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Chapter 6
Convergence of sequences of measurable functions

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6.1 Introduction

Let $\mathcal{S}$ be a σ-algebra of subsets of $\mathbb{R}$ and $\mathcal{I}$ – a proper σ-ideal included in $\mathcal{S}$. We shall say that some property holds $\mathcal{I}$-a.e. ($\mathcal{I}$-almost everywhere) if and only if the set of points for which this property does not hold belongs to $\mathcal{I}$. In this chapter we shall consider extended real functions defined $\mathcal{I}$-a.e. and finite $\mathcal{I}$-a.e. on $[0,1]$. The σ-algebra will be usually the family $\mathcal{L}$ of Lebesgue measurable sets or $\mathcal{B}$ – the family of sets having the Baire property. The σ-ideal associated with $\mathcal{L}$ will be the σ-ideal $\mathcal{N}$ of null sets in $\mathbb{R}$ and σ-ideal associated with $\mathcal{B}$ will be the σ-ideal $\mathcal{M}$ of first category sets (meager sets). The σ-algebra $\mathcal{B}or$ of Borel sets will not be used since the Lebesgue measure restricted to $\mathcal{B}or$ is not complete.

We shall say that two $\mathcal{I}$-a.e. finite extended real-valued functions f,g defined $\mathcal{I}$-a.e. on $[0,1]$ are equivalent if and only if $f(x)=g(x)$ $\mathcal{I}$-a.e. on $[0,1]$, i.e.

$$\{x\in[0,1]: f(x)\neq g(x)\}\in\mathcal{I}.$$

In this case we shall write $f \sim_{\mathcal{I}} g$.

Let $\mathcal{F}_{\mathcal{I}}$ be the family of all extended real-valued functions defined $\mathcal{I}$-a.e. on $[0,1]$, finite $\mathcal{I}$-a.e. and measurable with respect to $\mathcal{S}$. Denote by $\mathcal{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$ the quotient space. If $[f] \in \mathcal{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$, then there exists an $\mathcal{S}$-measurable real function g defined and finite everywhere on $[0,1]$ such that $g \sim_{\mathcal{I}} f$. Indeed, it is sufficient to put $g(x) = 0$ if $|f(x)| = +\infty$ or $f(x)$ is not defined and $g(x) = f(x)$ at remaining points. Therefore in the sequel we shall usually assume that all functions under considerations are finite-valued and defined everywhere on $[0,1]$. We shall consider different types of convergence of sequences of elements of $\mathcal{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$ using the symbols of functions rather than of equivalence classes. Also the limit of the sequence will be written as a function (although it will be always determined up to the equivalence). So we shall write $\lim_{n\to\infty} f_n = f$ $\mathcal{I}$-a.e. rather than $\lim_{n\to\infty}[f_n] = [f]$.

6.2 Convergence almost everywhere

Suppose now that $\mathcal{S} = \mathcal{L}$ and $\mathcal{I} = \mathcal{N}$, so we shall deal with Lebesgue measurable functions. Usually we shall write a.e. instead of $\mathcal{N}$-a.e. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued Lebesgue measurable functions defined on $[0,1]$ convergent a.e. to a function f (not necessarily a.e. finite), then it is well known that f is also Lebesgue measurable. Recall the theorem of D. Egorov (see, for example [2], p. 184 or [8], p. 143).

Theorem 6.1. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued Lebesgue measurable functions convergent a.e. to a.e. finite function f, then for each $\varepsilon > 0$ there exists a set $E_\varepsilon \in \mathcal{L}$, $E_\varepsilon \subset [0,1]$ such that $\lambda(E_\varepsilon) < \varepsilon$ and the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f on $[0,1] \setminus E_\varepsilon$.*

Remark 6.2. Obviously we can take E_ε as an open set. However, if $f_n(x) = x^n$ for $x \in [0,1]$, then it is not possible to find a set E_0 such that $\lambda(E_0) = 0$ and the convergence is uniform on $[0,1] \setminus E_0$. So the theorem of Egorov cannot be improved in this direction.

It can be reformulated in the following way:

Theorem 6.3. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued Lebesgue measurable functions convergent a.e. to a.e. finite function f, then there exists a sequence $\{A_m\}_{m\in\mathbb{N}}$ of sets from $\mathcal{L}$ such that $\lambda([0,1] \setminus \bigcup_{m=1}^{\infty} A_m) = 0$ and the convergence is uniform on each A_m.*

It is evident that Theorem 6.1 implies Theorem 6.3. Suppose now that the conclusion of Theorem 6.3 holds and take $\varepsilon > 0$. There exists $m_0 \in \mathcal{N}$ such that $\lambda([0,1] \setminus \bigcup_{m=1}^{m_0} A_m) < \varepsilon$ and the convergence on $\bigcup_{m=1}^{m_0} A_m$ is uniform, so Theorem 6.3 implies Theorem 6.1.

Remark 6.4. Observe that the theorem is not true without the assumption of measurability. Indeed, let $\{E_m\}_{m\in\mathbb{N}}$ be a sequence of pairwise disjoint subsets of $[0,1]$ such that $\lambda^*(E_m) = 1$ for each $m \in \mathbb{N}$ and $\bigcup_{m=1}^{\infty} E_m = [0,1]$. Put $f_n = \chi_{\bigcup_{i=n+1}^{\infty} E_i}$. It is easy to see that $f_n(x) \underset{n\to\infty}{\rightarrow} 0$ for each $x \in [0,1]$. If $A \subset [0,1]$ is a measurable set such that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on A to zero, then $A \subset \bigcup_{i=1}^{m} E_i$ for some $m \in \mathbb{N}$ and $\lambda(A) = 0$, since $\lambda_*(\bigcup_{i=1}^{m} E_i) = 0$ (where λ_* is an inner Lebesgue measure).

6.3 Convergence $\mathcal{I}$-a.e.

We shall generalize the notion of convergence in the following way: Suppose that $(X,\mathcal{S})$ is a measurable space, i.e. X is a non-empty set and $\mathcal{S}$ is a σ-algebra of subsets of X. Suppose also that $\mathcal{I} \subset \mathcal{S}$ is a proper σ-ideal of sets.

Definition 6.5 (see [10] or [12]). We shall say that a pair $(\mathcal{S},\mathcal{I})$ fulfills the condition (E) if for every set $D \in \mathcal{S} \setminus I$ and for each double sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of subsets belonging to $\mathcal{S}$ and satisfying the conditions: $B_{j,n} \subset B_{j,n+1}$ for each $j,n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} B_{j,n} = D$ for each $j \in \mathbb{N}$ there exists an increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and a sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $\bigcap_{p=1}^{\infty} B_{j_p,n_p} \notin \mathcal{I}$.

Definition 6.6. We shall say that a pair $(\mathcal{S},\mathcal{I})$ fulfills the countable chain condition (ccc) if every pairwise disjoint family of sets from $\mathcal{S} \setminus \mathcal{I}$ is at most denumerable.

Definition 6.7. We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$-measurable functions defined on X converges to a real-valued function f defined on X in the sense of Egorov if there exists a sequence $\{A_m\}_{m\in\mathbb{N}}$ of sets from $\mathcal{S}$ such that $X \setminus \bigcup_{m=1}^{\infty} A_m \in \mathcal{I}$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on each A_m, $m \in \mathbb{N}$.

Theorem 6.8 (see [12]). *Suppose that the pair $(\mathcal{S},\mathcal{I})$ fulfills ccc. Then the convergence $\mathcal{I}$-a.e. of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of a real-valued $\mathcal{S}$-measurable functions defined on X to a real-valued function f defined on X implies the convergence of $\{f_n\}_{n\in\mathbb{N}}$ to f in the sense of Egorov if and only if the pair $(\mathcal{S},\mathcal{I})$ fulfills the condition (E).*

Obviously the following theorem is true.

Theorem 6.9. *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued functions defined on X converges to a real-valued function f defined on X in the sense of Egorov, then $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{I}$-a.e. to f.*

Remark 6.10. Observe that in the above theorem the assumption of $\mathcal{S}$-measurability of f_n, $n \in \mathbb{N}$, is not necessary.

Now we shall present related kinds of convergence of sequences of functions.

Definition 6.11 (compare [8], p. 141 or [7]). We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$–measurable functions defined on X converges to a real-valued function f defined on X in the sense of Taylor if there exists a non-decreasing sequence $\{t_n\}_{n\in\mathbb{N}}$ tending to $+\infty$ such that the sequence $\{t_n \cdot (f_n - f)\}_{n\in\mathbb{N}}$ converges $\mathcal{I}$-a.e. to zero.

Definition 6.12 (compare [8], p. 141). We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$–measurable functions defined on X converges to a real-valued function f defined on X with the convergence regulator if there exists a non-negative extended real-valued function g defined on X and a sequence of positive numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ convergent to zero such that $|f_n(x) - f(x)| \le \alpha_n \cdot g(x)$ for each $n \in \mathbb{N}$ and $x \in X$.

Definition 6.13 (compare [12]). We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$–measurable functions defined on X converges to a real-valued function f defined on X in the sense of Yoneda if there exists a non-negative extended real-valued $\mathcal{S}$-measurable function d defined on X such that for each $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon \cdot d(x)$ for each $n > n(\varepsilon)$ and $x \in X$.

Theorem 6.14. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued $\mathcal{S}$-measurable functions defined on X and let f be a real-valued function defined on X. The following statements are equivalent:*
(a) $\{f_n\}_{n\in\mathbb{N}}$ *converges to f in the sense of Egorov,*
(b) $\{f_n\}_{n\in\mathbb{N}}$ *converges to f in the sense of Taylor,*
(c) $\{f_n\}_{n\in\mathbb{N}}$ *converges to f with the convergence regulator,*
(d) $\{f_n\}_{n\in\mathbb{N}}$ *converges to f in the sense of Yoneda.*

Proof. (a) $\Longrightarrow$ (b). Let $\{A_m\}_{m\in\mathbb{N}}$ be a sequence of sets from $\mathcal{S}$ such that $X \setminus \bigcup_{m=1}^{\infty} A_m \in \mathcal{I}$ and for each $m \in \mathbb{N}$ the convergence is uniform on A_m. For each $m \in \mathbb{N}$ choose an increasing sequence $\{n_{m,r}\}_{r\in\mathbb{N}}$ of positive integers such that $|f_n(x) - f(x)| < \frac{1}{r+1}$ for each $x \in A_m$ and each $n \geq n_{m,r}$. Let $\{n_m\}_{m\in\mathbb{N}}$ be an increasing sequence of positive integers such that $\lim_{r\to\infty} \frac{n_r}{n_{m,r}} = +\infty$ for each $m \in \mathbb{N}$ (for example $n_r = r - \max(n_{1r}, n_{2r}, \dots, n_{rr})$). Put

$$t_n = \begin{cases} 1 & \text{for } 1 \leq n < n_1, \\ \sqrt{r} & \text{for } n_{r-1} \leq n < n_r, \quad (r = 2, 3, \dots), \end{cases}$$

If $x \in \bigcup_{m=1}^{\infty} A_m$, then $x \in A_m$ for some $m \in \mathbb{N}$. Then there exists r_0 such that $n_r \geq n_{m,r}$ for $r \geq r_0$. If $n \geq n_{r_0}$, then there exists $r \geq r_0$ such that $n_r \leq n < n_{r+1}$. Hence for such n we have $|f_n(x) - f(x)| < \frac{1}{r+1} = \frac{1}{t_n^2}$, so $t_n \cdot |f_n(x) - f(x)| < \frac{1}{t_n}$. Finally $\{t_n \cdot (f_n - f)\}_{n\in\mathbb{N}}$ converges to zero on $\bigcup_{m=1}^{\infty} A_m$, which means $\mathcal{I}$-a.e. on X.

(b) $\implies$ (c) Put $g(x) = \sup_n t_n |f_n(x) - f(x)|$ for $x \in X$ and $\alpha_n = \frac{1}{t_n}$ for $n \in \mathbb{N}$. Obviously g is $\mathcal{S}$-measurable. By virtue of (b) g is $\mathcal{I}$-a.e. finite and the inequality $|f_n(x) - f(x)| \leq \frac{1}{t_n} \cdot g(x)$ is obvious.

(c) $\implies$ (d) Put $d(x) = g(x)$. Take $\varepsilon > 0$. There exists $n(\varepsilon)$ such that $\alpha_n < \varepsilon$ for $n > n(\varepsilon)$. Then, obviously, $|f_n(x) - f(x)| < \varepsilon \cdot d(x)$ for $n > n(\varepsilon)$ and $x \in X$.

(d) $\implies$ (a) Put $A_m = \{x \in X : d(x) \leq m\}$. Then we have $X \setminus \bigcup_{m=1}^{\infty} A_m \in \mathcal{I}$ and $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on each A_m. □

Theorem 6.15. *The pair* $(\mathcal{L}, \mathcal{N})$ *fulfills the condition* (E).

Proof. Take $D \in \mathcal{L} \setminus \mathcal{N}$ and such that $\lambda(D) < \infty$. Let $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ be a double sequence fulfilling both conditions. Put $j_p = p$ for each $p \in \mathbb{N}$ and choose n_p such that $\lambda(D \setminus B_{p,n_p}) < \frac{1}{3^p}\lambda(D)$. Then $\lambda(\bigcap_{p=1}^{\infty} B_{p,n_p}) > \frac{1}{2}\lambda(D) > 0$. □

Remark 6.16. If we choose n_p such that $\lambda(D \setminus B_{p,n_p}) < \frac{\varepsilon}{2^p}$ for $p \in \mathbb{N}$, where ε is fixed, then we obtain $\lambda(\bigcap_{p=1}^{\infty} B_{p,n_p}) > \lambda(D) - \varepsilon$. This choice is used when proving Egorov's theorem.

Theorem 6.17. *The pair* $(\mathcal{B}, \mathcal{M})$ *does not fulfill the condition* (E).

Proof. Let Q be the set of all rational numbers in $[0,1]$. Put $D = [0,1] \setminus Q$ and let $B_{j,n} = [0,1] \setminus (Q \cup A_{j,n})$, where $A_{j,n} = [0,1] \cap \bigcup_{i=1}^{j-1} (\frac{i}{j} - \frac{1}{n}, \frac{i}{j} + \frac{1}{n})$, for $n \in \mathbb{N}$, $j \in \mathbb{N} \setminus \{1\}$. If $\{j_p\}_{p\in\mathbb{N}}$ is an arbitrary increasing sequence of positive integers and $\{n_p\}_{p\in\mathbb{N}}$ is an arbitrary sequence of positive integers, then $\bigcap_{p=1}^{\infty} B_{j_p,n_p} = [0,1] \setminus (Q \cup \bigcup_{p=1}^{\infty} A_{j_p,n_p}) \in \mathcal{M}$, because $\bigcup_{p=1}^{\infty} A_{j_p,n_p}$ is an open set dense in $[0,1]$. □

6.4 Convergence in measure

Definition 6.18. We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions defined on $[0,1]$ converges in measure to a real-valued measurable function f defined on $[0,1]$ if

$$\lambda(\{x\in[0,1]:|f_n(x)-f(x)|>\varepsilon\}) \underset{n\to\infty}{\to} 0 \quad \text{for each } \varepsilon>0.$$

Recall well known facts:

Theorem 6.19. *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions defined on $[0,1]$ converges a.e. to a real-valued function f defined on $[0,1]$, then $\{f_n\}_{n\in\mathbb{N}}$ converges to f in measure.*

Remark 6.20. It is essential that $\lambda([0,1])<\infty$. In fact, the theorem holds in all finite measure spaces. If we take a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued functions defined on $[0,\infty)$ by the formula $f_n=\chi_{[n,+\infty)}$, $n\in\mathbb{N}$, then $f_n(x) \underset{n\to\infty}{\to} 0$ for each $x\in[0,\infty)$ and simultaneously $\{f_n\}_{n\in\mathbb{N}}$ does not converge to zero in measure. The assumption of measurability is also essential. Indeed, the sequence of functions from Remark 6.4 converges to zero everywhere without converging in measure.

Theorem 6.21 (F. Riesz, compare [4], Theorem 11.26). *If a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions defined on $[0,1]$ converges in measure to a real-valued measurable function f defined on $[0,1]$, then there exists an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers such that $\{f_{n_m}\}_{m\in\mathbb{N}}$ converges a.e. to f.*

The above theorem has a very nice version which will be useful for us:

Theorem 6.22. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued measurable functions defined on $[0,1]$ converges in measure to a real-valued measurable function f defined on $[0,1]$ if and only if for each subsequence $\{f_{n_m}\}_{m\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ which converges a.e. to f.*

Proof. Necessity follows immediately from the theorem of Riesz and from the obvious fact that convergence in measure is preserved by subsequences.

To prove the sufficiency suppose that $\{f_n\}_{n\in\mathbb{N}}$ does not converge to f in measure. Then there exist a pair of positive numbers ε and δ and an increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers such that $\lambda(\{x\in[0,1]:|f_{n_m}(x)-f(x)|>\varepsilon\})>\delta$ for each $m\in\mathbb{N}$. If $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ is a subsequence of $\{f_{n_m}\}_{m\in\mathbb{N}}$ convergent a.e. to f, then by virtue of Theorem 6.19 $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ converges to f

in measure. Hence $\lim_{p\to\infty}\lambda(\{x\in[0,1]:|f_{n_{m_p}}(x)-f(x)|>\varepsilon\})=0$, a contradiction. □

In the last theorem there is proved that the convergence in measure can be defined without measure, only using the notion of set of measure zero. It leads us to the notion of convergence with respect to the σ-ideal $\mathcal{I}$, which is obtained by changing the convergence a.e. (except on a set of measure zero) with the convergence $\mathcal{I}$-a.e., i.e. except on a set belonging to $\mathcal{I}$. □

6.5 Convergence with respect to the σ-ideal

Suppose again that $(X,\mathcal{S})$ is a measurable space and $\mathcal{I}\subset\mathcal{S}$ is a σ-ideal.

Definition 6.23 (see [10]). We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$-measurable functions defined on X converges to a real-valued $\mathcal{S}$-measurable function f defined on X with respect to the σ-ideal $\mathcal{I}$ if for each subsequence $\{f_{n_m}\}_{m\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{f_{n_{m_p}}\}_{p\in\mathbb{N}}$ which converges $\mathcal{I}$-a.e. to f. We shall use the denotation $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$.

Remark 6.24. If $f_n \underset{n\to\infty}{\to} f$ $\mathcal{I}$-a.e., then $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$. Obviously, the limit function with respect to $\mathcal{I}$ is determined up to equivalent functions.

It is not difficult to see that the following conditions are fulfilled (we shall formulate all conditions in terms of functions rather than of elements of the quotient space):

(L1) If $f_n=f$ for each $n\in\mathbb{N}$, then $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$;

(L2) If $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$, then $f_{n_m} \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$ for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers.

(L3) If the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not converge to f with respect to $\mathcal{I}$, then there exists a subsequence $\{f_{n_m}\}_{m\in\mathbb{N}}$, no subsequence of which converges to f with respect to $\mathcal{I}$.

So the family of real-valued $\mathcal{S}$-measurable functions defined on X (or, more precisely, the quotient space$\mathcal{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$) equipped with the convergence with respect to $\mathcal{I}$ is an $\mathcal{L}^*$ space (see [3], p. 90). In such a space one can define the closure operation assuming that f belongs to $\mathrm{cl}(A)$ if and only if there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions from A such that $f_n \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f$ (or, equivalently, there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions from A such that $f_n \underset{n\to\infty}{\to} f$ $\mathcal{I}-$a.e.).

This operation has always the following properties: $\mathrm{cl}(\emptyset) = \emptyset$, $A \subset \mathrm{cl}(A)$, $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for each A, B but the condition $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ need not hold. To assure this last equality it is necessary and sufficient that the following condition holds (see [3], p. 90):

(L4) If $\{f_j\}_{j\in\mathbb{N}}$, $\{f_{j,n}\}_{j,n\in\mathbb{N}}$ consist of real-valued $\mathcal{S}$-measurable functions defined on X and f is a real-valued $\mathcal{S}$-measurable function defined on X such that $f_j \underset{j\to\infty}{\overset{\mathcal{I}}{\to}} f$, $f_{j,n} \underset{n\to\infty}{\overset{\mathcal{I}}{\to}} f_j$ for each $j \in \mathbb{N}$, then there exist two sequences $\{j_p\}_{p\in\mathbb{N}}$, $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $f_{j_p,n_p} \underset{p\to\infty}{\overset{\mathcal{I}}{\to}} f$.

After a moment of reflection, choosing the subsequence from the column of limit functions and subsequences of the rows of functions, we can observe that the above condition is equivalent to the following one:

(L4)' If $\{f_j\}_{j\in\mathbb{N}}$, $\{f_{j,n}\}_{j,n\in\mathbb{N}}$ and f are as above and $f_j \underset{j\to\infty}{\to} f$ $\mathcal{I}$-a.e. and $f_{j,n} \underset{n\to\infty}{\to} f_j$ $\mathcal{I}$-a.e. for each $j \in \mathbb{N}$, then there exist two sequences $\{j_p\}_{p\in\mathbb{N}}$ and $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $f_{j_p,n_p} \underset{p\to\infty}{\to} f$ $\mathcal{I}$-a.e.

Remark 6.25. Observe that the condition (E) is equivalent to the condition (E') which requires that $\bigcap_{p=1}^{\infty} B_{j_p,n_p} \notin \mathcal{I}$ holds for $j_p = p$ for $p \in \mathbb{N}$.

Theorem 6.26 ([10], Theorem 1). *Suppose that the pair $(\mathcal{S},\mathcal{I})$ fulfills ccc. Then $\mathcal{F}_{\mathcal{I}}/\sim_{\mathcal{I}}$ is equipped with the Frechet topology generated by the convergence with respect to $\mathcal{I}$ (i.e. the closure operator fulfills all axioms of Kuratowski) if and only if the pair $(\mathcal{S},\mathcal{I})$ fulfills the condition* (E).

Below, we shall present several characterizations of the convergence $\mathcal{I}$-a.e. and the convergence with respect to the σ-ideal $\mathcal{I}$.

Theorem 6.27. *Suppose that the pair $(\mathcal{S},\mathcal{I})$ fulfills ccc. Then the sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$-measurable functions defined on X converges $\mathcal{I}$-a.e. to a real-valued $\mathcal{S}$-measurable function f defined on X if and only if for each $\varepsilon > 0$ the sequence $\{h_n^{\varepsilon}\}_{n\in\mathbb{N}}$ of functions defined in the following way: $h_n^{\varepsilon} = \chi_{E(n,\varepsilon)}$, where $E(n,\varepsilon) = \{x \in X : |f_n(x) - f(x)| > \varepsilon\}$, converges to zero with respect to σ-ideal $\mathcal{I}$.*

Theorem 6.28 (compare Lemma 4 in [10]). *Suppose that the pair $(\mathcal{S},\mathcal{I})$ fulfills ccc. Then the sequence $\{f_n\}_{n\in\mathbb{N}}$ of real-valued $\mathcal{S}$-measurable functions defined on X converges to zero with respect to $\mathcal{I}$ if and only if the following conditions are fulfilled:*

1. *for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers, for each set $D \in \mathcal{S}\setminus\mathcal{I}$ and for each $\varepsilon > 0$ there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ and a set $B \subset D$, $B \in \mathcal{S}\setminus\mathcal{I}$ such that* $\limsup_p f_{n_{m_p}}(x) < \varepsilon$ *for each $x \in B$.*
2. *for each increasing sequence $\{n_m\}_{m\in\mathbb{N}}$ of positive integers, for each set $D \in \mathcal{S}\setminus\mathcal{I}$ and for each $\varepsilon > 0$ there exists a subsequence $\{n_{m_p}\}_{p\in\mathbb{N}}$ and a set $B \subset D$, $B \in \mathcal{S}\setminus\mathcal{I}$ such that* $\liminf_p f_{n_{m_p}}(x) > -\varepsilon$ *for each $x \in B$.*

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued $\mathcal{S}$-measurable functions defined on X and f – a real-valued $\mathcal{S}$-measurable function defined on X. Put

$$E_n(\alpha) = \bigcup_{i=n}^{\infty}\{x \in X : |f_i(x) - f(x)| > \alpha\}.$$

Definition 6.29 (see [11]). We shall say that the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the vanishing restriction with respect to f if and only if $\bigcap_{n=1}^{\infty} E_n(\alpha) \in \mathcal{I}$ for all $\alpha > 0$.

Put $\phi_n(x) = \sup\{|f_i(x) - f(x)| : i \geq n, i \in \mathbb{N}\}$ for $n \in \mathbb{N}$.

Theorem 6.30 ([11], Theorem 2). *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued $\mathcal{S}$-measurable functions defined on X and f – a real-valued $\mathcal{S}$-measurable function defined on X, then the following conditions are equivalent:*

(i) the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f $\mathcal{I}$-a.e. on X;
(ii) the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the vanishing restriction with respect to f;
(iii) the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to zero with respect to $\mathcal{I}$.

Remark 6.31. As the pair $(\mathcal{L},\mathcal{N})$ fulfills ccc, it follows from Theorem 6.15 and Theorem 6.26 that the convergence in measure in a finite measure space yields the topology in $\mathcal{F}_{\mathcal{N}}/\sim\mathcal{N}$. This topology is metrizable (see for example [4], p. 182-183). Theorems 6.17 and 6.26 imply that the convergence in category (i.e. with respect to $\mathcal{M}$) does not generate a topology.

Remark 6.32. If $(X,\mathcal{S})$ is a measurable space and $\mathcal{I} \subset \mathcal{S}$ is a maximal σ-ideal, then the countable chain condition is fulfilled (every disjoint family in $\mathcal{S}\setminus\mathcal{I}$ can have at most one element). We shall prove that the condition (E) also holds. Let $D \in \mathcal{S}\setminus\mathcal{I}$ and let $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ be a double sequence of $\mathcal{S}$-measurable sets such that $B_{j,n} \subset B_{j,n+1}$ for $j,n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} B_{j,n} = D$ for $j \in \mathbb{N}$. Put $j_p = p$ for every natural p and choose n_p in such a way that $B_{p,n_p} \notin \mathcal{I}$ (it is possible for every p). Then $\bigcap_{p=1}^{\infty} B_{p,n_p} \notin \mathcal{I}$, because

$$X \setminus \bigcap_{p=1}^{\infty} B_{p,n_p} = \bigcup_{p=1}^{\infty}(X \setminus B_{p,n_p}) \in \mathcal{I}.$$

Remark 6.33. Let X be an arbitrary uncountable set, $\mathcal{S} = 2^X$ and $\mathcal{I} = \{\emptyset\}$. Then the convergence with respect to $\mathcal{I}$ is simply the pointwise convergence. It is well known that this kind of convergence does not yield a topology in this case. However the condition (E) is fulfilled. Indeed, let $D \in \mathcal{S} \setminus \mathcal{I}$ and let $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ be a double sequence of sets $B_{j,n} \subset B_{j,n+1}$ for each $j,n \in \mathbb{N}$, $\bigcup_{n=1}^{\infty} B_{j,n} = D$ for each $j \in \mathbb{N}$. Let $x_0 \in D$. It suffices to put $j_p = p$ and to choose n_p in such a way that $x_0 \in B_{p,n_p}$. Hence $x_0 \in \bigcap_{p=1}^{\infty} B_{p,n_p}$, so $\bigcap_{p=1}^{\infty} B_{p,n_p} \notin \mathcal{I}$.

Observe that the pair $(\mathcal{S},\mathcal{I})$ does not fulfill the countable chain condition. So this condition in Theorem 6.26 is important.

Below we shall give some examples showing that the convergence in measure (with respect to $\mathcal{N}$) differs essentially from the convergence in category (with respect to $\mathcal{M}$).

It is easy to construct a sequence of measurable and having the property of Baire functions, which is convergent in measure but not in category (or conversely). Indeed, let $A,B \subset [0,1]$ be a pair of sets such that A is of the first category, B is of Lebesgue measure zero and $A \cup B = [0,1]$ (see [5], p. 5). If $f_n = (-1)^n \chi_A$ for every n, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges $\mathcal{M}$-almost everywhere (so also in category) to the function $f \equiv 0$, but $\{f_n\}_{n\in\mathbb{N}}$ does not converge in measure. If we put $g_n = (-1)^n \chi_B$ for $n \in \mathbb{N}$, then we obtain a sequence convergent almost everywhere to $g \equiv 0$ but not convergent in category. Obviously, every f_n and g_n is measurable and has the Baire property.

It is a little more difficult to construct a sequence of continuous functions which is convergent in measure but not in category (or conversely), (compare [9], p. 310-311).

Let $A \subset [0,1]$ be a closed nowhere dense set of positive measure and let $\{(a_n,b_n)\}_{n\in\mathbb{N}}$ be a sequence of components of $[0,1] \setminus A$. Put

$$f_n(x) = \begin{cases} (-1)^n & \text{for } x \in A \cup \bigcup_{i=n+1}^{\infty} (a_i,b_i), \\ 0 & \text{for } x \in \bigcup_{i=1}^{n} [a_i + (b_i - a_i)2^{-n}, b_i - (b_i - a_i)2^{-n}], \\ \text{linear} & \text{on the intervals } [a_i, a_i + (b_i - a_i)2^{-n}] \text{ and} \\ & [b_i - (b_i - a_i)2^{-n}, b_i] \text{ for } i = 1,\dots,n. \end{cases}$$

It is not difficult to see that the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to zero for every $x \notin A$, so $\{f_n\}_{n\in\mathbb{N}}$ converges in category to $f \equiv 0$.

Simultaneously $\{f_n\}_{n\in\mathbb{N}}$ does not converge in measure, because this sequence does not fulfill the Cauchy condition in measure. Indeed, for $\varepsilon < 2$ we have the inclusion $\{x : |f_n(x) - f_{n+1}(x)| > \varepsilon\} \supset A$ for each $n \in \mathbb{N}$.

Let now

$$g_n(x) = \begin{cases} 1 & \text{for } x \in \bigcup_{i=1}^{n} \left[\frac{i}{n+1} - \frac{1}{2(n+1)^2}, \frac{i}{n+1} + \frac{1}{2(n+1)^2}\right], \\ 0 & \text{for } x \in \left[0, \frac{1}{n+1} - \frac{1}{(n+1)^2}\right] \cup \left[\frac{n}{n+1} + \frac{1}{(n+1)^2}, 1\right] \\ & \cup \bigcup_{i=1}^{n-1} \left[\frac{i}{n+1} + \frac{1}{(n+1)^2}, \frac{i+1}{n+1} - \frac{1}{(n+1)^2}\right], \\ \text{linear} & \text{on the intervals } \left[\frac{i}{n+1} - \frac{1}{(n+1)^2}, \frac{i}{n+1} - \frac{1}{2(n+1)^2}\right], \\ & \left[\frac{i}{n+1} + \frac{1}{2(n+1)^2}, \frac{i}{n+1} + \frac{1}{(n+1)^2}\right], \;\; i = 1, 2, \ldots, n. \end{cases}$$

It is not difficult to see that the sequence $\{g_n\}_{n\in\mathbb{N}}$ converges in measure to $g \equiv 0$. Indeed, the Lebesgue measure of the set $\{x : g_n(x) \neq 0\}$ is equal to $2n/(n+1)^2$. We shall prove that $\{g_n\}_{n\in\mathbb{N}}$ does not converge in category. Let $\{g_{m_n}\}_{n\in\mathbb{N}}$ be an arbitrary subsequence of $\{g_n\}_{n\in\mathbb{N}}$. We shall show that the set $\{x : \limsup_n g_{m_n}(x) = 1 \text{ and } \liminf_n g_{m_n} = 0\}$ is residual in $[0,1]$. Denote $A_n = \operatorname{int}(\{x : g_n(x) = 0\})$ and $B_n = \operatorname{int}(\{x : g_n(x) = 1\})$ for $n \in \mathbb{N}$. Then $\{x : \limsup_n g_{m_n} = 1\} \supset \bigcap_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} B_{m_n}$. The set $\bigcup_{n=k+1}^{\infty} B_{m_n}$ is open and dense in $[0,1]$. Hence $\bigcap_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} B_{m_n}$ is a residual set. Similarly one can prove that $\{x : \liminf_n g_{m_n}(x) = 0\} \supset \bigcap_{k=1}^{\infty} \bigcup_{n=k+1}^{\infty} A_{m_n}$ is a residual set. From the fact that $\{g_{m_n}\}_{n\in\mathbb{N}}$ was an arbitrary subsequence we conclude that $\{g_n\}_{n\in\mathbb{N}}$ does not converge in category.

6.6 Small translations of sets

It is well known (see, for example [1], p. 901-902) that if $A \subset [0,1]$, $A \in \mathcal{L}$, $\lambda(A) > 0$, then $\lim_{x\to 0} \lambda((A-x) \cap A \cap (A+x)) = \lambda(A)$. Obviously, also $\lim_{x\to 0} \lambda((A+x) \cap A) = \lambda(A)$. It means that if $\{x_n\}_{n\in\mathbb{N}}$ converges to zero, then the sequence $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ of characteristic functions converges in measure to χ_A. This statement cannot be improved.

Theorem 6.34 (see [13], Theorem 1). *There exists a set $A \subset [0,1]$, $A \in \mathcal{L}$, $\lambda(A) > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ convergent to 0 such that the sequence $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ does not converge almost everywhere to χ_A.*

The set A has been constructed in such a way that $\lambda(A \cap [a,b]) > 0$ and $\lambda([a,b] \setminus A) > 0$ for each $[a,b] \subset [0,1]$. Obviously, if $A \in \mathcal{N}$, then the sequence $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ converges to χ_A a.e.

Theorem 6.35 (see [13], Theorem 2). *If $A \subset [0,1]$, $A \in \mathcal{B}$, and $\{x_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence convergent to 0, then the sequence $\{\chi_{A+x_n}\}_{n\in\mathbb{N}}$ converges to χ_A $\mathcal{M}$-almost everywhere.*

The proof is based upon the representation $A = G \triangle P$, where G is open and $P \in \mathcal{M}$.

Similarly one can prove a slightly more general theorem:

Theorem 6.36 (see [6], Theorem 1). *If $A \subset [0,1]$, $A \in \mathcal{B}$ and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of continuous strictly increasing functions convergent uniformly to the identity function, then the sequence $\{\chi_{f_n(A)}\}_{n\in\mathbb{N}}$ converges to χ_A $\mathcal{M}$-almost everywhere.*

The situation is more complicated in the case of measurable sets:

Theorem 6.37 (see [6], Thoerem 2). *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of continuous increasing functions convergent uniformly to the identity function. Then $\lim_{n\to\infty} \lambda^*(A \triangle f_n(A)) = 0$ for each $A \subset [0,1]$, $A \in \mathcal{L}$ if and only if for the sequences of terms $\{g_n\}_{n\in\mathbb{N}}$ and $\{h_n\}_{n\in\mathbb{N}}$ from the Lebesgue decomposition of f_n (g_n is absolutely continuous and h_n is singular for $n \in \mathbb{N}$, both are nondecreasing) the following conditions are fulfilled:*

1. $\lim_{n\to\infty} h_n(1) = 0$ *(i.e. $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly to 0),*
2. *the sequence $\{g_n\}_{n\in\mathbb{N}}$ consists of uniformly absolutely continuous functions (i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $n \in \mathbb{N}$ and for each finite collection $[a_1,b_1],[a_2,b_2],\dots,[a_k,b_k]$ of nonoverlapping intervals contained in $[0,1]$ if $\sum_{i=1}^{k}(b_i - a_i) < \delta$, then $\sum_{i=1}^{k}(g_n(b_i) - g_n(a_i)) < \varepsilon$.*

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