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AN INFERENTIALLY MANY-VALUED
TWO-DIMENSIONAL NOTION OF ENTAILMENT

dedicated to Prof. Grzegorz Malinowski

Abstract

Starting from the notions of q-entailment and p-entailment, a two-dimensional notion of entailment is developed with respect to certain generalized q-matrices referred to as B-matrices. After showing that every purely monotonic single-conclusion consequence relation is characterized by a class of B-matrices with respect to q-entailment as well as with respect to p-entailment, it is observed that, as a result, every such consequence relation has an inferentially four-valued characterization. Next, the canonical form of B-entailment, a two-dimensional multiple-conclusion notion of entailment based on B-matrices, is introduced, providing a uniform framework for studying several different notions of entailment based on designation, antidesignation, and their complements. Moreover, the two-dimensional concept of a B-consequence relation is defined, and an abstract characterization of such relations by classes of B-matrices is obtained. Finally, a contribution to the study of inferential many-valuedness is made by generalizing Suszko's Thesis and the corresponding reduction to show that any B-consequence relation is, in general, inferentially four-valued.

1. Introduction

A (*logical*) *matrix* is usually defined as a pair $\langle \mathcal{A}, D \rangle$, where $\mathcal{A}$ is an algebra similar to a propositional language $\mathcal{L}$ (we shall identify the language $\mathcal{L}$ with its set of formulas), and D is a subset of $\mathcal{A}$'s carrier set A (see, e.g., [15, 29]). Intuitively, A (sometimes denoted as '$\mathcal{V}$') is a non-empty set of *truth-values*, and D is a set of *designated* truth-values. If $\mathcal{C}$ is the set of connectives of $\mathcal{L}$, then $\langle \mathcal{A}, D \rangle$ can be presented as a tuple $\langle \mathcal{V}, D, \{f_c \mid c \in \mathcal{C}\} \rangle$, where f_c is a function on $\mathcal{V}$ with the same arity as c. With a view towards obtaining a semantics for $\mathcal{L}$, an entailment relation is associated to a given matrix in a certain canonical way. For that purpose, a class of valuations is fixed, and often, in order to obtain a 'truth-functional semantics', the class $\mathsf{Hom}(\mathcal{L}, \mathcal{A})$ of all homomorphisms of $\mathcal{L}$ into $\mathcal{A}$ is considered (see [22]). If $\mathfrak{M} = \langle \mathcal{A}, D \rangle$ is a matrix, the single-conclusion entailment relation $\models_{\mathfrak{M}} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ induced by $\mathfrak{M}$ is defined as follows:

$$\Gamma \models_{\mathfrak{M}} \varphi \text{ iff } (\nu(\Gamma) \subseteq D \text{ implies } \nu(\varphi) \in D, \text{ for every } \nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A})),$$

where $\nu(\Gamma) = \{\nu(\psi) \mid \psi \in \Gamma\}$.

    If truth-functionality of the semantics is not required, the algebraic structure of $\mathcal{A}$ is not exploited in the same way, so that the first component of a matrix $\langle \mathcal{A}, D \rangle$ may just as well be any set $\mathcal{V}$, and $\mathsf{Hom}(\mathcal{L}, \mathcal{A})$ may be replaced by any collection S of functions from $\mathcal{L}$ into $\mathcal{V}$. Given such practice, the notion of a matrix can be broadened into a triple $\langle \mathcal{V}, D, \mathsf{S} \rangle$, as is implicitly done in [4]. In particular, if S is a singleton set, then $\langle \mathcal{V}, D, \mathsf{S} \rangle$ may be seen as a *semantical model based on the matrix* $\langle \mathcal{V}, D \rangle$.

    Let $\mathcal{M}$ be a class of matrices. The relation $\models_{\mathcal{M}} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ (entailment with respect to $\mathcal{M}$) is defined by setting $\Gamma \models_{\mathcal{M}} \varphi$ iff $\Gamma \models_{\mathfrak{M}} \varphi$ for all $\mathfrak{M} \in \mathcal{M}$. A relation $\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is said to be *Tarskian* if for every $\varphi, \psi \in \mathcal{L}$ and every $\Gamma, \Delta \subseteq \mathcal{L}$:

(Ref) $\Gamma \vdash \varphi$, whenever $\varphi \in \Gamma$

(Mon)   If $\Gamma \vdash \varphi$ then $\Gamma \cup \Delta \vdash \varphi$

(Trn)   If $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$ and $\Gamma \cup \Delta \vdash \psi$, then $\Gamma \vdash \psi$

Above, 'Ref', 'Mon' and 'Trn' stand, respectively, for *reflexivity*, *monotonicity*, and *transitivity* (or *closure*). It can readily be checked that every relation $\models_{\mathcal{M}}$ is a Tarskian consequence relation.

    Well-studied and important generalizations of the concept of a matrix are G. Malinowski's notion of a *q-matrix* [13] and S. Frankowski's notion

of a p-matrix [7, 8]. A q-matrix (*quasi matrix*) is a structure $\langle \mathcal{A}, \mathcal{D}^+, \mathcal{D}^- \rangle$, where $\mathcal{A}$ is an algebra similar to a propositional language $\mathcal{L}$, and where $\mathcal{D}^+$ and $\mathcal{D}^-$ are subsets of $\mathcal{A}$, and $\mathcal{D}^+ \cap \mathcal{D}^- = \varnothing$. Usually, $\mathcal{D}^+$ is referred to as the set of *designated values* and $\mathcal{D}^-$ as the set of *antidesignated values*. A p-matrix (*plausibility matrix*) is a structure $\langle \mathcal{A}, \mathcal{D}^+, \mathcal{D}^* \rangle$, where $\mathcal{A}$ is an algebra similar to a propositional language $\mathcal{L}$ and $D^+ \subseteq D^* \subseteq A$. The set $\mathcal{D}^*$ is usually referred to as the set of *plausible, non-antidesignated values*.

We adopt a compact notation that avoids superscripts and the bar-notation for set-theoretic complementation, introducing the symbols Y, ⅄, N, and И to denote, respectively, the sets of designated, non-designated ($\mathcal{V} \setminus$ Y), antidesignated, and non-antidesignated ($\mathcal{V} \setminus$ N) values. With a cognitive twist, they might be taken as representing acceptance, non-acceptance, rejection and non-rejection.[1]

If $\mathfrak{M} = \langle \mathcal{A}, \mathsf{Y}, \mathsf{N} \rangle$ is a q-matrix, the *q-entailment* relation $\models_{\mathfrak{M}}^q \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ induced by $\mathfrak{M}$ is defined with respect to a truth-functional semantics as follows:

$$\Gamma \models_{\mathfrak{M}}^q \varphi \text{ iff } (\nu(\Gamma) \cap \mathsf{N} = \varnothing \text{ implies } \nu(\varphi) \in \mathsf{Y}, \text{ for every } \nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A})).$$

If $\mathfrak{M} = \langle \mathcal{A}, \mathsf{Y}, \mathsf{И} \rangle$ is a p-matrix, the *p-entailment* relation $\models_{\mathfrak{M}}^p \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ induced by $\mathfrak{M}$ is defined with respect to a truth-functional semantics as follows:

$$\Gamma \models_{\mathfrak{M}}^p \varphi \text{ iff } (\nu(\Gamma) \subseteq \mathsf{Y} \text{ implies } \nu(\varphi) \in \mathsf{И}, \text{ for every } \nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A})).$$

These definitions are extended to classes of matrices exactly as in the case of the Tarskian notion of consequence.

Let $\mathcal{Q}$ be the class of all q-matrices, and $\mathcal{P}$ be the class of all p-matrices. Clearly, every q-matrix $\mathfrak{M} = \langle \mathcal{A}, \mathsf{Y}, \mathsf{N} \rangle$ uniquely determines a p-matrix $\mathfrak{M}_p = \langle \mathcal{A}, \mathsf{Y}, A \setminus \mathsf{N} \rangle$, and conversely, every p-matrix $\mathfrak{M} = \langle \mathcal{A}, \mathsf{Y}, \mathsf{И} \rangle$ uniquely determines a q-matrix $\mathfrak{M}_q = \langle \mathcal{A}, \mathsf{Y}, A \setminus \mathsf{И} \rangle$. The functions $(\cdot)_p$ and $(\cdot)_q$ are injective, for every $\mathfrak{M} \in \mathcal{Q}$, we have $\mathfrak{M}_{pq} = \mathfrak{M}$, and for every $\mathfrak{M} \in \mathcal{P}$, we have $\mathfrak{M}_{qp} = \mathfrak{M}$. The functions $(\cdot)_{q \circ p}$ and $(\cdot)_{p \circ q}$ are thus bijections, and every q-matrix (p-matrix) can be 'seen' as a p-matrix (q-matrix). Indeed, Frankowski [8] "for the sake of convenience" considers q-entailment

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[1] Malinowski [13] regards $\mathcal{D}^+$ as the set of *accepted* values and $\mathcal{D}^-$ as the set of *rejected* values. Since acceptance is usually associated with the attitude of belief, and rejection with the attitude of disbelief, Malinowski's understanding of $\mathcal{D}^+$ and $\mathcal{D}^-$ may be classified as doxastic.

over p-matrices, and in [24] p-entailment is defined over q-matrices. More-
over, Frankowski uses a deductive perspective on p-matrices to investigate
q-consequence, and Malinowski [19] observes that while q-entailment gener-
alizes the Tarskian notion of consequence by changing the notion of proof,
the generalization produced by p-entailment changes the form of rules of
inference. That being said, we will not study in the present paper the asso-
ciated proof theory of either of these notions of consequence, but will focus
instead on their semantical and their abstract characterizations.

The relations $\models^q_{\mathcal{M}}$ (resp. $\models^p_{\mathcal{M}}$), where $\mathcal{M}$ is a class of q-matrices (resp.
p-matrices) are examples of what Malinowski (resp. Frankowski) refer to as
'q-consequence relations' ('p-consequence relations'). A relation $\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$
is said to be a q-consequence relation if in addition to (Mon) the following
quasi closure axiom is respected for every $\Gamma \cup \{\psi\} \subseteq \mathcal{L}$:

(QTrn) $\Gamma \cup \{\varphi \mid \Gamma \vdash \varphi\} \vdash \psi$ implies $\Gamma \vdash \psi$

Quasi closure is a restricted form of '(cumulative) transitivity', and clearly
constitutes a weakened version of the Tarskian axiom (Trn). A relation
$\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is called a p-consequence relation if $\vdash$ satisfies reflexivity and
monotonicity. It can readily be checked that every q-entailment relation is a
q-consequence relation, and every p-entailment relation is a p-consequence
relation.

We will say that the language $\mathcal{L}$ has algebraic character in case it is the
term algebra generated by a set of propositional variables over a proposi-
tional signature. Endomorphisms of $\mathcal{L}$ are called substitutions. Given
one such substitution $\sigma : \mathcal{L} \longrightarrow \mathcal{L}$ and given $\Pi \subseteq \mathcal{L}$ we write $\sigma(\Pi)$ for
$\{\sigma(\pi) \mid \pi \in \Pi\}$. A relation $\vdash \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ is said to be substitution-invariant
(a.k.a. 'structural') if for $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, and every endomorphism σ of $\mathcal{L}$,
the following axiom is respected:

(SI) $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$

It is well known that every substitution-invariant Tarskian consequence
relation is characterized by a class of matrices (cf. [28]). In addition, Mali-
nowski [13] proved that every substitution-invariant q-consequence relation
is characterized by a class of q-matrices, and Frankowski [7] proved that
every substitution-invariant p-consequence relation is characterized by a
class of p-matrices.

In the present paper we will deal with certain generalized q-matrices
which we shall refer to as 'B-matrices'. For B-matrices the restriction on q-
matrices according to which no value is both designated and antidesignated

is abandoned.[2] This generalization is less straightforward for p-matrices, what makes it attractive to consider p-entailment and other forms of entailment over B-matrices.

A (*logical*) B-*matrix* for $\mathcal{L}$ is a structure $\langle \mathcal{V}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$, where $\mathsf{Y} \subseteq \mathcal{V}$, $\mathsf{N} \subseteq \mathcal{V}$, and the *semantics* S is a collection of mappings $\nu : \mathcal{L} \longrightarrow \mathcal{V}$ called *valuations*. In case $\mathcal{L}$ has algebraic character and $\mathcal{V}$ is an algebra of the same similarity type as $\mathcal{L}$, one may naturally consider a *truth-functional* semantics S defined by the collection $\mathsf{Hom}(\mathcal{L}, \mathcal{V})$ of all homomorphisms from $\mathcal{L}$ into $\mathcal{V}$. Given a family $\mathcal{M} = \{\mathfrak{M}_i\}_{i \in I}$ of B-matrices, we will associate to it the semantics $\mathsf{S}_{\mathcal{M}}$ given by $\bigcup_{i \in I} \mathsf{S}_i$.

As is well-known, the semantic characterization of Tarskian consequence relations in terms of matrices gives room to the so-called *Suszko Reduction* (cf. [3]), which shows that every Tarskian consequence relation may be alternatively characterized by a class of semantical models with two-element carriers. Roman Suszko [25] proposed indeed to distinguish between 'algebraic valuations', which are homomorphic and which he also called *reference assignments*, and 'logical valuations', which are not necessarily homomorphic. From that perspective, what is normally called a κ-valued logic may then be called a *referentially many-valued* logic. The so-called *Suszko's Thesis* (see [15, Ch. 4]) consists in the claim that every referentially many-valued logic can be given a 'bivalent description', namely a characterization in terms of so-called logical valuations whose codomains have at most two 'logical values', the True and the False. As a practical application of that idea, the Suszko Reduction, seen as the technical counterpart of Suszko's Thesis, has nowadays been given a fully algorithmic implementation that applies to any finite-valued logic, and this has been used to provide uniform classic-like analytic deductive counterparts to all such logics (cf. [5]).

Grzegorz Malinowski is especially well-known for his investigation of *inferential many-valuedness* (see [13, 14, 16, 17, 18, 19, 20]). Such enterprise consists in pushing the frontiers of Suszko's Thesis in order to accommodate

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[2]In [26], the set of designated values D of a matrix is required to be non-empty, and in [6], D is required to be a non-empty, proper subset of $\mathcal{V}$. Following the definition of matrices in [6], in [24, p.174] it is assumed that in a (generalized) q-matrix $\langle \mathcal{V}, D^+, D^-, \{f_c \mid c \in \mathcal{C}\} \rangle$, the sets D^+ and D^- are distinct, non-empty, proper subsets of $\mathcal{V}$. With a view towards defining useful entailment relations induced by a matrix or by a (generalized) q-matrix, these restrictions are quite natural and reasonable; for the general characterization of consequence relations, however, such restrictions do not apply.

for other notions of consequence that do not in general allow for a bivalent description. Malinowski [14] proved that every q-consequence relation has a characterization by a class of q-matrices with three-element carrier sets, and showed that the original version of Suszko's Thesis did not apply in general to q-consequence. Such characterization is said to be *inferentially three-valued*, insofar as it makes use of the three sets Y, N, and $\mathcal{V} \setminus (\mathsf{Y} \cup \mathsf{N})$ into which the carrier set of a q-matrix may be partitioned. Frankowski [9] makes an analogous observation for p-consequence relations: to characterize the latter as inferentially three-valued, he makes use of the three sets Y, $\mathcal{V} \setminus \mathsf{И}$ and $\mathsf{И} \setminus \mathsf{Y}$ into which the carrier of a p-matrix may be partitioned.

    As we will see, in the case of B-matrices, the distinguished sets Y and N do not, in general, give rise to a partition of the set $\mathcal{V}$ of truth values in a similar fashion, and their corresponding (four-valued) inferential characterization must be attained using a different strategy. Whilst, on the one hand, a logical matrix displays only *one* distinguished subset of $\mathcal{V}$, namely $\mathcal{D}$, and a second subset of $\mathcal{V}$ is given by the complement of $\mathcal{D}$, on the other hand a q-matrix or, more generally, a B-matrix, displays *two* distinguished sets, Y and N. In [24], starting from such a generalized perspective on the notion of a logical matrix, special attention is paid to the following four, in general pairwise distinct, notions of entailment, with respect to a given generalized q-matrix $\mathfrak{M}$:

t-ent.: $\Gamma \models^t_{\mathfrak{M}} \varphi$ iff $(\nu(\Gamma) \subseteq \mathsf{Y}$ implies $\nu(\varphi) \in \mathsf{Y}$, for all $\nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A}))$

f-ent.: $\Gamma \models^f_{\mathfrak{M}} \varphi$ iff $(\nu(\Gamma) \subseteq \mathsf{И}$ implies $\nu(\varphi) \in \mathsf{И}$, for all $\nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A}))$

q-ent.: $\Gamma \models^q_{\mathfrak{M}} \varphi$ iff $(\nu(\Gamma) \subseteq \mathsf{И}$ implies $\nu(\varphi) \in \mathsf{Y}$, for all $\nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A}))$

p-ent.: $\Gamma \models^p_{\mathfrak{M}} \varphi$ iff $(\nu(\Gamma) \subseteq \mathsf{Y}$ implies $\nu(\varphi) \in \mathsf{И}$, for all $\nu \in \mathsf{Hom}(\mathcal{L}, \mathcal{A}))$

We shall here build on that perspective and generalize it in various respects, in particular by using the distinguished sets Y and N of a B-matrix to originate the four logical values represented by the sets $\mathsf{\lambda} \cap \mathsf{N}$, $\mathsf{\lambda} \cap \mathsf{И}$, $\mathsf{Y} \cap \mathsf{N}$, and $\mathsf{Y} \cap \mathsf{И}$. In the present paper we will show how several distinct notions of entailment, including all the ones mentioned above, may be defined with the use of such distinguished sets, on top of the thereby defined 'logical values'.

    In what follows, it is first shown that every purely monotonic single-conclusion consequence relation is characterized by a class of B-matrices with respect to q-entailment as well as with respect to p-entailment, and it is observed that, as a result, every purely monotonic single-conclusion

consequence relation has an inferentially four-valued semantics. Next, the notion of entailment is generalized so as to obtain a *two-dimensional* notion of B-entailment, based on B-matrices, that subsumes the above defined notions of *t*-, *f*-, *q*- and *p*-entailment. In a multiple-conclusion setting, sixteen notions of entailment are studied in detail, from both an abstract viewpoint and an inferential viewpoint. It is shown that these notions collapse into four classes, in terms of their abstract characterizations. The Tarskian notion of consequence is also generalized to the two-dimensional setting by introducing the notion of a B-consequence relation that subsumes, among others, the notions of *q*- and *p*-consequence. Moreover, an abstract characterization of B-consequence relations by classes of B-matrices is presented. Next, the Suszko Reduction is generalized to show that any B-consequence relation has, in general, an inferentially four-valued characterization. Finally, for any given specific B-entailment relation, it is shown that it may accommodate in a natural way up to nine one-dimensional notions of entailment of different kinds.

2. Abstract characterization of single-conclusion purely monotonic consequence relations

In this section we show that every *purely monotonic consequence relation* $\mathsf{C} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ —namely, a relation respecting axiom (Mon)— is characterized by a class of B-matrices with respect to *q*-entailment as well as by a class of B-matrices with respect to *p*-entailment. Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$ and $\mathsf{C} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$, we shall write $\mathsf{C}(\Gamma)$ for $\{\varphi \mid (\Gamma, \varphi) \in \mathsf{C}\}$. Note that in terms of the latter unary operation on $2^{\mathcal{L}}$, monotonicity means simply that $\mathsf{C}(\Phi) \subseteq \mathsf{C}(\Phi \cup \Psi)$.

    Let $\mathsf{C} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ be a purely monotonic consequence relation. For every $\Gamma \subseteq \mathcal{L}$, the tuple $\mathfrak{M}_{\Gamma}^{q} = \langle \mathcal{L}, \mathsf{C}(\Gamma), \mathcal{L} \setminus \Gamma, \{\mathrm{id}\} \rangle$, where id is the identity mapping on $\mathcal{L}$, is a B-matrix. We call $\mathfrak{M}_{\Gamma}^{q}$ the *Lindenbaum* B-*matrix of* Γ *with respect to q-entailment* and set $\mathcal{B}_{\mathsf{C}}^{q} = \{\mathfrak{M}_{\Gamma}^{q} \mid \Gamma \subseteq \mathcal{L}\}$.

THEOREM 1. *Every purely monotonic consequence relation* C *is characterized by some class of Lindenbaum* B-*matrices with respect to q-entailment.*

PROOF: We show that C is characterized by $\mathcal{B}_{\mathsf{C}}^{q}$.
($\Rightarrow$) Let $\varphi \in \mathsf{C}(\Gamma)$, let $\mathfrak{M}_{\Delta}^{q}$ be an arbitrary B-matrix from $\mathcal{B}_{\mathsf{C}}^{q}$, and suppose that $\Gamma \cap (\mathcal{L} \setminus \Delta) = \varnothing$ and hence $\Gamma \subseteq \Delta$. By monotonicity, we know that $\mathsf{C}(\Gamma) \subseteq \mathsf{C}(\Delta)$. Therefore, $\Gamma \models_{\mathfrak{M}_{\Delta}^{q}}^{q} \varphi$. Since Δ was arbitrary, it follows that $\Gamma \models_{\mathcal{B}_{\mathsf{C}}^{q}}^{q} \varphi$.

($\Leftarrow$) Suppose that $\Gamma \models^q_{\mathcal{B}^q_{\mathsf{C}}} \varphi$. We then have $\Gamma \models^q_{\mathfrak{M}^q_{\Delta}} \varphi$ for every $\mathfrak{M}^q_{\Delta} \in \mathcal{B}^q_{\mathsf{C}}$. In particular, $\Gamma \models^q_{\mathfrak{M}^q_{\Gamma}} \varphi$. Since $\Gamma \cap (\mathcal{L} \setminus \Gamma) = \varnothing$, we conclude that $\varphi \in \mathsf{C}(\Gamma)$. $\square$

Let $\mathsf{C} \subseteq 2^{\mathcal{L}} \times \mathcal{L}$ be a purely monotonic consequence relation. For every $\Gamma \subseteq \mathcal{L}$, the tuple $\mathfrak{M}^p_{\Gamma} = \langle \mathcal{L}, \Gamma, \mathcal{L} \setminus \mathsf{C}(\Gamma), \{\mathsf{id}\} \rangle$ is a B-matrix. We call $\mathfrak{M}^p_{\Gamma}$ the *Lindenbaum* B-*matrix of* Γ *with respect to p-entailment* and set $\mathcal{B}^p_{\mathsf{C}} = \{\mathfrak{M}^p_{\Gamma} \mid \Gamma \subseteq \mathcal{L}\}$.

THEOREM 2. *Every purely monotonic consequence relation* C *is characterized by some class of Lindenbaum* B-*matrices with respect to p-entailment.*

PROOF: We show that C is characterized by $\mathcal{B}^p_{\mathsf{C}}$.
($\Rightarrow$) Let $\varphi \in \mathsf{C}(\Gamma)$, let $\mathfrak{M}^p_{\Delta}$ be an arbitrary B-matrix from $\mathcal{B}^p_{\mathsf{C}}$, and suppose that $\Gamma \subseteq \Delta$. By monotonicity, we know that $\mathsf{C}(\Gamma) \subseteq \mathsf{C}(\Delta)$. Therefore $\varphi \notin \mathcal{L} \setminus \mathsf{C}(\Delta)$ and thus $\Gamma \models^p_{\mathfrak{M}^p_{\Delta}} \varphi$. Since Δ was arbitrary, it follows that $\Gamma \models^p_{\mathcal{B}^p_{\mathsf{C}}} \varphi$.
($\Leftarrow$) Suppose that $\Gamma \models^p_{\mathcal{B}^p_{\mathsf{C}}} \varphi$. Then $\Gamma \models^p_{\mathfrak{M}^p_{\Delta}} \varphi$ for every $\mathfrak{M}^p_{\Delta} \in \mathcal{B}^p_{\mathsf{C}}$. In particular, $\Gamma \models^p_{\mathfrak{M}^p_{\Gamma}} \varphi$. Since Γ is the set of designated values of $\mathfrak{M}^p_{\Gamma}$, we have $\varphi \notin \mathcal{L} \setminus \mathsf{C}(\Gamma)$, and hence $\varphi \in \mathsf{C}(\Gamma)$. $\square$

If we have homomorphic valuations in mind, a few adjustments in the above characterizations are in order. In particular, the Lindenbaum matrices $\mathfrak{M}^q_{\Gamma}$ and $\mathfrak{M}^p_{\Gamma}$ are in such case redefined so that instead of $\mathsf{S} = \{\mathsf{id}\}$ we take $\mathsf{S} = \mathsf{Hom}(\mathcal{L}, \mathcal{L})$.

THEOREM 3. *Every substitution-invariant purely monotonic consequence relation* C *is characterized by some class of* B-*matrices with respect to q-entailment, with homomorphic valuations.*

PROOF: We show again that C is characterized by $\mathcal{B}^q_{\mathsf{C}}$.
($\Rightarrow$) Let $\varphi \in \mathsf{C}(\Gamma)$, let $\mathfrak{M}^q_{\Delta} = \langle \mathcal{L}, \mathsf{C}(\Delta), \mathcal{L} \setminus \Delta, \mathsf{Hom}(\mathcal{L}, \mathcal{L}) \rangle$ be an arbitrary B-matrix from $\mathcal{B}^q_{\mathsf{C}}$, and let σ be an arbitrary endomorphism of $\mathcal{L}$ for which $\sigma(\Gamma) \cap (\mathcal{L} \setminus \Delta) = \varnothing$. Then $\sigma(\Gamma) \subseteq \Delta$. By substitution-invariance, $\sigma(\varphi) \in \mathsf{C}(\sigma(\Gamma))$, and by monotonicity, $\mathsf{C}(\sigma(\Gamma)) \subseteq \mathsf{C}(\Delta)$. Thus, $\sigma(\varphi) \in \mathsf{C}(\Delta)$. Therefore, $\Gamma \models^q_{\mathfrak{M}^q_{\Delta}} \varphi$. Since Δ was arbitrary, it follows that $\Gamma \models^q_{\mathcal{B}^q_{\mathsf{C}}} \varphi$.
($\Leftarrow$) Suppose that $\Gamma \models^q_{\mathcal{B}^q_{\mathsf{C}}} \varphi$. Then $\Gamma \models^q_{\mathfrak{M}^q_{\Delta}} \varphi$ for every $\mathfrak{M}^q_{\Delta} \in \mathcal{B}^q_{\mathsf{C}}$. In particular, $\Gamma \models^q_{\mathfrak{M}^q_{\Gamma}} \varphi$. Since the identity mapping id on $\mathcal{L}$ is an endomorphism of $\mathcal{L}$, and $\Gamma \cap (\mathcal{L} \setminus \Gamma) = \varnothing$, it follows that $\varphi \in \mathsf{C}(\Gamma)$. $\square$

THEOREM 4. *Every substitution-invariant purely monotonic consequence relation* C *is characterized by some class of* B*-matrices with respect to p-entailment, with homomorphic valuations.*

PROOF: We show again that C is characterized by $\mathcal{B}_\mathsf{C}^p$.

($\Rightarrow$) Let $\varphi \in \mathsf{C}(\Gamma)$, let $\mathfrak{M}_\Delta^p = \langle \mathcal{L}, \Delta, \mathcal{L} \setminus \mathsf{C}(\Delta), \mathsf{Hom}(\mathcal{L}, \mathcal{L}) \rangle$ be an arbitrary B-matrix from $\mathcal{B}_\mathsf{C}^p$, and let σ be an arbitrary endomorphism of $\mathcal{L}$ with $\sigma(\Gamma) \subseteq \Delta$. By substitution-invariance, $\sigma(\varphi) \in \mathsf{C}(\sigma(\Gamma))$, and by monotonicity, $\mathsf{C}(\sigma(\Gamma)) \subseteq \mathsf{C}(\Delta)$. Therefore $\sigma(\varphi) \notin \mathcal{L} \setminus \mathsf{C}(\Delta)$ and thus $\Gamma \models_{\mathfrak{M}_\Delta^q}^q \varphi$. Since Δ was arbitrary, we conclude that $\Gamma \models_{\mathcal{B}_\mathsf{C}^p}^q \varphi$.

($\Leftarrow$) Suppose that $\Gamma \models_{\mathcal{B}_\mathsf{C}^p}^p \varphi$. Then $\Gamma \models_{\mathfrak{M}_\Delta^p}^p \varphi$ for every $\mathfrak{M}_\Delta^p \in \mathcal{B}_\mathsf{C}^p$. In particular, $\Gamma \models_{\mathfrak{M}_\Gamma^p}^p \varphi$. Since $\mathsf{C}(\Gamma) \cap (\mathcal{L} \setminus \mathsf{C}(\Gamma)) = \varnothing$, we obtain $\varphi \in \mathsf{C}(\Gamma)$. $\square$

Given the above characterizations, it is possible to upgrade the machinery behind the so-called Suszko Reduction, as will be done in Section 5, to show that every monotonic relation $\mathsf{C} \subseteq 2^\mathcal{L} \times \mathcal{L}$ has an at most four-valued (in general non-truth-functional) semantics. Let $\mathfrak{M} = \langle \mathcal{V}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$ be a B-matrix. It is enough then to build out of this a B-matrix $\mathfrak{M}'$ which is indistinguishable from $\mathfrak{M}$ from the viewpoint of q- as well as of p-entailment, by setting $\mathfrak{M}' = \langle \{F, N, B, T\}, \{B, T\}, \{F, B\}, \{\nu_4 \mid \nu \in\} \rangle$ where

$$\nu_4(\varphi) = \begin{cases} F & \text{if } \nu(\varphi) \in \mathsf{X} \cap \mathsf{N} \\ N & \text{if } \nu(\varphi) \in \mathsf{X} \cap \mathsf{И} \\ B & \text{if } \nu(\varphi) \in \mathsf{Y} \cap \mathsf{N} \\ T & \text{if } \nu(\varphi) \in \mathsf{Y} \cap \mathsf{И} \end{cases}$$

It is not difficult to see that for any $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}$, we have $\Gamma \models_{\mathfrak{M}}^q \varphi$ iff $\Gamma \models_{\mathfrak{M}'}^q \varphi$ and $\Gamma \models_{\mathfrak{M}}^p \varphi$ iff $\Gamma \models_{\mathfrak{M}'}^p \varphi$ (for a more general and detailed version of this result, check the proof of Theorem 11).

3. A uniform framework for the study of diverse forms of entailment

We now introduce a two-dimensional, B-matrix-based notion of semantical consequence. Consider a B-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$. The semantical notion of B-*entailment* canonically induced by the semantics S is defined by setting:

$$\frac{\Phi_{11}}{\Phi_{21}}\Big|\frac{\Phi_{12}}{\Phi_{22}} \text{ is S-}valid \text{ iff } \begin{array}{l}\text{there is no } \nu \in \mathsf{S} \text{ such that}\\ \nu(\Phi_{11}) \subseteq \mathsf{И} \text{ and } \nu(\Phi_{12}) \subseteq \mathsf{入} \text{ and } \quad (\mathsf{B}\text{-}entailment)\\ \nu(\Phi_{21}) \subseteq \mathsf{Y} \text{ and } \nu(\Phi_{22}) \subseteq \mathsf{N}\end{array}$$

where each Φ_{mn} denotes an arbitrary subset of $\mathcal{L}$. Rather than saying that $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is S-valid, sometimes we say that $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is *valid for* $\mathfrak{M}$, and we omit the reference to the semantics or to the matrix if the context suffices to disambiguate it.

From the above definition of distinguished sets, it is straightforward to note that B-entailment always enjoys, in particular, the following properties concerning S-validity:

(In$^\mathsf{y}$) $\frac{\varnothing}{\Phi}\big|\frac{\Psi}{\varnothing}$ is valid, whenever $\Phi \cap \Psi \neq \varnothing$

(In$^\mathsf{n}$) $\frac{\Phi}{\varnothing}\big|\frac{\varnothing}{\Psi}$ is valid, whenever $\Phi \cap \Psi \neq \varnothing$

(C1$^\mathsf{y}$)   if both $\frac{\Phi_{11}}{\Phi_{21}\cup\{\varphi\}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ and $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}\cup\{\varphi\}}{\Phi_{22}}$ are valid,
           then $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is valid

(C1$^\mathsf{n}$)   if both $\frac{\Phi_{11}\cup\{\varphi\}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ and $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}\cup\{\varphi\}}$ are valid,
           then $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is valid

Any expression of the form $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ will henceforth be called a B-*statement*. In case both Φ_{11} and Φ_{22} are empty, we will write the corresponding B-statement as $\frac{\cdot\cdot}{\Phi_{21}}\big|\frac{\Phi_{12}}{\cdot\cdot}$, and call it a T-*statement* (mnemonic: $\mathsf{Y} \Rightarrow \mathsf{Y}$). In addition, and using a similar notational convention, the expression $\frac{\Phi_{11}}{\cdot\cdot}\big|\frac{\cdot\cdot}{\Phi_{22}}$ will be called an F-*statement* (mnemonic: $\mathsf{И} \Rightarrow \mathsf{И}$), the expression $\frac{\Phi_{11}}{\cdot\cdot}\big|\frac{\Phi_{12}}{\cdot\cdot}$ will be called a Q-*statement* (mnemonic: $\mathsf{И} \Rightarrow \mathsf{Y}$), and the expression $\frac{\cdot\cdot}{\Phi_{21}}\big|\frac{\cdot\cdot}{\Phi_{22}}$ will be called a P-*statement* (mnemonic: $\mathsf{Y} \Rightarrow \mathsf{И}$).

In general, we will say about a B-matrix that it *allows for gappy reasoning* in case $\mathsf{入} \cap \mathsf{И} \neq \varnothing$ (equivalently, $\mathsf{入} \nsubseteq \mathsf{N}$ or $\mathsf{И} \nsubseteq \mathsf{Y}$), and say that it *allows for glutty reasoning* in case $\mathsf{Y} \cap \mathsf{N} \neq \varnothing$ (equivalently, $\mathsf{N} \nsubseteq \mathsf{入}$ or $\mathsf{Y} \nsubseteq \mathsf{И}$). It is easy to check that the following properties are respected whenever glutty reasoning is *not* allowed for:

(In$^\text{u}$) $\frac{\varnothing}{\Phi}\big|\frac{\varnothing}{\Psi}$ is valid, whenever $\Phi \cap \Psi \neq \varnothing$

(C1$^\text{u}$)   if both $\frac{\Phi_{11}\cup\{\varphi\}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ and $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}\cup\{\varphi\}}{\Phi_{22}}$ are valid,
    then $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is valid

and the following properties are respected whenever gappy reasoning is *not* allowed for:

(In$^\text{a}$) $\frac{\Phi}{\varnothing}\big|\frac{\Psi}{\varnothing}$ if valid, whenever $\Phi \cap \Psi \neq \varnothing$

(C1$^\text{a}$)   if both $\frac{\Phi_{11}}{\Phi_{21}\cup\{\varphi\}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ and $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}\cup\{\varphi\}}$ are valid,
    then $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ is valid

Note that a B-matrix with a semantics S allows for gappy reasoning iff the Q-statement $\frac{\varphi}{..}\big|\frac{\varphi}{..}$ fails to be S-valid, for some $\varphi \in \mathcal{L}$, and it allows for glutty reasoning iff the P-statement $\frac{..}{\varphi}\big|\frac{..}{\varphi}$ fails to be S-valid, for some $\varphi \in \mathcal{L}$.

Fixed a B-matrix $\mathfrak{M}$, and the collection $\mathfrak{S}$ of all T-statements validated by its semantics, we associate to $\mathfrak{M}$ a one-dimensional gt-*entailment relation* $\models^{\text{gt}} \subseteq 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ by setting $\Phi \models^{\text{gt}} \Psi$ iff $\frac{..}{\Phi}\big|\frac{\Psi}{..}$ is in $\mathfrak{S}$. Along the same lines, we define a one-dimensional gf-*entailment relation* $\models^{\text{gf}} \subseteq 2^{\mathcal{L}} \times 2^{\mathcal{L}}$ from the collection of all F-statements validated by the semantics of $\mathfrak{M}$. Similarly, we define a gq-*entailment relation* and a gp-*entailment relation*, respectively, from the collection of all Q-statements and the collection of all P-statements validated by the semantics of $\mathfrak{M}$. For each such notion of gx-entailment we define a one-dimensional gx\u-*entailment* relation from a collection of X-statements together with the assumption (expressed by an appropriate collection of P-statements, as pointed out above) that $\mathfrak{M}$ does not allow for glutty reasoning, and define a gx\a-*entailment* relation from a collection of X-statements together with the assumption (expressed by a collection of Q-statements) that $\mathfrak{M}$ does not allow for gappy reasoning. Analogously, a gx\ua-*entailment* relation will be defined from a collection of X-statements together with the assumption that $\mathfrak{M}$ allows neither for glutty nor for gappy reasoning. A gt\ua-entailment relation over $\mathfrak{M}$ will here more simply be called a t-*entailment* relation over $\mathfrak{M}$, and a gf\ua-entailment relation over $\mathfrak{M}$ will be called an f-*entailment* relation over $\mathfrak{M}$. In addition, a gq\u-entailment relation over $\mathfrak{M}$ will here more simply be called a q-*entailment* relation over $\mathfrak{M}$, and a gp\u-entailment relation over $\mathfrak{M}$ will be called a p-*entailment* relation over $\mathfrak{M}$. To simplify notation, we shall also use wq instead of gq\ua, and wp instead of gp\ua. Finally, gp\a-entailment relations over $\mathfrak{M}$ may be said to be 'dual' to q-entailment (they

consist in collections of P-statements disallowing gaps, instead of collections of Q-statements disallowing gluts), and will henceforth be referred to as d-*entailment* relations. Analogously, gq\a-entailment relations over $\mathfrak{M}$ dualize p-entailment, and will henceforth be referred to as b-*entailment* relations.[3] Please refer to Table 1 for a compilation of the above definitions and notational conventions.[4]

It is easy to attest that a B-matrix allowing for gappy reasoning does not in general give support to (In^a) and (C1^a). To check this, it suffices to consider a B-matrix such that $Y \cup N \neq \mathcal{V}$ and consider a semantics containing a valuation ν and some $\varphi \in \mathcal{L}$ such that $\nu(\varphi) \in \curlywedge \cap \text{И}$. The failure of (In^a) —and the ensuing failure of $\varphi \models^q \varphi$, in general— justifies why q-entailment is often said to be 'non-reflexive', while the failure of (C1^a) —and the fact that $\Phi \models^p \Psi$ does not necessarily follow from $\Phi \cup \{\varphi\} \models^p \Psi$ and $\Phi \models^p \Psi \cup \{\varphi\}$— justifies why p-entailment is said to be 'non-transitive'. For analogous reasons, d-entailment also fails, in general, to be reflexive, and b-entailment also fails, in general, to be transitive. At any rate, in case a B-matrix identifies designatedness with non-antidesignatedness (i.e., in case it takes $Y = \text{И}$) and identifies antidesignatedness with non-designatedness (i.e., it takes $N = \curlywedge$), then it should be clear that the properties called (In^x) and (C1^x), for $x \in \{y, n, a, u\}$, are *all* enjoyed by the corresponding B-entailment relation, and there is in such a situation no difference in semantic status to be found between T-, F-, Q- and P-statements.

4. Consequence in one and in two dimensions

Following Shoesmith & Smiley's [23], a symmetrical one-dimensional generalization of the Tarskian notion of consequence is given by a 2-place relation $\cdot \Vdash \cdot$ on subsets of $\mathcal{L}$ subject to the following axioms:

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[3]The attentive reader will have noticed that q-entailment and p-entailment generalize to a multiple-conclusion environment the notions of *q*-entailment and *p*-entailment introduced in Section 1. Moreover, it is worth noting that the notion of d-entailment, as dual to q-entailment, was introduced in a single-conclusion environment by Malinowski in [17].

[4]As mnemonics, we let 'g' stand for 'generalized', 'w' for 'weakened', 'a' for 'gaps' and 'u' for 'gluts'.

x-entailment	X-statements	logical values	matrix geometry
t	T: $\frac{\cdots}{\Phi}\big\vert\frac{\Psi}{\cdots}$ (Y⇒Y)	neither gaps nor gluts	Y=И N=λ
f	F: $\frac{\Phi}{\cdots}\big\vert\frac{\cdots}{\Psi}$ (И⇒И)		
wq	Q: $\frac{\Phi}{\cdots}\big\vert\frac{\Psi}{\cdots}$ (И⇒Y)		
wp	P: $\frac{\cdots}{\Phi}\big\vert\frac{\cdots}{\Psi}$ (Y⇒И)		
gt\u	T	no gluts: $\frac{\cdots}{\varphi}\big\vert\frac{\cdots}{\varphi}$	Y N
gf\u	F		
q	Q		
p	P		
gt\a	T	no gaps: $\frac{\varphi}{\cdots}\big\vert\frac{\varphi}{\cdots}$	Y N
gf\a	F		
b (dual-p)	Q		
d (dual-q)	P		
gt	T	may allow for both gappy and glutty reasoning	Y N
gf	F		
gq	Q		
gp	P		

Table 1. Some one-dimensional notions of entailment over B-matrices

(Over) $\Phi \Vdash \Psi$, whenever $\Phi \cap \Psi \neq \varnothing$

(1Ext)   if $\Phi \Vdash \Psi$, then $\Phi \cup \Phi' \Vdash \Psi \cup \Psi'$

(CTrn)   given $\Pi \subseteq \mathcal{L}$, if $\Phi \cup \Sigma \Vdash \Psi \cup (\Pi \setminus \Sigma)$ for every $\Sigma \subseteq \Pi$, then $\Phi \Vdash \Psi$

Above, 'Over', 'Ext' and 'CTrn' stand, respectively, for *overlap*, *extendability* and (*cumulative*) *transitivity*. Whenever there is need to avoid ambiguity, instead of simply writing $\Phi \Vdash \Psi$ we shall say that $\big[\Phi \; ; \; \Psi\big]$ *holds according to* $\cdot \Vdash \cdot$. We omit the reference to the consequence relation if the context suffices to disambiguate it.

While, on the one hand, it is an easy exercise to check that t-, f-, gt-, gf-, gt\u-, gt\a-, gf\u-, gf\a-, wq-, and wp-entailment relations all respect the above axioms, on the other hand it is well known that any relation respecting these axioms may be characterized by an appropriate class of matrices. We revise such result in the context of B-consequence and B-entailment by showing first that for each particular symmetrical Tarskian consequence relation C there is a t-entailment relation that characterizes it in terms of families of B-matrices. For that matter we will say that a subset Δ of $\mathcal{L}$ is t-*closed* if $\left[\Delta \; ; \; (\mathcal{L} \setminus \Delta)\right]$ fails to hold (according to C), and we will associate to such t-closed set the semantical model given by a B-matrix $\mathfrak{M}^{\mathsf{t}} = \langle \mathcal{L}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$ where $\mathsf{Y} = \Delta$, $\mathsf{N} = \mathcal{L} \setminus \Delta$, and S contains just the semantical model given by the identity mapping id on $\mathcal{L}$. We call *Lindenbaum* t-*bundle* $\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}$ the family of B-matrices associated to all the t-closed subsets of $\mathcal{L}$. Then:

THEOREM 5. *Any* t-*consequence relation* C *is sound and complete with respect to* $\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}$, *that is,* $\left[\Phi \; ; \; \Psi\right]$ *holds according to* C *iff* $\Phi \models^{\mathsf{t}} \Psi$ *is* $\mathsf{S}_{\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}}$-*valid.*

PROOF: ($\Rightarrow$) Suppose $\Phi \models^{\mathsf{t}} \Psi$ is not $\mathsf{S}_{\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}}$-valid. By the definition of t-entailment, this is to say that $\frac{\ddots}{\Phi}\big|\frac{\Psi}{\ddots}$ is not $\mathsf{S}_{\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}}$-valid. So, there must be some B-matrix $\langle \mathcal{L}, \mathsf{Y}, \mathcal{L} \setminus \mathsf{Y}, \{\mathsf{id}\} \rangle$ in the Lindenbaum t-bundle $\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}$ for which $\frac{\ddots}{\Phi}\big|\frac{\Psi}{\ddots}$ fails to be valid. By the definition of S-validity, and taking into account that id is the identity mapping on $\mathcal{L}$, it follows that $\Phi \subseteq \mathsf{Y}$ and $\Psi \subseteq \mathcal{L} \setminus \mathsf{Y}$. By the very definition of $\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}$, we know that Y is t-closed, thus $\left[\mathsf{Y} \; ; \; (\mathcal{L} \setminus \mathsf{Y})\right]$ fails to hold according to C. Using (1Ext), we then conclude that $\left[\Phi \; ; \; \Psi\right]$ fails to hold according to C.

($\Leftarrow$) Suppose now that $\left[\Phi \; ; \; \Psi\right]$ fails to hold according to C. By (CTrn), we know that there must be some $\mathsf{Y} \subseteq \mathcal{L}$ such that $\left[\Phi \cup \mathsf{Y} \; ; \; \Psi \cup (\mathcal{L} \setminus \mathsf{Y})\right]$ fails to hold according to C. By (Over), it follows that $\Phi \subseteq \mathsf{Y}$ and $\Psi \subseteq \mathcal{L} \setminus \mathsf{Y}$. Thus, by the definition of t-entailment, we see that $\Phi \models^{\mathsf{t}} \Psi$ is not valid, that is, the T-statement $\frac{\ddots}{\Phi}\big|\frac{\Psi}{\ddots}$ is not valid according to the B-matrix $\mathfrak{M}^{\mathsf{t}} = \langle \mathcal{L}, \mathsf{Y}, \mathcal{L} \setminus \mathsf{Y}, \{\mathsf{id}\} \rangle$. So, *a fortiori*, this same T-statement also fails to be valid according to $\mathcal{B}^{\mathsf{t}}_{\mathsf{C}}$. □

The very same result holds for the gt-, gt\a- and gt\u-entailment relations, for which the notion of t-closure applies equally well. It is easy to adapt that result for f-entailment. Indeed, call f-*closed* any subset Δ of $\mathcal{L}$ such that

$[(\mathcal{L} \setminus \Delta) \; ; \; \Delta]$ fails to hold (according to a given consequence relation C), associate to such f-closed set a B-matrix $\mathfrak{M}^{\mathsf{f}} = \langle \mathcal{L}, \mathcal{L} \setminus \Delta, \Delta, \{\mathsf{id}\} \rangle$, and let the corresponding *Lindenbaum f-bundle* be the family of B-matrices associated to the f-closed subsets of $\mathcal{L}$. The necessary adjustments in the proof of Theorem 5 are then immediate. Again, the very same result holds for the gt-, gf\a- and gf\u-entailment relations. In addition, either result may be adapted to the case of wq- and wp-entailment relations — here, given that the latter entailment relations are neither gappy nor glutty, one may use the fact that $\mathsf{Y} = \mathsf{N}$ and $\mathsf{N} = \lambda$.

We prove next an analogous result for q-entailment and its dual. To axiomatize q-entailment we will make use of extendability and transitivity. Accordingly, a q-*consequence relation* will be a relation subject to axioms (1Ext) and (CTrn). It is an easy exercise to check that q-entailment relations respect both these axioms. Given a q-consequence relation C, and two disjoint subsets Δ_1 and Δ_2 of $\mathcal{L}$, we will say that $\langle \Delta_1, \Delta_2 \rangle$ is a q-*closed pair* if $[\mathcal{L} \setminus \Delta_2 \; ; \; \mathcal{L} \setminus \Delta_1]$ fails to hold (according to C), and we will associate to such q-closed pair a B-matrix $\mathfrak{M}^{\mathsf{q}} = \langle \mathcal{L}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$ where $\mathsf{Y} = \Delta_1$, $\mathsf{N} = \Delta_2$, and S contains just the identity mapping id on $\mathcal{L}$. We call *Lindenbaum q-bundle* $\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}$ the family of B-matrices associated to all the q-closed pairs of $\mathcal{L}$. We can then prove that:

THEOREM 6. *Any* q-*consequence relation* C *is sound and complete with respect to* $\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}$, *that is,* $[\Phi \; ; \; \Psi]$ *holds according to* C *iff* $\Phi \models^{\mathsf{q}} \Psi$ *is* $\mathsf{S}_{\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}}$-*valid.*

PROOF: ($\Rightarrow$) Suppose $\Phi \models^{\mathsf{q}} \Psi$ is not $\mathsf{S}_{\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}}$-valid. By the definition of q-entailment, this is to say that $\frac{\Phi}{\because} \Big| \frac{\Psi}{\because}$ is not $\mathsf{S}_{\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}}$-valid. So, in the Lindenbaum q-bundle $\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}$ there must be some B-matrix $\langle \mathcal{L}, \mathsf{Y}, \mathsf{N}, \{\mathsf{id}\} \rangle$ where $\mathsf{Y} \cap \mathsf{N} = \varnothing$ for which $\frac{\Phi}{\because} \Big| \frac{\Psi}{\because}$ fails to be valid. By the definition of B-entailment, it follows that $\Phi \subseteq \mathsf{N}$ and $\Psi \subseteq \lambda$. By the very definition of $\mathcal{B}_{\mathsf{C}}^{\mathsf{q}}$, we know that $\langle \mathsf{Y}, \mathsf{N} \rangle$ is q-closed, thus $[\mathsf{N} \; ; \; \lambda]$ fails to hold according to C. Then, using (1Ext) we conclude that $[\Phi \; ; \; \Psi]$ fails to hold according to C.

($\Leftarrow$) Suppose now that $[\Phi \; ; \; \Psi]$ fails to hold according to C. By (CTrn) we know that there must be some set $\mathsf{N} \subseteq \mathcal{L}$ such that $[\Phi \cup \mathsf{N} \; ; \; \Psi \cup \lambda]$, where $\lambda = \mathcal{L} \setminus \mathsf{N}$, fails to hold according to C. Note that $\mathcal{L} \setminus (\Psi \cup \lambda)$ and $\mathcal{L} \setminus (\Phi \cup \mathsf{N})$ are disjoint, thus $\langle \Psi \cup \lambda, \Phi \cup \mathsf{N} \rangle$ is a q-closed pair according to C. By the definition of q-entailment, we then see that $\Phi \models^{\mathsf{q}} \Psi$ is not

valid, that is, the Q-statement $\frac{\Phi}{\vdots}\Big|\frac{\Psi}{\vdots}$ is not valid according to the B-matrix $\mathfrak{M}^{\mathsf{q}} = \langle \mathcal{L}, \mathcal{L}\setminus(\Psi\cup\lambda), \mathcal{L}\setminus(\Phi\cup\mathcal{M}), \{\mathsf{id}\}\rangle$. So, *a fortiori*, this same Q-statement also fails to be valid according to $\mathcal{B}^{\mathsf{q}}_{\mathsf{C}}$. □

It is easy to adapt this result for d-entailment. In that case, given a q-consequence relation C, we might call d-*closed* any pair $\langle\Delta_1,\Delta_2\rangle$ of sets such that $\Delta_1 \cup \Delta_2 = \mathcal{L}$ and $\big[\Delta_1 \; ; \; \Delta_2\big]$ fails to hold (according to C), associate to such d-closed pair a B-matrix $\mathfrak{M}^{\mathsf{d}} = \langle\mathcal{L}, \Delta_1, \Delta_2, \{\mathsf{id}\}\rangle$, and let the corresponding *Lindenbaum* d-*bundle* be the family of B-matrices associated to the d-closed subsets of $\mathcal{L}$. The necessary adjustments in the proof of Theorem 6 are then immediate.

To prove an analogous result for p-entailment ($\mathsf{Y} \Rightarrow \mathcal{M}$) and its dual, we first define a p-*consequence relation* as a relation subject to axioms (Over) and (1Ext). It is an easy exercise to check that p-entailment respects axioms (Over) and (1Ext). Given a p-consequence relation C, and two disjoint subsets Δ_1 and Δ_2 of $\mathcal{L}$, we will say that $\langle\Delta_1,\Delta_2\rangle$ is a p-*closed pair* if $\big[\Delta_1 \; ; \; \Delta_2\big]$ fails to hold (according to C), and we associate to such p-closed pair a B-matrix $\mathfrak{M}^{\mathsf{p}} = \langle\mathcal{L}, \mathsf{Y}, \mathsf{N}, \mathsf{S}\rangle$ where $\mathsf{Y} = \Delta_1$, $\mathsf{N} = \Delta_2$, and $\mathsf{S} = \{\mathsf{id}\}$. We call *Lindenbaum* p-*bundle* $\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}$ the family of B-matrices associated to all the p-closed pairs of $\mathcal{L}$. We can then prove that:

THEOREM 7. *Any* p-*consequence relation* C *is sound and complete with respect to* $\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}$, *that is,* $\big[\Phi \; ; \; \Psi\big]$ *holds according to* C *iff* $\Phi \models^{\mathsf{p}} \Psi$ *is* $\mathsf{S}_{\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}}$-*valid.*

PROOF: ($\Rightarrow$) Suppose $\Phi \models^{\mathsf{p}} \Psi$ is not $\mathsf{S}_{\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}}$-valid. By the definition of p-entailment, this is to say that $\frac{\vdots}{\Phi}\Big|\frac{\vdots}{\Psi}$ is not $\mathsf{S}_{\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}}$-valid. So, in the Lindenbaum p-bundle $\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}$ there must be some B-matrix $\langle\mathcal{L}, \mathsf{Y}, \mathsf{N}, \{\mathsf{id}\}\rangle$ where $\mathsf{Y} \cap \mathsf{N} = \varnothing$ for which $\frac{\vdots}{\Phi}\Big|\frac{\vdots}{\Psi}$ fails to be valid. By the definition of B-entailment, it follows that $\Phi \subseteq \mathsf{Y}$ and $\Psi \subseteq \mathsf{N}$. By the very definition of $\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}$, we know that $\langle\mathsf{Y},\mathsf{N}\rangle$ is p-closed, thus $\big[\mathsf{Y} \; ; \; \mathsf{N}\big]$ fails to hold according to C. Using (1Ext), we conclude that $\big[\Phi \; ; \; \Psi\big]$ fails to hold according to C.

($\Leftarrow$) Suppose now that $\big[\Phi \; ; \; \Psi\big]$ fails to hold according to C. By (Over), it follows that Φ and Ψ are disjoint, thus $\langle\Phi,\Psi\rangle$ is a p-closed pair according to C. By the definition of p-entailment, this implies that $\Phi \models^{\mathsf{p}} \Psi$ fails to be valid, given that the P-statement $\frac{\vdots}{\Phi}\Big|\frac{\vdots}{\Psi}$ fails to be valid according to the B-matrix $\mathfrak{M}^{\mathsf{p}} = \langle\mathcal{L}, \Phi, \Psi, \{\mathsf{id}\}\rangle$. So, *a fortiori*, this same P-statement also fails to be valid according to $\mathcal{B}^{\mathsf{p}}_{\mathsf{C}}$. □

To adapt this result for b-entailment, given a p-consequence relation C, we might call b-*closed* any pair $\langle \Delta_1, \Delta_2 \rangle$ of sets such that $\Delta_1 \cup \Delta_2 = \mathcal{L}$ and $\big[\mathcal{L} \setminus \Delta_2 \; ; \; \mathcal{L} \setminus \Delta_1 \big]$ fails to hold (according to C), associate to such b-closed pair a B-matrix $\mathfrak{M}^{\mathsf{b}} = \langle \mathcal{L}, \Delta_1, \Delta_2, \{\mathsf{id}\} \rangle$, and let the corresponding *Lindenbaum* b-*bundle* be the family of B-matrices associated to all the b-closed subsets of $\mathcal{L}$. The necessary adjustments in the proof of Theorem 7 are then immediate.

Finally, we shall prove an analogous result that applies to gq-, gp-, gd- and gb-entailment. All these forms of entailment will be seen to be characterized by the notion of gq-*consequence* axiomatized simply by (1Ext). To start with, it is easy to check that these forms of entailment do indeed respect (1Ext). Now, given a gq-consequence relation C, and two arbitrary subsets Δ_1 and Δ_2 of $\mathcal{L}$, we will say that $\langle \Delta_1, \Delta_2 \rangle$ is a gq-*closed pair* if $\big[(\mathcal{L} \setminus \Delta_2) \; ; \; (\mathcal{L} \setminus \Delta_1) \big]$ fails to hold (according to C), and we will associate to such gq-closed pair a B-matrix $\mathfrak{M}^{\mathsf{gq}} = \langle \mathcal{L}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$ where $\mathsf{Y} = \Delta_1$, $\mathsf{N} = \Delta_2$, and S contains just the identity mapping id on $\mathcal{L}$. We call *Lindenbaum* gq-*bundle* $\mathcal{B}_C^{\mathsf{gq}}$ the family of B-matrices associated to all the gq-closed pairs of $\mathcal{L}$. We can then prove that:

THEOREM 8. *Any* gq-*consequence relation* C *is sound and complete with respect to* $\mathcal{B}_C^{\mathsf{gq}}$, *that is,* $\big[\Phi \; ; \; \Psi \big]$ *holds according to* C *iff* $\Phi \models^{\mathsf{gq}} \Psi$ *is* $\mathsf{S}_{\mathcal{B}_C^{\mathsf{gq}}}$-*valid.*

PROOF: ($\Rightarrow$) Suppose $\Phi \models^{\mathsf{gq}} \Psi$ is not $\mathsf{S}_{\mathcal{B}_C^{\mathsf{gq}}}$-valid. By the definition of gq-entailment, this is to say that $\frac{\Phi}{\because} \big| \frac{\Psi}{\because}$ is not $\mathsf{S}_{\mathcal{B}_C^{\mathsf{gq}}}$-valid. So, in the Lindenbaum gq-bundle $\mathcal{B}_C^{\mathsf{gq}}$ there must be some B-matrix $\langle \mathcal{L}, \mathsf{Y}, \mathsf{N}, \{\mathsf{id}\} \rangle$ for which $\frac{\Phi}{\because} \big| \frac{\Psi}{\because}$ fails to be valid. By the definition of B-entailment, it follows that $\Phi \subseteq \mathsf{N}$ and $\Psi \subseteq \mathsf{Y}$. By the very definition of $\mathcal{B}_C^{\mathsf{gq}}$, we know that $\langle \mathsf{Y}, \mathsf{N} \rangle$ is gq-closed, thus $\big[\mathsf{N} \; ; \; \mathsf{Y} \big]$ fails to hold according to C. Using (1Ext), we conclude that $\big[\Phi \; ; \; \Psi \big]$ fails to hold according to C.

($\Leftarrow$) Suppose now that $\big[\Phi \; ; \; \Psi \big]$ fails to hold according to C. So, $\langle \mathcal{L} \setminus \Psi, \mathcal{L} \setminus \Phi \rangle$ is a gq-closed pair. By the definition of gq-entailment, this implies that $\Phi \models^{\mathsf{gq}} \Psi$ fails to be valid, given that the Q-statement $\frac{\Phi}{\because} \big| \frac{\Psi}{\because}$ fails to be valid according to the B-matrix $\mathfrak{M}^{\mathsf{gq}} = \langle \mathcal{L}, \mathcal{L} \setminus \Psi, \mathcal{L} \setminus \Phi, \{\mathsf{id}\} \rangle$. So, *a fortiori*, this same Q-statement also fails to be valid according to $\mathcal{B}_C^{\mathsf{gq}}$. $\square$

The latter applies to gb-entailment without modifications, given that in such case we are dealing again with an arbitrary collection of Q-statements.

To adapt the latter result for arbitrary collections of P-statements that are characteristic both of gp-entailment and of gd-entailment, we proceed as follows. Given a gq-consequence relation C, we might call gp-*closed* any pair $\langle \Delta_1, \Delta_2 \rangle$ of $\mathcal{L}$ such that $\begin{bmatrix} \Delta_1 & ; & \Delta_2 \end{bmatrix}$ fails to hold (according to C), associate to such gp-closed pair a B-matrix $\mathfrak{M}^b = \langle \mathcal{L}, \Delta_1, \Delta_2, \{\mathsf{id}\} \rangle$, and let the corresponding *Lindenbaum* gp-*bundle* be the family of B-matrices associated to the gp-closed subsets of $\mathcal{L}$. The necessary adjustments in the proof of Theorem 8 are then immediate. The result applies to gd-entailment without modifications. Note also that such results generalize Theorems 1 and 2 from Section 2.

Following [2], we now introduce a two-dimensional generalization of the standard multiple-conclusion notion of consequence given by (Over), (1Ext) and (CTrn). The canonical notion of B-*consequence* is a 2×2-place relation $\stackrel{.}{\cdot}\|\stackrel{.}{\cdot}$ on subsets of $\mathcal{L}$ subject to the following axioms:

(Over^y) $\quad \frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$, whenever $\Phi_{21} \cap \Phi_{12} \neq \varnothing$

(Over^n) $\quad \frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$, whenever $\Phi_{11} \cap \Phi_{22} \neq \varnothing$

(2Ext) $\quad$ if $\frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$, then $\frac{\Phi_{11} \cup \Psi_{11}}{\Phi_{21} \cup \Psi_{21}} \big\| \frac{\Phi_{12} \cup \Psi_{12}}{\Phi_{22} \cup \Psi_{22}}$

(CTrn^y) $\quad$ given $\Pi \subseteq \mathcal{L}$,
$\qquad$ if $\frac{\Phi_{11}}{\Phi_{21} \cup \Sigma} \big\| \frac{\Phi_{12} \cup (\Pi \backslash \Sigma)}{\Phi_{22}}$ for every $\Sigma \subseteq \Pi$, then $\frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$

(CTrn^n) $\quad$ given $\Pi \subseteq \mathcal{L}$,
$\qquad$ if $\frac{\Phi_{11} \cup (\Pi \backslash \Sigma)}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22} \cup \Sigma}$ for every $\Sigma \subseteq \Pi$, then $\frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$

Whenever there is need to avoid ambiguity, instead of simply writing $\frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$ we shall say that $\begin{bmatrix} \Phi_{11} & ; & \Phi_{21} \\ \Phi_{12} & ; & \Phi_{22} \end{bmatrix}$ *holds according to* $\stackrel{.}{\cdot}\|\stackrel{.}{\cdot}$. The verification that B-entailment is a form of B-consequence, i.e., that it respects all the above axioms, is an easy exercise.

An important property of consequence relations defined over languages with algebraic character is the so-called *substitution-invariance* (a.k.a. 'structurality'), that can here be represented by the following axiom:

(SI) $\quad$ for every endomorphism σ of $\mathcal{L}$, if $\frac{\Phi_{11}}{\Phi_{21}} \big\| \frac{\Phi_{12}}{\Phi_{22}}$, then $\frac{\sigma(\Phi_{11})}{\sigma(\Phi_{21})} \big\| \frac{\sigma(\Phi_{12})}{\sigma(\Phi_{22})}$

It is easy to see that any B-matrix based on a truth-functional semantics respects the axiom (SI).

We proceed to show that any particular B-consequence relation C may be given an adequate semantics in terms of B-entailment. For that matter we will say that the pair $\langle \Psi_y, \Psi_n \rangle$ of subsets of $\mathcal{L}$ is B-*closed* if $\left[\begin{smallmatrix} \mathcal{L} \setminus \Psi_n \ ; \ \Psi_y \\ \mathcal{L} \setminus \Psi_y \ ; \ \Psi_n \end{smallmatrix} \right]$ fails to hold (according to C), and we will associate to such a B-closed pair a B-matrix $\mathfrak{M}^B = \langle \mathcal{L}, Y, N, S \rangle$ where $Y = \Psi_y$, $N = \Psi_n$, and S contains just the identity mapping id on $\mathcal{L}$. We call *Lindenbaum* B-*bundle* $\mathcal{B}_C^B$ the family of B-matrices associated to all the B-closed pairs of subsets of $\mathcal{L}$. The following result shows that any B-consequence relation may be fully characterized by its associated Lindenbaum B-bundle.

THEOREM 9. *Any* B-*consequence relation* C *is sound and complete with respect to* $\mathcal{B}_C^B$, *that is,* $\left[\begin{smallmatrix} \Phi_{11} \ ; \ \Phi_{21} \\ \Phi_{12} \ ; \ \Phi_{22} \end{smallmatrix} \right]$ *holds according to* C *iff* $\frac{\Phi_{11}}{\Phi_{21}} | \frac{\Phi_{12}}{\Phi_{22}}$ *is* $S_{\mathcal{B}_C^B}$-*valid.*

PROOF: ($\Rightarrow$) Suppose $\frac{\Phi_{11}}{\Phi_{21}} | \frac{\Phi_{12}}{\Phi_{22}}$ is not $S_{\mathcal{B}_C^B}$-valid. This means that there is some B-matrix $\langle \mathcal{L}, Y, N, \{id\} \rangle$ in the Lindenbaum B-bundle $\mathcal{B}_C^B$ for which $\frac{\Phi_{11}}{\Phi_{21}} | \frac{\Phi_{12}}{\Phi_{22}}$ fails to be valid. By the definition of S-validity, and taking into account that id is the identity mapping on $\mathcal{L}$, it follows that $\Phi_{21} \subseteq Y$, $\Phi_{11} \subseteq \mathcal{L} \setminus N$, $\Phi_{22} \subseteq N$ and $\Phi_{12} \subseteq \mathcal{L} \setminus Y$. Given that, by the very definition of $\mathcal{B}_C^B$, the pair $\langle Y, N \rangle$ is B-closed, we know that $\left[\begin{smallmatrix} \mathcal{L} \setminus N \ ; \ Y \\ \mathcal{L} \setminus Y \ ; \ N \end{smallmatrix} \right]$ fails to hold. Using (2Ext), we conclude then that $\left[\begin{smallmatrix} \Phi_{11} \ ; \ \Phi_{21} \\ \Phi_{12} \ ; \ \Phi_{22} \end{smallmatrix} \right]$ fails to hold.

($\Leftarrow$) Suppose now that $\left[\begin{smallmatrix} \Phi_{11} \ ; \ \Phi_{21} \\ \Phi_{12} \ ; \ \Phi_{22} \end{smallmatrix} \right]$ fails to hold. By (CTrn^y), we know that there must be some $Y \subseteq \mathcal{L}$ such that $\left[\begin{smallmatrix} \Phi_{11} & ; \ \Phi_{21} \cup Y \\ \Phi_{12} \cup (\mathcal{L} \setminus Y) & ; \ \Phi_{22} \end{smallmatrix} \right]$ fails to hold. By (Over^y), it follows that $\Phi_{21} \subseteq Y$ and $\Phi_{12} \subseteq \mathcal{L} \setminus Y$. Analogously, using (CTrn^n) and (Over^n) we conclude that $\Phi_{11} \subseteq \mathcal{L} \setminus N$ and $\Phi_{22} \subseteq N$. Thus, it follows by the definition of B-entailment that the B-statement $\frac{\Phi_{11}}{\Phi_{21}} | \frac{\Phi_{12}}{\Phi_{22}}$ is not valid according to the B-matrix $\langle \mathcal{L}, Y, N, \{id\} \rangle$, so *a fortiori* it also fails to be valid according to $\mathcal{B}_C^B$. □

An important specialization of the above result may be proved in case we associate B-consequence to a language and a semantics structured in the appropriate ways:

THEOREM 10. *Any substitution-invariant* B-*consequence relation* C *is characterizable by the Lindenbaum* B-*bundle of truth-functional* B-*matrices* $\mathcal{B}_C^B$.

PROOF: ($\Rightarrow$) Suppose $\frac{\Phi_{11}}{\Phi_{21}} | \frac{\Phi_{12}}{\Phi_{22}}$ is not $S_{\mathcal{B}_C^B}$-valid. This means that there is some truth-functional B-matrix $\langle \mathcal{L}, Y, N, Hom(\mathcal{L}, \mathcal{L}) \rangle$ in the Lindenbaum B-

bundle $\mathcal{B}_\mathsf{C}^\mathsf{B}$ for which $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ fails to be valid. Note that in this B-matrix valuations are simply identified with substitutions. By definition of S-validity, it follows that there is some substitution $\sigma \in \mathsf{Hom}(\mathcal{L},\mathcal{L})$ such that $\sigma(\Phi_{21}) \subseteq \mathsf{Y}$, $\sigma(\Phi_{11}) \subseteq \mathcal{L} \setminus \mathsf{N}$, $\sigma(\Phi_{22}) \subseteq \mathsf{N}$ and $\sigma(\Phi_{12}) \subseteq \mathcal{L} \setminus \mathsf{Y}$. Given that the pair $\langle \mathsf{Y}, \mathsf{N} \rangle$ is B-closed, by the definition of $\mathcal{B}_\mathsf{C}^\mathsf{B}$, we know that $\left[\begin{smallmatrix} \mathcal{L}\setminus\mathsf{N} \ ; \ \mathsf{Y} \\ \mathcal{L}\setminus\mathsf{Y} \ ; \ \mathsf{N} \end{smallmatrix}\right]$ fails to hold. Using (2Ext), we conclude that $\left[\begin{smallmatrix} \sigma(\Phi_{11}) \ ; \ \sigma(\Phi_{21}) \\ \sigma(\Phi_{12}) \ ; \ \sigma(\Phi_{22}) \end{smallmatrix}\right]$ fails to hold. Finally, from (SI) it follows that $\left[\begin{smallmatrix} \Phi_{11} \ ; \ \Phi_{21} \\ \Phi_{12} \ ; \ \Phi_{22} \end{smallmatrix}\right]$ fails to hold.

($\Leftarrow$) This direction follows closely the proof of Theorem 9($\Leftarrow$). Indeed, note that id is an endomorphism of $\mathcal{L}$, and invoke the definition of validity for B-entailment. $\square$

Substitution-invariant versions of Theorems 5, 6, 7 and 8 may be easily obtained by following a similar line of reasoning as in Theorem 10.

5.   Inferential many-valuedness

Generalizing Suszko's Thesis, one may now show that a B-consequence relation is, in general, inferentially four-valued. For that purpose, consider the following set $\mathcal{V}_4 = \{F, N, B, T\}$ of truth-values. Given a B-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$, let $\flat : \mathcal{V} \longrightarrow \mathcal{V}_4$ be defined by setting:

$$\flat(w) = \begin{cases} F & \text{if } w \in \mathsf{A} \cap \mathsf{N} \\ N & \text{if } w \in \mathsf{A} \cap \mathsf{M} \\ B & \text{if } w \in \mathsf{Y} \cap \mathsf{N} \\ T & \text{if } w \in \mathsf{Y} \cap \mathsf{M} \end{cases}$$

On top of this definition, consider the B-matrix $\mathfrak{M}_4 = \langle \mathcal{V}_4, \{B, T\}, \{F, B\}, \mathsf{S}_4 \rangle$, where $\mathsf{S}_4 = \{\flat \circ \nu \mid \nu \in \mathsf{S}\}$. Then it is not hard to check that:

THEOREM 11. *$\mathfrak{M}$ and $\mathfrak{M}_4$ characterize the same logic, that is,* $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ *is* S-*valid iff it is* S_4-*valid.*

PROOF: Obviously, any valuation ν in S that witnesses the invalidity of $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ according to $\mathfrak{M}$ translates into a valuation $\nu_4 = \flat \circ \nu$ that witnesses the invalidity of $\frac{\Phi_{11}}{\Phi_{21}}\big|\frac{\Phi_{12}}{\Phi_{22}}$ according to $\mathfrak{M}_4$. For the converse direction, let ν be a valuation in $\mathfrak{M}_4$ such that $\nu(\Phi_{21}) \subseteq \{B, T\}$, $\nu(\Phi_{11}) \subseteq \{N, T\}$, $\nu(\Phi_{22}) \subseteq \{F, B\}$ and $\nu(\Phi_{12}) \subseteq \{F, N\}$. By definition of $\mathfrak{M}_4$, we know that

$\nu = \flat \circ \nu_{\mathfrak{M}}$ for some $\nu_{\mathfrak{M}} \in \mathsf{S}$. Let $\mathcal{L}^T \subseteq \Phi_{21} \cup \Phi_{11}$ be defined as $\nu^{-1}(\{T\})$ (the inverse image of T under ν), let $\mathcal{L}^B \subseteq \Phi_{21} \cup \Phi_{22}$ be defined as $\nu^{-1}(\{B\})$, let $\mathcal{L}^N \subseteq \Phi_{11} \cup \Phi_{12}$ be defined as $\nu^{-1}(\{N\})$, and let $\mathcal{L}^F \subseteq \Phi_{22} \cup \Phi_{12}$ be defined as $\nu^{-1}(\{F\})$.

Now, given $\varphi \in \Phi_{21}$, we have to show that $\nu_{\mathfrak{M}}(\varphi) \in \mathsf{Y}$. Note that $\nu(\varphi)$ belongs to $\nu(\Phi_{21})$ (the direct image of Φ_{21} under ν), so by the assumption that $\nu(\Phi_{21}) \subseteq \{B, T\}$, it follows that $\nu(\varphi) \in \{B, T\}$, thus $\varphi \in \mathcal{L}^B \cup \mathcal{L}^T = \nu^{-1}(\{B, T\})$. But by the definition of $\flat$, we know that $\flat(\nu_{\mathfrak{M}}(\varphi)) = \nu(\varphi) \in \{B, T\}$ iff $\nu_{\mathfrak{M}}(\varphi) \in \mathsf{Y}$. We reason in an analogous way to check that $\nu_{\mathfrak{M}}(\varphi) \in \mathsf{И}$ for $\varphi \in \Phi_{11}$, that $\nu_{\mathfrak{M}}(\varphi) \in \mathsf{N}$ for $\varphi \in \Phi_{22}$, and that $\nu_{\mathfrak{M}}(\varphi) \in \mathsf{Λ}$ for $\varphi \in \Phi_{12}$. $\qquad\square$

Note that when gappy reasoning is not allowed for (i.e., in case $\mathsf{Λ} \cap \mathsf{И} = \varnothing$), then $\mathcal{V}_4$ reduces to $\mathcal{V}_3^\top = \{F, B, T\}$, and when glutty reasoning is not allowed for (i.e., in case $\mathsf{Y} \cap \mathsf{N} = \varnothing$), then $\mathcal{V}_4$ reduces to $\mathcal{V}_3^\bot = \{F, N, T\}$. Finally, in case neither gappy nor glutty reasoning are allowed for, then $\mathcal{V}_4$ reduces to $\mathcal{V}_2 = \{F, T\}$. Considering the definitions summarized in Table 1 (check in particular its last column), one may accordingly say that, in principle, from an inferential viewpoint:

(M1) t- and f-entailment are inferentially two-valued;
the same applies to wq- and wp-entailment
(M2) q-, d-, p-, b-entailment are all inferentially three-valued
the same applies to $\mathsf{gt\backslash u}$-, $\mathsf{gt\backslash a}$-, $\mathsf{gf\backslash u}$- and $\mathsf{gf\backslash a}$-entailment
(M3) q- and p-entailment may allow for gaps;
b- and d-entailment may allow for gluts
(M4) all generalized notions of entailment
(gx-entailment, for $\mathsf{x} \in \{\mathsf{t}, \mathsf{f}, \mathsf{q}, \mathsf{p}\}$),
are inferentially four-valued

As we have seen in the previous sections, several in principle distinct one-dimensional consequence relations may be defined from any given B-consequence relation. We have also just shown, above, that such consequence relations can be endowed with semantics based on at most four inferential values, and so we may hereupon use the latter to go about investigating the former. Given a specific B-consequence relation C, by the t-*aspect of* C we will refer to all the T-statements that hold in C together with the assumptions that neither gaps nor gluts are present (notation: $\models_\mathsf{C}^\mathsf{t}$). Analogously, the gt-*aspect of* C will refer to all the T-statements that hold in C, without the assumptions about gaps and gluts (notation:
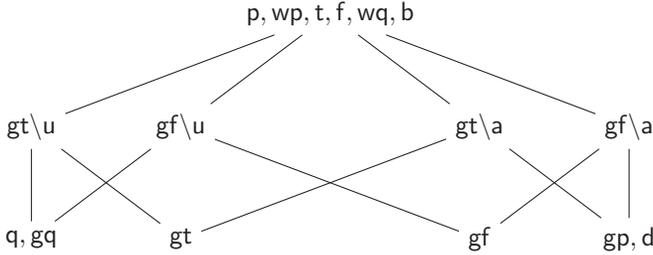
Fig. 1: Aspects of a given B-entailment relation

$\models_{\mathsf{C}}^{\mathsf{gt}}$). We may similarly define the x-*aspect of* C for each of the forms of x-entailment described in Table 1. From that perspective, we explore in what follows the set-theoretic inter-relations between the various notions of entailment introduced before.

THEOREM 12. *Let* C *be a* B-*consequence relation over a language* $\mathcal{L}$*. Then, the Hasse diagram in Figure* 1 *represents all connections in terms of strict set-theoretic inclusion between the various aspects of* C*. In other words:*

(1) *If* x_1 *is above* x_2*, then* $\models_{\mathsf{C}}^{\mathsf{x}_2} \subseteq \models_{\mathsf{C}}^{\mathsf{x}_1}$
(2) *If* x_1 *is above* x_2*, then* $\models_{\mathsf{C}}^{\mathsf{x}_2} \not\supseteq \models_{\mathsf{C}}^{\mathsf{x}_1}$
(3) *If* x_1 *and* x_2 *are not comparable, then* $\models_{\mathsf{C}}^{\mathsf{x}_1} \not\subseteq \models_{\mathsf{C}}^{\mathsf{x}_2}$ *and* $\models_{\mathsf{C}}^{\mathsf{x}_2} \not\subseteq \models_{\mathsf{C}}^{\mathsf{x}_1}$

PROOF: For the first part of the proof, let $\mathfrak{M}_{\mathbb{4}} = \langle \{F, N, B, T\}, \{B, T\},$ $\{F, B\}, \mathsf{S} \rangle$ be an inferentially four-valued characterization of C. Checking the equalities ($\models_{\mathsf{C}}^{\mathsf{q}} = \models_{\mathsf{C}}^{\mathsf{gq}}$, etc) is an easy exercise using the definitions summarized in Table 1, and is left to the reader. To check that $\models_{\mathsf{C}}^{\mathsf{q}} \subseteq$ $\models_{\mathsf{C}}^{\mathsf{gt}\backslash\mathsf{u}}$, let us suppose that $\Phi \models_{\mathsf{C}}^{\mathsf{gt}\backslash\mathsf{u}} \Psi$ is not S-valid. In that case, there must be some $\nu \in \mathsf{S}$ such that (a) $\nu(\Phi) \subseteq \{B, T\} \setminus \{B\} = \{T\}$ and (b) $\nu(\Psi) \subseteq \{F, N\} \setminus \{B\}$. But from (a) it follows that (c) $\nu(\Phi) \subseteq \{N, T\} = \{N, T\} \setminus \{B\}$. So, from (c) and (b) we conclude that $\Phi \models_{\mathsf{C}}^{\mathsf{q}} \Psi$ is not S-valid. Next, to check that $\models_{\mathsf{C}}^{\mathsf{gt}\backslash\mathsf{u}} \subseteq \models_{\mathsf{C}}^{\mathsf{t}}$, we suppose this time that $\Phi \models_{\mathsf{C}}^{\mathsf{t}} \Psi$ is not S-valid. This means that there must be some $\nu \in \mathsf{S}$ such that (a') $\nu(\Phi) \subseteq \{B, T\} \setminus \{N, B\} = \{T\}$ and (b') $\nu(\Psi) \subseteq \{F, N\} \setminus \{N, B\} = \{F\}$. From (b') we conclude that (c') $\nu(\Psi) \subseteq \{F, N\} = \{F, N\} \setminus \{B\}$. Then, from (a') and (c') it follows that $\Phi \models_{\mathsf{C}}^{\mathsf{gt}\backslash\mathsf{u}} \Psi$ is not S-valid. The remaining ten inclusions may be checked in a similar way, and the corresponding exercise is again left to the reader.

For the second part of the proof, let C be a B-consequence relation containing the six following unary 4-valued connectives, characterized by their respective truth-tables:

	$\circ_1$	$\circ_2$	$\circ_3$	$\circ_4$	$\circ_5$	$\circ_6$
F	F	F	F	F	F	F
N	N	F	B	B	N	F
B	F	N	B	F	N	B
T	N	N	N	B	B	B

Let's denote by $J(\mathsf{x}, n)$ the judgment of the form $\varphi \models^{\mathsf{x}}_{\mathsf{C}} \circ_n \varphi$. We will show that for each choice of $\mathsf{x}_1, \mathsf{x}_2 \in \{\mathsf{t}, \mathsf{q}, \mathsf{d}, \mathsf{gt}, \mathsf{gf}, \mathsf{gt}\backslash\mathsf{u}, \mathsf{gt}\backslash\mathsf{a}, \mathsf{gf}\backslash\mathsf{u}, \mathsf{gf}\backslash\mathsf{a}\}$, with $\mathsf{x}_1 \neq \mathsf{x}_2$, there is some $n \in \{1, 2, 3, 4, 5, 6\}$ such that $J(\mathsf{x}_1, n)$ is S-valid while $J(\mathsf{x}_2, n)$ is *not* S-valid (or the other way round). Note first that: (i) $J(\mathsf{x}, 1)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{q}, \mathsf{gf}, \mathsf{gf}\backslash\mathsf{u}, \mathsf{gf}\backslash\mathsf{a}\}$, (ii) $J(\mathsf{x}, 2)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{d}, \mathsf{gt}\backslash\mathsf{a}, \mathsf{gf}\backslash\mathsf{a}\}$, (iii) $J(\mathsf{x}, 3)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{gt}\backslash\mathsf{a}, \mathsf{gf}\backslash\mathsf{u}, \mathsf{gf}\backslash\mathsf{a}\}$, (iv) $J(\mathsf{x}, 4)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{q}, \mathsf{gt}\backslash\mathsf{u}, \mathsf{gf}\backslash\mathsf{u}\}$, (v) $J(\mathsf{x}, 5)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{gt}\backslash\mathsf{u}, \mathsf{gt}\backslash\mathsf{a}\}$, and (vi) $J(\mathsf{x}, 6)$ is S-valid iff $\mathsf{x} \in \{\mathsf{t}, \mathsf{gt}, \mathsf{gt}\backslash\mathsf{u}, \mathsf{gt}\backslash\mathsf{a}\}$. One may now evaluate the converses of the twelve inclusions from the first part of the proof. For instance, to conclude that $\models^{\mathsf{q}}_{\mathsf{C}} \not\supseteq \models^{\mathsf{gt}\backslash\mathsf{u}}_{\mathsf{C}}$ one may invoke either item (v) or item (vi), and to conclude that $\models^{\mathsf{gt}\backslash\mathsf{u}}_{\mathsf{C}} \not\supseteq \models^{\mathsf{t}}_{\mathsf{C}}$ items (i) or (ii) or (iii) will do the job. The other ten cases are left as exercise to the reader.

For the third part of the proof, containing the remaining fourty pairwise comparisons between one-dimensional entailment relations of various kinds induced by a given two-dimensional entailment relation, one may again use the connectives from the second part of the proof. Items (i)–(vi) will suffice for the reader to complete the argument. □

6. Summary and outlook

To sum up, let us recall that in the present paper we have first shown that every purely monotonic single-conclusion consequence relation is semantically characterized by a certain class of generalized q-matrices, which we call B-matrices. We next introduced a two-dimensional multiple-conclusion notion of entailment based on B-matrices, which provides a uniform framework for studying several different notions of entailment based on designation, antidesignation, and their complements. Let us underline that

we take the two-dimensional presentation to be quite useful in comparison with a linear presentation, because it nicely supports seeing affinities, for instance, between different forms of 'reflexivity' of entailment, such as (In^y), (In^n), (In^u) and (In^a), which impose distinct semantic constraints on the geometry of the underlying matrices. Moreover, the generalization to a multiple-conclusion framework emphasizes the symmetries between the four positions of a B-statement, and the multiplicity of inferential values in B-matrices allows one to accommodate *at the same time* not only an understanding of entailment as the preservation of some property from the premises to the conclusion of an inference, but also other, non-Tarskian conceptions of semantical consequence such as *p*- and *q*-entailment. Here, we defined the two-dimensional concept of a B-consequence relation, and presented an abstract characterization of B-consequence relations by classes of B-matrices, and eventually it was also shown that any B-consequence relation is, in general, inferentially four-valued. Our study is not alone in that quest: our result about inferential four-valuedness, applied to the multiple-conclusion one-dimensional framework, may be seen as a particular case of a result from Humberstone, in [11], where the author analyzes the situation in which the consequence relations are allowed to involve sets of formulas from two different languages, both associated to logics characterized in terms of t-entailment; Ripley and French, in [10], also investigate inferential many-valuedness, its connections with *q*- and *p*-entailment, and the abstract characterizations of the consequence relations thereby involved, including the purely monotonic case, with an approach based on the well-known Galois connection between semantics and the abstract notion of consequence. Such investigations make clear that logic should not be restricted to the study of Tarski-type, Scott-type, or Shoesmith-Smiley-type consequence relations. The present effort should be seen thus as a contribution to the discussion about the concept of entailment and, hence, also the understanding of logic as a discipline.

It is worth briefly highlighting here some of the principal novelties brought by the present study, as well as pointing out some possible directions for further investigation. The multiple-conclusion versions of *q*-entailment and *p*-entailment, here dubbed q-entailment and p-entailment, have first been introduced in this paper, together with their duals given by the notions of d-entailment and b-entailment. Figure 2 groups the one-dimensional notions of entailment hereby defined in terms of their abstract characterizations: (i) t- and gt-entailment, and their duals f- and

$$\text{gq}, \text{gp}$$

$$\text{b}, \text{p} \qquad\qquad\qquad\qquad \text{d}, \text{q}$$

$$\text{wq}, \text{gt}\backslash\text{u}, \text{gt}\backslash\text{a}, \text{gt}, \text{t}, \text{f}, \text{gf}, \text{gf}\backslash\text{a}, \text{gf}\backslash\text{u}, \text{wp}$$
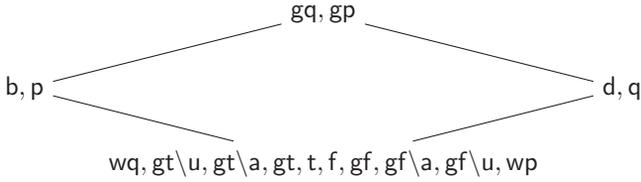
Fig. 2: Classes of consequence relations, from an abstract viewpoint

gf-entailment have been seen to be characterized by axioms (Over), (1Ext) and (CTrn), which are known to provide the most natural generalization of the Tarskian single-conclusion notion of consequence, and the same applies to other notions of entailment introduced in the present paper; (ii) q-entailment and its dual d-entailment have been seen to be characterized by (1Ext) and (CTrn); (iii) p-entailment and its dual b-entailment have been seen to be characterized by (Over) and (1Ext); in addition, (iv) gq- and its dual gp-entailment have been seen to be characterized by (1Ext), that is, by 'pure monotonicity'. The Hasse diagram in Figure 2 shows how the classes of entailment relations of each kind are organized according to set-theoretic inclusion: we have seen, for instance, that each consequence relation characterized in terms of t-entailment is also characterizable in terms of f-entailment, and vice versa, we have seen that each of the latter may be seen as particular cases of some consequence relation characterized in terms of q-entailment, and so forth.

From the viewpoint of the reduction results presented in Section 5, one should note in particular that: (i) the t-aspect of a given B-consequence relation does not in general coincide with its gt-aspect, for the latter allows for two extra inferential values and consequently more valuations based on them; (ii) the gt-aspect of a given B-consequence relation treats as a gap what its gf-aspect treats as a glut, and vice versa, and so they might not coincide; (iii) the q-aspect and the p-aspect of a given B-consequence relation do not in general coincide, even though both allow for gappy reasoning, and a similar thing might be said about the d-aspect and the b-aspect of a given B-consequence relation, with the difference that the two latter aspects allow for glutty instead of gappy reasoning; (iv) the gq-aspect and the gp-aspect of a given B-consequence do not in general coincide, in spite of both being in principle logically four-valued and of both respecting the same abstract axioms. In terms of the sixteen kinds of entailment relation studied in the

present paper, Figure 1 shows how some two-dimensional notions of consequence may indeed exhibit up to nine different aspects. Of course, one does not need to rest content with those specific aspects: other interesting notions of entailment are in principle definable by yet other combinations of distinguished sets of logical values. One might take it that the different one-dimensional aspects of a given two-dimensional notion of consequence play a role similar to the one played by the 'zero-dimensional' notion of tautology with respect to the usual one-dimensional Tarskian consequence relations. It is worth noticing, at any rate, that the distinct aspects of a specific B-consequence represent logics on their own right, and might be taken to vindicate a variety of *logical pluralism* in which logics of different kinds may be said to 'cohabitate' the same generalized logical structure.

We have presented B-matrices as a natural generalization of q-matrices, and by explicitly adding a semantics to the notion of logical matrix we have made it clear that the abstract characterization results apply very naturally even in the case of consequence relations in which substitution-invariance is not at issue. From the viewpoint of B-consequence we have also seen that q-entailment and p-entailment have much more in common than they might originally have appeared to have. For instance, both of these non-Tarskian notions of entailment respect the forms of overlap and cumulative transitivity represented at the diagonals of the two-dimensional syntactical representation of B-consequence. An explanation of why q-entailment appears to fail 'reflexivity', at the one-dimensional level, is to be found at the two-dimensional level, where q-consequence is seen to be defined by a collection of Q-statements but respects a particular form of overlap that is given only by a P-statement. Explanations of why q-entailment only respects a weakened form of 'transitivity', and also of why p-entailment appears to fail 'transitivity' may also be found at the two-dimensional level — again p-consequence is defined by a collection of P-statements, but the appropriate additional notion of transitivity respected by it is only expressible as a Q-statement.

The two-dimensional presentation was also adopted in [12] to define predicate sequent systems for partial logics whose semantics is inferentially three-valued, and in [1] to define a propositional sequent system for an inferentially four-valued version of the logic of First Degree Entailment. In the present paper we have only dealt with *specific* logics in proving the second half of Theorem 12. As a matter of fact, it may be shown that any (non-deterministic) connective of any arity may be characterized in terms of

appropriate collections of B-statements. We shall however leave the details of that result, as well as a general proof-theoretic study of B-entailment, as matter for future work.

It has generally been held in the literature that at least one of the 'logical values' obtained by Suszko Reduction represents a distinguished set of values in the corresponding notion of matrix, by way of an appropriate (generalized) characteristic function that applies to the algebraic values. Malinowski [21, p. 1], for example explains that

> two facets of many-valuedness — referential and inferential — are unravelled. The first, fits the standard approach and it results in multiplication of semantic correlates of sentences, and not logical values in a proper sense. The second many-valuedness is a metalogical property of inference and refers to partition of the matrix universe into more than two disjoint subsets, used in the definition of inference.

For the generalized version of q-entailment introduced here, we have upgraded the inferential reduction by exploring a very different strategy in order to obtain the logical values out of the distinguished subsets used in the corresponding definition of B-entailment. This way one sees that, in general, the distinguished sets of the logical matrices need not be mapped onto logical values; rather, the logical values play a direct role in defining the carrier of the matrix obtained through the reduction, but only play an indirect role in defining the notion of entailment. Starting from B-consequence one obtains inferentially three-valued and inferentially two-valued notions of entailment by excluding some logical values through the addition of appropriate forms of the axiom (Over). A similar strategy might be used to go beyond four logical values: for instance, to obtain five logical values one could naturally add an additional independent distinguished set to the definition of B-matrix and appropriately add a further dimension to the corresponding notion of entailment; then, to exclude three out of the eight logical values thereby induced, one would again add appropriate variations of the axiom (Over).

The traditional proof of the logical two-valuedness of Tarskian consequence relations relies on the division of the set of truth-values into a set of designated values and its complement. Given that these sets uniquely determine one another, the Suszko Reduction may actually be claimed to demonstrate the 'logical mono-valuedness' of Tarskian consequence rela-

tions, as has been remarked in [27, 24]. Since the four subsets $\mathsf{Y} \cap \mathsf{N}$, $\mathsf{Y} \cap \mathsf{И}$, $\mathsf{У} \cap \mathsf{N}$, and $\mathsf{У} \cap \mathsf{И}$ that are singled out from a given B-matrix $\mathfrak{M} = \langle \mathcal{V}, \mathsf{Y}, \mathsf{N}, \mathsf{S} \rangle$ to obtain the B-matrix $\mathfrak{M}_4 = \langle \mathcal{V}_4, \{B, T\}, \{F, B\}, \mathsf{S}_4 \rangle$ are uniquely determined by Y, N and their complements, through set-theoretic intersection, it might now be held that B-consequence relations are, in general, not only inferentially four-valued, but actually 'logically bi-valued'.

According to G. Malinowski [20], "[g]etting logical n-valuedness for $n > 3$ is tempting" and Malinowski identifies as a first step in that direction a division of the matrix universe into more than three mutually disjoint subsets. This might suggest identifying the logical values with mutually disjoint subsets of $\mathcal{V}$. The disjointness requirement, however, creates a problem for B-matrices because the set $\mathsf{Y} \cap \mathsf{N}$ may in general be nonempty. Nonetheless, the idea of identifying the logical values with subsets of $\mathcal{V}$ points at an alternative direction into which the notion of B-entailment can be generalized. Along the lines of R. Wojcicki's [28, 29] notion of a ramified (or general) matrix, a generalized B-matrix for a language $\mathcal{L}$ could be defined as a tuple $\langle \mathcal{V}, D_1, \ldots, D_n, \mathsf{S} \rangle$, where $\mathcal{V}$ is a set, $D_i \subseteq \mathcal{V}$ for every $i \in \{1, \ldots, n\}$, and S is a collection of mappings $\nu : \mathcal{L} \longrightarrow \mathcal{V}$. Then again, set-theoretic combinations of the distinguished subsets D_i and their complements might be used to define notions of entailment that reach far beyond the ones considered in the present paper.

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