Bulletin of the **S**ection of **L**ogic
Volume 47/3 (2018), pp. 201–214

http://dx.doi.org/10.18778/0138-0680.47.3.05

Mateusz Klonowski

A POST-STYLE PROOF OF COMPLETENESS THEOREM FOR SYMMETRIC RELATEDNESS LOGIC S*

Abstract

One of the logic defined by Richard Epstein in a context of an analysis of subject matter relationship is Symmetric Relatedness Logic S. In the monograph [2] we can find some open problems concerning relatedness logic, a Post-style completeness theorem for logic S is one of them. Our paper introduces a solution of this metalogical issue.

Keywords: Normal forms, Post-style proof of completeness, Relatedness logic, Relating logic

1. The Epstein's logics

In the case of most of non-classical interpretations of conditionals two aspects are considered as substantial:

1. logical values of an antecedent and a consequent
2. a relationship between an antecedent and a consequent.

The analysis of conditionals introduced by Richard Epstein in [2] are based on ways of understanding of relationships postulated by 2. Different concepts allow for the presentation of different implications. In order to define truth conditions for these logical connectives some binary relations based on a set of formulas with some constraints are introduced. But Epstein introduces also a different approach. He considers some functions which

*The author would like to thank the National Science Centre of Poland for the support of this research through grant No.: UMO-2015/19/N/HS1/02401 and Tomasz Jarmużek for his remarks and suggestions.

assigns to each formula some set (set-assignments). Such functions are intended to enable a notion of a content (subject matter) on a formal ground to be represented. In this case it is important to consider relations between contents of propositions, for instance one might be contained in another. These two approaches are proved to be equivalent in the case of some logics introduced by Epstein. He considered two families of logics defined by classes of models with only one relation. Namely relatedness logics and dependence logics [2, pp. 61–84, 115–143].

Epstein defined two relatedness logic. Symmetric Relatedness Logic S is one of them. Among problems analysed in [2] there is a question about a Post-style proof of completeness theorem for logic S. In this paper a solution of this metalogical issue will be presented.

The Epstein's approach via set-assigments appeared to be quite fruitful for expressing many intensional logics like modal logics, intuitionistic logic, many-valued logics or paraconsistent logics [2, pp. 145–287]. That is why Stanisław Krajewski proposed to treat the analysis of Epstein as a bigger project concerning logics of two aspects of propositions: their logical value and contents (see [4, pp. 17–18]). The general concepts and the most important results of such approach has been presented in [1], [2], [4] and also [5].

A much different line is examined in [3] by Jarmużek and Kaczkowski. In this case authors consider a logic defined by models with one binary relation without any extra constraints. However, in this approach only two intensional connectives: implication and conjunction were examined.

Now, more extensive research on this kind of logic, but with the language consists of all Boolean connectives and intensional counterpart of binary Boolean connectives, is being done, since any binary connective can be interpreted by logical valuation of components and binary relation. Presently, we can distinguish various relational conditions that may determine subclasses of the class of all binary relations of formulas, and consequently define numerous logics of the considered kind. Any of such logic is called relating logic.[1] In consequence, Epstein approach is a special case of relating logics program, since Epstein's logics are special cases of relating logics. Note that the converse dependence does not hold. It is

[1] The ideas concerning relating logic were developed during a logic seminar held in Toruń, led by Tomasz Jarmużek and they are in various forms being studied, examined and developed by Torunian PhD students, participating in that seminar.

also worth noticing that an analysis of relating logics seems to be promising for a philosophical interpretation of relating connectives like causal or temporal ones. Such issue should be a subject of the further investigations concerning applications of relating logics.

2. Language of relatedness logic

Formulas of relatedness logic are build by means of propositional letters $p_1, p_2, \ldots$, three logical connectives $\neg, \wedge, \rightarrow$ and parentheses $), ($. A set of propositional letters is denoted by Pl. A set For of formulas is the smallest set $\Sigma \subseteq \mathsf{Pl}$ such that: if $A \in \Sigma$, then $\neg A \in \Sigma$ and if $A, B \in \Sigma$, then $(A \wedge B), (A \rightarrow B) \in \Sigma$. We will omit the outermost parentheses. In the case of formulas build by an iteration of $\wedge$ we shall agree to associate to the left and write, for instance, $A \wedge B \wedge C$ instead of $(A \wedge B) \wedge C$. In some cases we use parentheses $], [$ in order to make some formulas and metalogical expressions more readable. Additionally to simplify some of formalism we introduce the following abbreviations for every $A, B, A_1, \ldots, A_n \in \mathsf{For}$ $(n \geq 2)$:

$$
\begin{aligned}
&A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A) \\
&A_1 \vee \ldots \vee A_n := \neg(\neg A_1 \wedge \ldots \wedge \neg A_n) \\
&A \looparrowright B := A \rightarrow (B \rightarrow B) \\
&A \supset B := \neg(A \wedge \neg B) \\
&A \equiv B := \neg(A \wedge \neg B) \wedge \neg(B \wedge \neg A).
\end{aligned}
$$

By the complexity of a given formula we understand an output of function $c\colon \mathsf{For} \longrightarrow \mathbb{N}$ defined in a standard way, wherein $c(A) = 0$, if $A \in \mathsf{Pl}$. A notion of subformula of a given formula is determined by function $\mathsf{sub}\colon \mathsf{For} \longrightarrow \mathcal{P}(\mathsf{For})$ also defined in a standard way. In order to refer to propositional letters of a given formula we use the following set $\mathsf{pl}(A) = \mathsf{sub}(A) \cap \mathsf{Pl}$, for every $A \in \mathsf{For}$.

3. Notion of relatedness

According to Epstein's analysis of relatedness there are at least two good candidates for formal attributes of a content relationship. The first one is reflexivity, motivated by the obvious fact that any content is identical with itself. The second one is to be independent from logical connectives, which is motivated by the fact that connectives are syncategorematic. Another intuitive attribute might be symmetry. In this way, we come to the concept of symmetric relatedness relation:

Definition 3.1. *Relation* $R \subseteq \mathsf{For} \times \mathsf{For}$ *is symmetric relatedness relation (for short:* srr*) iff* R *fulfils the following conditions for every* $A, B, C \in \mathsf{For}$*:*

$$R(A, A) \tag{re}$$
$$R(A, \neg B) \textit{ iff } R(A, B) \tag{srr1}$$
$$R(A, B \wedge C) \textit{ iff } R(A, B \to C) \tag{srr2}$$
$$R(A, B \wedge C) \textit{ iff } [R(A, B) \textit{ or } R(A, C)] \tag{srr3}$$
$$R(A, B) \textit{ iff } R(B, A). \tag{sym}$$

In the monograph [2, pp. 65–68] it is presented how by means of srr one can express contents relationships recognised as relationships between propositions due to a common subject matter. For a simple illustration of a such relationship let us consider the following propositions:

1. If John is interested in logic, then John knows Post's proof of completeness for Classical Propositional Logic
2. John considers a notion of normal forms for formulas of Classical Propositional Logic.

There are many subject matters which are shared by 1 and 2, one of them might be expressed as *metalogical properties of Classical Propositional Logic.*

The next fact determines a way of extending reflexive and symmetric relations defined on Pl to srr (see [2, pp. 67–68]).

Fact 3.2. *Let* $\mathrm{Q} \subseteq \mathsf{Pl} \times \mathsf{Pl}$ *be reflexive and symmetric relation. Let* $R \subseteq \mathsf{For} \times \mathsf{For}$ *be an extension of* Q *on* For *defined for every* $A, B \in \mathsf{For}$ *in the following way:*

$$R(A, B) \textit{ iff } \exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B)} \mathrm{Q}(x, y). \tag{$\star$}$$

Then R *is* srr*.*

Proof: Assume all hypothesis. Let $A, B, C \in \mathsf{For}$.

- Ad. (re). Let $a \in \mathsf{pl}(A)$. By reflexivity of Q, $\mathrm{Q}(a, a)$. Therefore, by ($\star$), $\mathrm{R}(A, A)$.

- Ad. (srr1). We have: $\mathrm{R}(A, \neg B)$, by ($\star$), iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(\neg B)} \mathrm{Q}(x, y)$, by equality $\mathsf{pl}(\neg A) = \mathsf{pl}(A)$ and classical logic, iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B)} \mathrm{Q}(x, y)$, by ($\star$), iff $\mathrm{R}(A, B)$.
- Ad. (srr2). We have: $\mathrm{R}(A, B \wedge C)$, by ($\star$), iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B \wedge C)}$ $\mathrm{Q}(x, y)$, by equality $\mathsf{pl}(B \wedge C) = \mathsf{pl}(B \to C)$ and classical logic, iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B \to C)} \mathrm{Q}(x, y)$, by ($\star$), iff $\mathrm{R}(A, B \to C)$.
- Ad. (srr3). We have: $\mathrm{R}(A, B \wedge C)$, by ($\star$), iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B \wedge C)}$ $\mathrm{Q}(x, y)$, by equality $\mathsf{pl}(B \wedge C) = \mathsf{pl}(B) \cup \mathsf{pl}(C)$, iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B) \cup \mathsf{pl}(C)} \mathrm{Q}(x, y)$, by definition of union and classical logic, iff $[\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B)} \mathrm{Q}(x, y)$ or $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(C)} \mathrm{Q}(x, y)]$, by ($\star$), iff $[\mathrm{R}(A, B)$ or $\mathrm{R}(A, C)]$.
- Ad. (sym). We have: $\mathrm{R}(A, B)$, by ($\star$), iff $\exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B)} \mathrm{Q}(x, y)$, by symmetry of Q and classical logic, iff $\exists_{y \in \mathsf{pl}(B)} \exists_{x \in \mathsf{pl}(A)} \mathrm{Q}(y, x)$, by ($\star$), iff $\mathrm{R}(B, A)$.

It is easy to see that an extension received by condition ($\star$) is unique.

4. Symmetric Relatedness Logic S

Definition 4.1. *A model of relatedness logic based on* srr *(or simply a model) is the following ordered pair* $\langle v, R \rangle$ *such that:*

- $v \in \{1, 0\}^{\mathsf{Pl}}$ *is a valuation of propositional letters*
- $R \subseteq \mathsf{For} \times \mathsf{For}$ *is* srr.

A class of models is denoted by $\mathcal{M}$. Relation R (resp. valuation v) of model $\mathfrak{M} \in \mathcal{M}$ is denoted by $\mathrm{R}_{\mathfrak{M}}$ (resp. $v_{\mathfrak{M}}$). Now we define a notion of a truth in a model:

Definition 4.2. *Let* $\mathfrak{M} \in \mathcal{M}$ *and* $A \in \mathsf{For}$. *A is a truth in* $\mathfrak{M}$ *(for short:* $\mathfrak{M} \models A$*) iff for every* $B, C \in \mathsf{For}$*:*

$$
\begin{array}{ll}
v_{\mathfrak{M}}(A) = 1, & \text{if } A \in \mathsf{Pl} \\
\mathfrak{M} \not\models B, & \text{if } A := \neg B \\
\mathfrak{M} \models B \;\&\; \mathfrak{M} \models C, & \text{if } A := B \wedge C \\
[\mathfrak{M} \not\models B \text{ or } \mathfrak{M} \models C] \;\&\; R_{\mathfrak{M}}(B, C), & \text{if } A := B \to C.
\end{array}
$$

For every $\Sigma \subseteq \mathsf{For}$ and $\mathfrak{M} \in \mathcal{M}$ in the case $\forall_{A \in \Sigma} \mathfrak{M} \models A$ we will write $\mathfrak{M} \models \Sigma$.

It is easy to observe that by Definition 4.2 the following abbreviation $A \vee B$, $A \supset B$, $A \equiv B$ denote respectively extensionally interpreted disjunction, conditional and biconditional.

Let us notice that formula $A \looparrowright B$ plays a special role in Epstein's investigations. It enables to express srr on the ground of the language of relatedness logic (see [2, pp. 77–78]):

FACT 4.3. *Let* $\mathfrak{M} \in \mathcal{M}$ *and* $A, B \in \mathsf{For}$. *Then:* $\mathfrak{M} \models A \looparrowright B$ *iff* $R_{\mathfrak{M}}(A, B)$.

PROOF: Assume all hypothesis

„$\Longrightarrow$" Let $\mathfrak{M} \models A \looparrowright B$, so $\mathfrak{M} \models A \to (B \to B)$. Hence, $\mathrm{R}_{\mathfrak{M}}(A, B \to B)$. Thus, by ($\mathsf{srr2}$), ($\mathsf{srr3}$) we get $\mathrm{R}_{\mathfrak{M}}(A, B)$.

„$\Longleftarrow$" Let $\mathrm{R}_{\mathfrak{M}}(A, B)$. Hence, by ($\mathsf{srr2}$), ($\mathsf{srr3}$), $\mathrm{R}_{\mathfrak{M}}(A, B \to B)$. By ($\mathsf{re}$) and because either $\mathfrak{M} \models B$ or $\mathfrak{M} \not\models B$, we get $\mathfrak{M} \models B \to B$. Hence, either $\mathfrak{M} \not\models A$ or $\mathfrak{M} \models B \to B$. Therefore, $\mathfrak{M} \models A \to (B \to B)$. Thus, $\mathfrak{M} \models A \looparrowright B$.

DEFINITION 4.4. *Let* $\Sigma \cup \{A\} \subseteq \mathsf{For}$. *Then:*

- A *is a semantic consequence of* Σ *in* S *(nota.:* $\Sigma \models_S A$*) iff* $\forall_{\mathfrak{M} \in \mathcal{M}}(\mathfrak{M} \models \Sigma \Longrightarrow \mathfrak{M} \models A)$.
- A *is a tautology in* S *(nota.:* $\models_S A$*) iff* $\varnothing \models_{\mathcal{M}} A$.

In the next section we remind Hilbert-style formulation of S.

5. Axiomatization of logic S

Axiom schemata of logic S are the following formulas, for every $A, B, C \in \mathsf{For}$ (see [2, p. 80]):

$$A \looparrowright A \tag{ax_1}$$

$$(B \looparrowright A) \to (A \looparrowright B) \tag{ax_2}$$

$$(A \looparrowright \neg B) \leftrightarrow (A \looparrowright B) \tag{ax_3}$$

$$(A \looparrowright (B \to C)) \leftrightarrow ((A \looparrowright B) \vee (A \looparrowright C)) \tag{ax_4}$$

$$(A \looparrowright (B \wedge C)) \leftrightarrow (A \looparrowright (B \to C)) \tag{ax_5}$$

$$(A \wedge B) \to A \tag{ax_6}$$

$$A \to (B \to (A \wedge B)) \tag{ax_7}$$

$$(A \wedge B) \to (B \wedge A) \tag{ax_8}$$

$$A \leftrightarrow \neg\neg A \tag{ax_9}$$

$$(A \to B) \leftrightarrow (\neg(A \wedge \neg B) \wedge (A \looparrowright B)) \tag{ax_{10}}$$

$$A \to (\neg(A \land B) \to \neg B) \tag{ax_{11}}$$

$$\neg(A \land B) \to (\neg(C \land \neg B) \to \neg(A \land C)) \tag{ax_{12}}$$

$$\neg((A \to B) \land (A \land \neg B)). \tag{ax_{13}}$$

Schemata (ax_1)–(ax_5) are intended to give a syntactic characterization of srr. The rest of schemata characterize logical connectives in logic S. The only rule of inference is *modus ponens*:

$$\frac{A, A \to B}{B}. \tag{MP}$$

We have the standard definition of the relation of syntactic consequence for S:

Definition 5.1. *Let* $\Sigma \cup \{A\} \subseteq \mathsf{For}$. *Then:*

- *A is a syntactic consequence of* Σ *in S (nota.:* $\Sigma \vdash_S A$*) iff there is a finite sequence of formulas* $B_1, \ldots, B_n$ *such that* $B_n = A$ *and for every* $i \leq n$ *at least one of the following conditions holds: (1)* $B_i :=$ $(\mathsf{ax}_1), \ldots, (\mathsf{ax}_{13})$*, (2)* $B_i \in \Sigma$ *or (3)* $\exists_{j,k<i} B_k := B_j \to B_i$.
- *A is a thesis in S (nota.:* $\vdash_S A$*) iff* $\varnothing \vdash_S A$.

One of the metalogical problems of logic S raised by Epstein concerns a proof of completeness by means of Post's method [2, s. 81]. He noticed, however, that a non-constructive proof of completeness might be received by a simple modifcation of a proof presented for Dependence Logic D [2, pp. 81, 126–129].

Let us notice that for every axiom schemata A, $\models_{\mathrm{S}} A$ and for every $A, B \in \mathsf{For}$, $A, A \to B \models_{\mathrm{S}} B$. Hence we have, the following fact:

Fact 5.2 (Theorem of weak soundenss for S). *Let* $A \in \mathsf{For}$. *Then:* $\vdash_S A \implies \models_S A$.

Let $\vdash_{\mathrm{CPL}}$ be the relation of syntactic consequence for $\{\neg, \land\}$-fragment of Classical Propositional Logic. According to an observation of Epstein (see [2, pp. 74–75]) we should be able to prove the following fact:

Fact 5.3. *Let* $A \in \mathsf{For}$. *Then:* $\vdash_{CPL} A \implies \vdash_S A$.

Let us notice that the following formulas are these in logic S:

$$A \supset (A \land A) \tag{1}$$

$$(A \land B) \supset A \tag{2}$$

$$(A \supset B) \supset (\neg(B \land C) \supset \neg(C \land A)). \tag{3}$$

Moreover the following rule of *modus ponens* for $\supset$ is derivable:

$$\frac{A, A \supset B}{B}. \tag{4}$$

Formulas (1)–(3) with rule (4) (for formulas A, B, C build only by means of $\neg, \wedge$) enable to determine relation $\vdash_{\mathrm{CPL}}$ (see [6, pp. 12–46, 54–76]).

6. Normal forms of formulas

The set of literals is defined in a standard way $\mathsf{Li} := \mathsf{Pl} \cup \mathsf{nPl}$, where $\mathsf{nPl} := \{\neg A \in \mathsf{For} \mid A \in \mathsf{Pl}\}$. Additionally we define a set of related propositional letters $\mathsf{rPl} = \{A \looparrowright B \in \mathsf{For} \mid A, B \in \mathsf{Pl}\}$ and a set of non-related propositional letters $\mathsf{nrPl} = \{\neg(A \looparrowright B) \in \mathsf{For} \mid A, B \in \mathsf{Pl}\}$.

Definition 6.1. *$A \in \mathsf{For}$ is elementary disjunction (for short: ed) in the following cases:*

(1) $A \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$

(2) $A := B \vee C$, where B is ed, and $C \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$.

Remark 6.2. *Let us notice that by Definition 6.1 A is ed iff $A := B_1 \vee \ldots \vee B_n$ ($n \in \mathbb{N}$), where for any $i \leq n$, $B_i \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. The equivalence might be also used in order to define ed.*

A conjunctive normal form is defined in a standard way:

Definition 6.3. *$A \in \mathsf{For}$ is in conjunctive normal form (for short: cnf) in the following cases:*

(1) A is ed

(2) $A := B \wedge C$, where B is in cnf and C is ed.

Remark 6.4. *Similarly to Remark 6.2, let us notice that by Definition 6.3 A is in cnf iff $A := B_1 \wedge \ldots \wedge B_n$ ($n \in \mathbb{N}$), where for any $i \leq n$, B_i is ed. The equivalence might be also used in order to define cnf.*

Let us define a function that enables to refer to an «opposite formula» of any:

Definition 6.5. *Let $'\colon \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl} \longrightarrow \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ be a function such that, for every $A \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ we put:*

$$A' = \begin{cases} \neg A, & \text{if } A \in \mathsf{Pl} \cup \mathsf{rPl} \\ B, & \text{if } A \in \mathsf{nPl} \cup \mathsf{nrPl} \ \&\ A := \neg B. \end{cases}$$

Let us notice that:

- A' is ed
- $\vdash_{\mathrm{S}} \neg A \equiv A'$
- $\vdash_{\mathrm{S}} (\neg A \nrightarrow C) \equiv (A' \nrightarrow C)$, for every $C \in \mathsf{For}$
- $A'' = A$.

We also define a function that enables to refer to an «antecedent» or «consequent» of the given formula:

Definition 6.6. *Let* $^a : \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl} \longrightarrow \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$, $^c : \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl} \longrightarrow \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ *be functions such that, for every* $A \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ *we put:*

$$A^a = \begin{cases} A, & \text{if } A \in \mathsf{Pl} \\ A', & \text{if } A \in \mathsf{nPl} \\ B, & \text{if } A \in \mathsf{rPl} \cup \mathsf{nrPl} \ \& \ [A := B \nrightarrow C \text{ or } A := \neg(B \nrightarrow C)]. \end{cases}$$

$$A^c = \begin{cases} A, & \text{if } A \in \mathsf{Pl} \\ A', & \text{if } A \in \mathsf{nPl} \\ B, & \text{if } A \in \mathsf{rPl} \cup \mathsf{nrPl} \ \& \ [A := C \nrightarrow B \text{ or } A := \neg(C \nrightarrow B)]. \end{cases}$$

Let us notice that:

- A^a, A^c are ed
- $\vdash_{\mathrm{S}} (A \nrightarrow C) \equiv ((A^a \nrightarrow C) \vee (A^b \nrightarrow C))$, for every $C \in \mathsf{For}$.

Fact 6.7. *Let* $A \in \mathsf{For}$. *Then, there is* $B \in \mathsf{For}$ *in* cnf *such that:* $\vdash_S A \equiv B$ *and for every* $C \in \mathsf{For}$, $\vdash_S (A \nrightarrow C) \equiv (B \nrightarrow C)$.

Proof: We use induction on complexity of formulas.

Basis. Let $A \in \mathsf{For}$ and $c(A) = 0$. Then by Definition 6.1 A is ed, hence by Definition 6.3 is in cnf. By Fact 5.3, for every $C \in \mathsf{For}$ we have, $\vdash_{\mathrm{S}} C \equiv C$.

Inductive hypothesis. Let $n \in \mathbb{N}$. Suppose for every $C \in \mathsf{For}$, if $c(C) \leq n$, then the fact holds for C.

Inductive step. Let $A \in \mathsf{For}$ and $c(A) = n + 1$. Then:

- Let $A := \neg D$. By the inductive hypothesis for some $B \in \mathsf{For}$ which is in cnf we have that: $\vdash_{\mathrm{S}} D \equiv B$ (1) and for every $C \in \mathsf{For}$, $\vdash_{\mathrm{S}} (D \nrightarrow C) \equiv (B \nrightarrow C)$ (2).
  B is in cnf. Hence, by Remark 6.2 and 6.4: $B := (B_{1_1} \vee \ldots \vee B_{n_1}) \wedge \ldots \wedge (B_{1_m} \vee \ldots \vee B_{n_m})$, where for every $i \leq n$ and $j \leq m$, $B_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. Let $\overline{B} := (B'_{1_1} \vee B'_{1_2} \vee \ldots \vee B'_{1_m}) \wedge (B'_{2_1} \vee B'_{2_2} \vee \ldots \vee B'_{2_m}) \wedge \ldots \wedge (B'_{n_1} \vee B'_{n_2} \vee \ldots \vee B'_{n_m})$. Hence, $\overline{B}$ is in cnf.

Let us notice that, by Fact 5.3: $\vdash_{\mathrm{S}} \neg B \equiv [(B'_{1_1} \wedge \ldots \wedge B'_{n_1}) \vee \ldots \vee (B'_{1_m} \wedge \ldots \wedge B'_{n_m})]$ (3) and $\vdash_{\mathrm{S}} [(B'_{1_1} \wedge \ldots \wedge B'_{n_1}) \vee \ldots \vee (B'_{1_m} \wedge \ldots \wedge B'_{n_m})] \equiv \overline{B}$ (4). Hence, by Fact 5.3 (transitivity of $\equiv$), (1), (3), (4) and (MP) we get: $\vdash_{\mathrm{S}} \neg D \equiv \overline{B}$.
Let us notice that, for every $C \in \mathsf{For}$: $\vdash_{\mathrm{S}} [(D \nrightarrow C) \equiv (B \nrightarrow C)] \supset [(\neg D \nrightarrow C) \equiv (\neg B \nrightarrow C)]$ (5) and $\vdash_{\mathrm{S}} (\neg B \nrightarrow C) \equiv (\overline{B} \nrightarrow C)$ (6). Hence, by Fact 5.3 (transitivity of $\equiv$), (2), (5), (6) and (MP) we get: $\vdash_{\mathrm{S}} (\neg D \nrightarrow C) \equiv (\overline{B} \nrightarrow C)$.

- Let $A := D \wedge E$. By the inductive hypothesis for some $B_0, B_1 \in \mathsf{For}$ which are in cnf we have that: $\vdash_{\mathrm{S}} D \equiv B_0$ (1), for every $C \in \mathsf{For}$, $\vdash_{\mathrm{S}} (D \nrightarrow C) \equiv (B_0 \nrightarrow C)$ (2), $\vdash_{\mathrm{S}} E \equiv B_1$ (3) and for every $C \in \mathsf{For}$, $\vdash_{\mathrm{S}} (E \nrightarrow C) \equiv (B_1 \nrightarrow C)$ (4).
B_0, B_1 are in cnf. Hence, by Remark 6.2 and 6.4: $B_0 := (C_{1_1} \vee \ldots \vee C_{n_1}) \wedge \ldots \wedge (C_{1_m} \vee \ldots \vee C_{n_m})$, where for every $i \leq n$ and $j \leq m$, $C_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ and $B_1 := (D_{1_1} \vee \ldots \vee D_{k_1}) \wedge \ldots \wedge (D_{1_l} \vee \ldots \vee D_{k_l})$, where for every $i \leq k$ and $j \leq l$, $D_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. Let $B_2 := (C_{1_1} \vee \ldots \vee C_{n_1}) \wedge \ldots \wedge (C_{1_m} \vee \ldots \vee C_{n_m}) \wedge (D_{1_1} \vee \ldots \vee D_{k_1}) \wedge \ldots \wedge (D_{1_l} \vee \ldots \vee D_{k_l})$. Hence, B_2 is in cnf.
Let us notice that, by Fact 5.3: $\vdash_{\mathrm{S}} (D \equiv B_0) \supset [(E \equiv B_1) \supset ((D \wedge E) \equiv (B_0 \wedge B_1))]$ (5) and $\vdash_{\mathrm{S}} (B_0 \wedge B_1) \equiv B_2$ (6). Hence, by Fact 5.3 (transitivity of $\equiv$), (1), (3), (5), (6) and (MP) we get: $\vdash_{\mathrm{S}} (D \wedge E) \equiv B_2$.
Let us notice that, for every $C \in \mathsf{For}$: $\vdash_{\mathrm{S}} [(D \nrightarrow C) \equiv (B_0 \nrightarrow C)] \supset [((E \nrightarrow C) \equiv (B_1 \nrightarrow C)) \supset (((D \wedge E) \nrightarrow C) \equiv ((B_0 \wedge B_1) \nrightarrow C))]$ (7) and $\vdash_{\mathrm{S}} ((B_0 \wedge B_1) \nrightarrow C) \equiv (B_2 \nrightarrow C)$ (8). Therefore, by Fact 5.3 (transitivity of $\equiv$), (2), (4), (7), (8) and (MP) we get: $\vdash_{\mathrm{S}} ((D \wedge E) \nrightarrow C) \equiv (B_2 \nrightarrow C)$.

- Let $A := D \rightarrow E$. By the inductive hypothesis for some $B_0, B_1 \in \mathsf{For}$ which are in cnf we have that: $\vdash_{\mathrm{S}} D \equiv B_0$ (1), for every $C \in \mathsf{For}$, $\vdash_{\mathrm{S}} (D \nrightarrow C) \equiv (B_0 \nrightarrow C)$ (2), $\vdash_{\mathrm{S}} E \equiv B_1$ (3) and for every $C \in \mathsf{For}$, $\vdash_{\mathrm{S}} (E \nrightarrow C) \equiv (B_1 \nrightarrow C)$ (4).
B_0, B_1 are in cnf. Hence, by Remark 6.2 and 6.4: $B_0 := (C_{1_1} \vee \ldots \vee C_{n_1}) \wedge \ldots \wedge (C_{1_m} \vee \ldots \vee C_{n_m})$, where for every $i \leq n$ and $j \leq m$, $C_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. Also by Remark 6.2 and 6.4: $B_1 := (D_{1_1} \vee \ldots \vee D_{k_1}) \wedge \ldots \wedge (D_{1_l} \vee \ldots \vee D_{k_l})$, where for every $i \leq k$ and $j \leq l$, $D_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. Let $\overline{B_0} := (C'_{1_1} \vee \ldots \vee C'_{1_m}) \wedge \ldots \wedge (C'_{n_1} \vee \ldots \vee C'_{n_m})$. Hence, $\overline{B_0}$ is in cnf. Let $B_2 := (C'_{1_1} \vee \ldots \vee C'_{1_m} \vee D_{1_1} \vee \ldots \vee D_{k_1}) \wedge$

$(C'_{1_1} \vee \ldots \vee C'_{1_m} \vee D_{1_2} \vee \ldots \vee D_{k_2}) \wedge \ldots \wedge (C'_{n_1} \vee \ldots \vee C'_{n_m} \vee D_{1_l} \vee \ldots \vee D_{k_l})$. Formula B_2 is also in cnf.

Let us notice that, by the Fact 5.3: $\vdash_S (\neg B_0 \vee B_1) \equiv (\overline{B_0} \vee B_1)$ (5) and $\vdash_S (\overline{B_0} \vee B_1) \equiv B_2$ (6). Hence, by Fact 5.3 (transitivity of $\equiv$), (5), (6) and (MP) we get: $\vdash_S (\neg B_0 \vee B_1) \equiv B_2$ (7).

For every $i \leq n$ and $j \leq m$, $C_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$ and for every $i \leq k$ and $j \leq l$, $B_{i_j} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. Let $B_3 = (C^a_{1_1} \nleftrightarrow D^a_{1_1}) \vee (C^a_{1_1} \nleftrightarrow D^c_{1_1}) \vee (C^c_{1_1} \nleftrightarrow D^a_{1_1}) \vee (C^c_{1_1} \nleftrightarrow D^c_{1_1}) \vee \ldots \vee (C^a_{n_m} \nleftrightarrow D^a_{k_l}) \vee (C^a_{n_m} \nleftrightarrow D^c_{k_l}) \vee (C^c_{n_m} \nleftrightarrow D^a_{k_l}) \vee (C^c_{n_m} \nleftrightarrow D^c_{k_l})$. Hence, B_3 is ed.

Let us notice that: $\vdash_S (B_0 \nleftrightarrow B_1) \equiv B_3$ (8). Hence, by Fact 5.3, (7), (8) and (MP) we get: $\vdash_S [(\neg B_0 \vee B_1) \wedge (B_0 \nleftrightarrow B_1)] \equiv (B_2 \wedge B_3)$ (9).

Let $B_4 := (C'_{1_1} \vee \ldots \vee C'_{1_m} \vee D_{1_1} \vee \ldots \vee D_{k_1}) \wedge (C'_{1_1} \vee \ldots \vee C'_{1_m} \vee D_{1_2} \vee \ldots \vee D_{k_2}) \wedge \ldots \wedge (C'_{n_1} \vee \ldots \vee C'_{n_m} \vee D_{1_l} \vee \ldots \vee D_{k_l}) \wedge [(C^a_{1_1} \nleftrightarrow D^a_{1_1}) \vee (C^a_{1_1} \nleftrightarrow D^c_{1_1}) \vee (C^c_{1_1} \nleftrightarrow D^a_{1_1}) \vee (C^c_{1_1} \nleftrightarrow D^c_{1_1}) \vee \ldots \vee (C^a_{n_m} \nleftrightarrow D^a_{k_l}) \vee (C^a_{n_m} \nleftrightarrow D^c_{k_l}) \vee (C^c_{n_m} \nleftrightarrow D^a_{k_l}) \vee (C^c_{n_m} \nleftrightarrow D^c_{k_l})]$. Therefore, B_4 is in cnf.

Let us notice that, by Fact 5.3: $\vdash_S (B_2 \wedge B_3) \equiv B_4$ (10). Hence, by Fact 5.3 (transitivity of $\equiv$), (9), (10) and (MP) we get: $\vdash_S [(\neg B_0 \vee B_1) \wedge (B_0 \nleftrightarrow B_1)] \equiv B_4$ (11). We also have that, for every $C \in \mathsf{For}$, $\vdash_S ((B_0 \wedge B_1) \nleftrightarrow C) \equiv (B_4 \nleftrightarrow C)$ (12).

Let us notice that: $\vdash_S [(E \nleftrightarrow B_0) \equiv (B_1 \nleftrightarrow B_0)] \supset [(B_0 \nleftrightarrow E) \equiv (B_0 \nleftrightarrow B_1)]$ (13). Hence, by Fact 5.3 (transitivity of $\equiv$), (2), (4), (13), and (MP) we get: $\vdash_S (D \nleftrightarrow E) \equiv (B_0 \nleftrightarrow B_1)$ (14).

Let us notice that: $\vdash_S [(D \nleftrightarrow E) \equiv (B_0 \nleftrightarrow B_1)] \supset [(D \equiv B_0) \supset ((E \equiv B_1) \supset ((D \to E) \equiv ((\neg B_0 \vee B_1) \wedge (B_0 \nleftrightarrow B_1))))]$ (15). Hence, by (1), (3), (14), (15) and (MP): $\vdash_S (D \to E) \equiv ((\neg B_0 \vee B_1) \wedge (B_0 \nleftrightarrow B_1))$ (16). Hence, by Fact 5.3 (transitivity of $\equiv$), (11), (16) and (MP) we get: $\vdash_S (D \to E) \equiv B_4$.

Let us notice that, by Fact 5.3: $\vdash_S [(D \nleftrightarrow C) \equiv (B_0 \nleftrightarrow C)] \supset [((E \nleftrightarrow C) \equiv (B_1 \nleftrightarrow C)) \supset ((D \nleftrightarrow C) \vee (E \nleftrightarrow C)) \equiv ((B_0 \nleftrightarrow C) \vee (B_1 \nleftrightarrow C)))]$ (16). For every $C \in \mathsf{For}$ we have: $\vdash_S ((D \to E) \nleftrightarrow C) \equiv ((D \nleftrightarrow C) \vee (E \nleftrightarrow C))$ (17) and $\vdash_S ((B_0 \nleftrightarrow C) \vee (B_1 \nleftrightarrow C)) \equiv ((B_0 \wedge B_1) \nleftrightarrow C)$ (18). Hence, by Fact 5.3 (transitivity of $\equiv$), (2), (4), (16), (17), (18) and (MP) we get: $\vdash_S ((D \to E) \nleftrightarrow C) \equiv ((B_0 \wedge B_1) \nleftrightarrow C)$ (19). Hence, by Fact 5.3 (transitivity of $\equiv$), (12), (19) and (MP) we get: $\vdash_S ((D \to E) \nleftrightarrow C) \equiv (B_4 \nleftrightarrow C)$.

LEMMA 6.8. *Let $A := B_1 \vee \ldots \vee B_n$ be* ed *($n \in \mathbb{N}$). Then: $\models_S A$ iff at least* *one of the following conditions hold:*

(1) $B_k := p_i$ *and* $B_l := \neg p_i$, *for some* $k, l \leq n$ *and* $i \in \mathbb{N}$

(2) $B_k := p_i \nleftrightarrow p_i$, *for some* $k \leq n$ *and* $i \in \mathbb{N}$

(3) $B_k := p_i \nleftrightarrow p_j$ *and* $B_l := \neg(p_i \nleftrightarrow p_j)$, *for some* $k, l \leq n$ *and* $i, j \in \mathbb{N}$.

Proof: Assume all hypothesis.

„$\Longrightarrow$" Suppose that non of the conditions (1)–(3) holds ($\bigstar$). We define a model $\mathfrak{M} = \langle v, \mathrm{R} \rangle$ in the following way:

1. Let $i \in \mathbb{N}$ we put:

$$v(p_i) = \begin{cases} 1, & \text{if } \neg p_i \in \mathsf{sub}(A), \\ 0, & \text{if } p_i \in \mathsf{sub}(A). \end{cases}$$

2. Let $i, j \in \mathbb{N}$ and $i \neq j$. Let $\mathrm{Q} \subseteq \mathsf{Pl} \times \mathsf{Pl}$ be the smallest relation which fulfils the following conditions:
   - $p_i \nleftrightarrow p_j \in \mathsf{sub}(A) \Longrightarrow {\sim}\mathrm{Q}(p_i, p_j)$
   - $\neg(p_i \nleftrightarrow p_j) \in \mathsf{sub}(A) \Longrightarrow \mathrm{Q}(p_i, p_j)$
   - $\mathrm{Q}(p_i, p_i)$
   - $\mathrm{Q}(p_i, p_j)$ iff $\mathrm{Q}(p_j, p_i)$.

   Q is obviously reflexive and symmetric. We extend Q on For in the following way for every $A, B \in \mathsf{For}$:

$$\mathrm{R}(A, B) \text{ iff } \exists_{x \in \mathsf{pl}(A)} \exists_{y \in \mathsf{pl}(B)} \mathrm{Q}(x, y).$$

By Fact 3.2 R is srr. Let $i, j \in \mathbb{N}$ and $i \neq j$. Let us consider the following cases:

- Suppose $B_k := p_i$, for some $k \leq n$. By the definition of $v_{\mathfrak{M}}$ we get $\mathfrak{M} \not\models_{\mathrm{S}} p_n$.
- Suppose $B_k := \neg p_n$, for some $k \leq n$. By the definition of $v_{\mathfrak{M}}$ we get $\mathfrak{M} \not\models_{\mathrm{S}} \neg p_n$.
- By ($\bigstar$) it is excluded that: $B_k = p_i$ and $B_l := \neg p_i$, for some $k, l \leq n$.
- Suppose $B_k := p_i \nleftrightarrow p_j$, for some $k \leq n$. By the definition of $\mathrm{R}_{\mathfrak{M}}$ we get $\mathfrak{M} \not\models_{\mathrm{S}} p_i \nleftrightarrow p_j$.
- Suppose $B_k := \neg(p_i \nleftrightarrow p_j)$, for some $k \leq n$. By the definition of $\mathrm{R}_{\mathfrak{M}}$ we get $\mathfrak{M} \not\models_{\mathrm{S}} \neg(p_i \nleftrightarrow p_j)$.
- By ($\bigstar$) it is excluded that: $B_k := p_i \nleftrightarrow p_j$, for some $k \leq n$ and it is excluded that: $B_k := p_i \nleftrightarrow p_j$ and $B_l := \neg(p_i \nleftrightarrow p_j)$, for some $k, l \leq n$.

Therefore, $\mathfrak{M} \not\models_{\mathrm{S}} A$, so by Definition 4.4 $\not\models_{\mathrm{S}} A$.

„$\Longleftarrow$" Suppose that at least one of the conditions (1)–(3) holds. Let $\mathfrak{M} \in \mathcal{M}$. If condition (1) or (3) holds then, by Definition 4.2, $\mathfrak{M} \models A$. If condition (2) holds and $B_k := p_i \nleftrightarrow p_i$, for some $k \leq n$ and $i \in \mathbb{N}$. By (re) $\mathrm{R}_{\mathfrak{M}}(p_i, p_i)$. Therefore, by Fact 4.3, $\mathfrak{M} \models p_i \nleftrightarrow p_i$. Hence, by Definition 4.2, $\mathfrak{M} \models A$. Therefore, $\mathfrak{M} \models_{\mathrm{S}} A$. Thus, by Definition 4.4, $\models_{\mathrm{S}} A$.

LEMMA 6.9. *Let* $A := B_1 \wedge \ldots \wedge B_n$ *be in* cnf *(*$n \in \mathbb{N}$*). Then:* $\models_S A$ *iff* $\models_S B_k$*, for every* $k \leq n$.

PROOF: Assume all hypothesis. By Definition 4.2: $\models_{\mathrm{S}} B_1 \wedge \ldots \wedge B_n$ iff $\models_{\mathrm{S}} B_k$, for every $k \leq n$.

7. Completeness theorem for logic S

THEOREM 7.1 (Completeness theorem for logic S). *Let* $A \in \mathsf{For}$*. Then:* $\models_S A \Longrightarrow \vdash_S A$.

PROOF: Let $A \in \mathsf{For}$. Suppose $\models_{\mathrm{S}} A$ (1). By Fact 6.7 for some $B \in \mathsf{For}$ in cnf we have $\vdash_{\mathrm{S}} A \equiv B$ (2). By Fact 5.2 we get $\models_{\mathrm{S}} A \equiv B$ (3). Hence, by (1) and (3), $\models_{\mathrm{S}} B$. Moreover, B is in cnf. Let $n \in \mathbb{N}$ and $B := B_1 \wedge \ldots \wedge B_n$, where for every $i \leq n$, B_i is ed. Let $i \leq n$, by Lemma 6.9, $\models_{\mathrm{S}} B_i$. We also have that $B_i := C_{1_i} \vee \ldots \vee C_{m_i}$, for some $m \in \mathbb{N}$, and for every $k \leq m$, $C_{k_i} \in \mathsf{Li} \cup \mathsf{rPl} \cup \mathsf{nrPl}$. By Lemma 6.8 at least one of the following conditions holds:

(a) $C_{k_i} := p_j$ and $C_{l_i} := \neg p_j$, for some $k, l \leq m$ and $j \in \mathbb{N}$

(b) $C_{k_i} := p_j \nleftrightarrow p_j$, for some $k \leq m$ and $j \in \mathbb{N}$

(c) $C_{k_i} := p_j \nleftrightarrow p_h$ and $C_{l_i} := \neg(p_j \nleftrightarrow p_h)$, for some $k, l \leq m$ and $i, h \in \mathbb{N}$.

Suppose condition (a) holds and $B_i := p_j \vee \neg p_j \vee C$, where C is not important part of B_i. Let us notice that $\vdash_{\mathrm{S}} p_j \vee \neg p_j$ (4). Moreover, for every $D, E \in \mathsf{For}$ we have that $\vdash_{\mathrm{S}} D \to (D \vee E)$ (5). Hence, by (4) and (5), $\vdash_{\mathrm{S}} B_i$.

Suppose condition (b) holds and $B_i := p_j \nleftrightarrow p_j \vee C$, where C is not important part of B_i. Let us notice that $\vdash_{\mathrm{S}} p_j \nleftrightarrow p_j$. We reason as in the case of condition (a).

Suppose condition (c) holds and $B_i := (p_j \nleftrightarrow p_h) \vee \neg(p_j \nleftrightarrow p_h) \vee C$, where C is not important part of B_i. Let us notice that $\vdash_{\mathrm{S}} (p_j \nleftrightarrow p_h) \vee \neg(p_j \nleftrightarrow p_h)$. We reason as in the case of condition (a).

Hence, $\vdash_S B_i$, for every $i \leq n$ (6). Let us also note that $\vdash_S D \to (E \to (D \wedge E))$ (7). By (6) and (7) we get $\vdash_S B_1 \wedge \ldots \wedge B_n$. Hence, $\vdash_S B$. And therefore, by (2), $\vdash_S A$.

References

[1] R. L. Epstein, *Relatedness and Implication*, **Philosophical Studies**, Vol. 36:2 (1979), pp. 137–173.

[2] R. L. Epstein, (with the assistance and collaboration of: W. A. Camielli, I. M. L. D'Ottaviano, S. Krajewski, R. D. Maddux), **The Semantic Foundtations of Logic. Volume 1: Propositional Logics**, Springer Science+Business Media, Dordrecht (1990).

[3] T. Jarmużek and B. Kaczkowski, *On some Logic with a Relation Imposed on Formulae: Tableau System $\mathcal{F}$*, **Bulletin of the Section of Logic**, Vol. 43:1/2 (2014), pp. 53–72.

[4] S. Krajewski, *One or Many Logics? (Epstein's Set-Assignement Semantics for Logical Calculi)*, **The Journal of Non-Classical Logic** 8:1 (1991), pp. 7–33.

[5] S. Krajewski, *On Relatedness Logic of Richard L. Epstein*, **Bulletin of the Section of Logic**, Vol. 11:1/2 (1982), pp. 24–30.

[6] J. B. Rosser, **Logic for Mathematicians**, McGraw-Hill, New York (1953).

[7] D. Walton, *Philosophical Basis of Relatedness Logic*, **Philosophical Studies**, Vol. 36:2 (1979), pp. 115–136.

Department of Logic
Nicolaus Copernicus University
ul. Fosa Staromiejska 1a, 87-100, Toruń, Poland
e-mail: `matklon@doktorant.umk.pl`