) = \beta(b_1) \ominus x$. Now, let $y = b_1 \ominus f_1(d)$. Then

$$\beta(y) = \beta((b_1' \oplus f_1(d))') = (\beta(b_1' \oplus f_1(d)))' = (\beta(b_1') \oplus \beta(f_1(d)))' = (\beta(b_1') \oplus \beta(b_1) \ominus x)' = (1 \ominus x)' = x.$$

Therefore, $\beta$ is an $A$-epimorphism and so $\beta$ is an $A$-isomorphism.
$(ii), (iii)$ The proofs are routine. □

THEOREM 4.9. *Let $I$ be an injective $A$-module, $B$ be an $A$-ideal of $I$ and $E$ be a maximal essential extension of $B$ such that $E \subseteq I$. Then $E$ is an injective $A$-module.*

PROOF: Let

$$D = \{H : \ H \ is \ an \ A - ideal \ of \ I, H \cap E = \{0\}\}$$

Since $\{0\} \in D$, we have $D \neq \emptyset$. By Zorn's Lemma, $D$ has maximal element $H'$. Then $H' \cap E = \{0\}$. Now, consider the mapping $\pi : I \longrightarrow \frac{I}{H'}$. If $\delta = \pi \mid_E$, then $\delta$ is an $A$-monomorphism. We show that $\delta$ is an essential monomorphism. Consider $A$-ideal $\frac{K}{H'}$ of $\frac{I}{H'}$, where $H' \subset K$ (It is not possible $K = H'$). Then there is $0 \neq b \in K \cap E$ and $b \notin H'$ and so $\delta(b) = \frac{b}{H'} \neq \frac{0}{H'}$. It means that $\delta(E) \cap \frac{K}{H'} \neq \{0\}$ and so $\delta$ is an essential extension of $E$. Since $E$ can not accept any essential $A$-monomorphism except trivial $A$-monomorphism, $\delta : E \longrightarrow \frac{I}{E'}$ is an $A$-isomorphism. Now, consider the exact sequence

$$0 \to H' \overset{\subseteq}{\to} I \overset{\delta^{-1}\pi}{\to} E \to 0$$

If $f : E \longrightarrow I$ be conclusion mapping, then $\delta^{-1}\pi f(a) = \delta^{-1}\pi(a) = \delta^{-1}(\frac{a}{H'}) = a$, for every $a \in I$. Hence $\delta^{-1}\pi f = I_E$ and so by Lemma 4.8 $(iii)$, the above sequence is a split exact sequence. It results that $I \simeq E \oplus H'$. Since $I$ is an injective $A$-module, $E$ is an injective $A$-module, too. □

## 5. Conclusion

The categorical equivalence between $MV$-algebras and $lu$-groups leads to the problem of defining a product operation on $MV$-algebras, in order to obtain structures corresponding to $l$-rings. In fact, by defining $MV$-modules, $MV$-algebras were extended. Hence, $MV$-modules are fundamental notions in algebra. IN 2016, free $MV$-modules were defined [10]. We introduced injective $MV$-modules and obtained some essential properties in this field. The obtained results encourage us to continue this long way. It seems that one can introduces notion of projective $MV$-module and obtain the relationship between free $MV$-modules and projective (or injective) $MV$-modules. In fact, there are many questions in this field that should be verified.

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Department of Mathematics,
Farhangian University,
Kerman, Iran
e-mail: siminsaidi@yahoo.com

Department of Mathematics,
Shahid Beheshti University,
Tehran, Iran
e-mail: borzooei@sbu.ac.ir