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Chapter 4
Continuity connected with ψ-density

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4.1 Introduction

The notion of approximately continuous functions was introduced at the beginning of the XXth century. In 1915 in [5], A. Denjoy stated that a real valued function f is approximately continuous at a point x_0 if and only if there exists a measurable set $A \subset \mathbb{R}$ such that

$$\lim_{h\to 0^+} \frac{m(A\cap[x_0-h,x_0+h])}{2h} = 1 \text{ and } f(x_0) = \lim_{\substack{x\to x_0\\ x\in A}} f(x), \tag{4.1}$$

where m denotes the Lebesgue measure. The point x_0 which fulfills the first from the above equalities is called a density point of a set A. Denjoy discovered that approximately continuous functions are of Baire class 1 and have Darboux property. He also proved that a real function defined on $\mathbb{R}$ is measurable if and only if it is approximately continuous almost everywhere on $\mathbb{R}$. The first definition did not involve the concept of density topology, which appeared later in the paper [14]. In 1961 ([13]) C. Goffman and D. Waterman examined approximately continuous transformations from euclidean space into an arbitrary metric space. They defined the density topology (denoted by $\mathscr{T}_d$) finer then the natural topology $\mathscr{T}_o$ and proved that the approximately continuous transformations are Darboux Baire 1 and are continuous as $f\colon (\mathbb{R},\mathscr{T}_d) \to (\mathbb{R},\mathscr{T}_o)$. In the same year it was shown ([12]) that $\mathscr{T}_d$ is the coarsest topology relative to which approximately continuous functions are continuous, because the density topology on $\mathbb{R}$ is completely regular (but not normal).

By putting density topology on the domain and on the range of a function we can obtain another class of continuous functions: density continuous functions, which were deeply examined by K. Ciesielski, L. Larson and K. Ostaszewski (for instance [3], [4], [20]), J. Niewiarowski ([19]) and A. Bruckner ([1]).

In 1910 ([16]) H. Lebesgue proved that for any Lebesgue measurable set $A \subset \mathbb{R}$ the equality

$$\lim_{h\to 0^+} \frac{m\left(A\cap[x-h,x+h]\right)}{2h} = 1 \tag{4.2}$$

holds for all points $x \in A$ except for the set of Lebesgue measure zero. We denote by $\mathscr{L}$ the family of all Lebesgue measurable sets on $\mathbb{R}$. Equivalently we can say that $m\left(A\Delta\Phi_d\left(A\right)\right) = 0$ for any $A \in \mathscr{L}$ (Δ stands for symmetric difference), where

$$\Phi_d\left(A\right) = \left\{x\in\mathbb{R}\colon \lim_{h\to 0^+} \frac{m\left(A\cap[x-h,x+h]\right)}{2h} = 1\right\}.$$

The family

$$\mathscr{T}_d = \{A \in \mathscr{L}\colon A \subset \Phi_d\left(A\right)\}$$

forms a topology called the density topology. $(\mathbb{R},\mathscr{T}_d)$ is a Baire space and $\mathscr{T}_d$ is invariant under translations and multiplications by nonzero numbers. The families of meager sets and nowhere dense sets in $(\mathbb{R},\mathscr{T}_d)$ coincide and both are equal to the family of Lebesgue null sets. Any set of positive inner measure has nonempty interior in $\mathscr{T}_d$.

In 1959 ([21]) S. J. Taylor solved the problem presented by S. Ulam in The Scottish Book. Taylor's results were the contribution to the development of ψ-density topology and another classes of continuous functions, with density and ψ-density on the domain and the range. Let us present his two main theorems.

Theorem 4.1 ([21, Theorem 3]). *For any Lebesgue measurable set $A \subset \mathbb{R}$ there exists a function $\psi\colon (0,\infty) \to (0,\infty)$ which is continuous, nondecreasing and $\lim\limits_{x\to 0^+} \psi(x) = 0$ such that*

$$\lim_{m(I)\to 0} \frac{m(A' \cap I)}{m(I)\psi(m(I))} = 0$$

for almost all $x \in A$, where I is any interval containing x (A' stands for the complement of A).

Theorem 4.2 ([21, Theorem 4], compare with [22, Theorem 0.2]). *For any function $\psi\colon (0,\infty) \to (0,\infty)$ which is continuous, nondecreasing, $\lim_{x\to 0^+} \psi(x) = 0$, and for any real number $\alpha \in (0,1)$, there exists a perfect set $E \subset [0,1]$ such that $m(E) = \alpha$ and*

$$\limsup_{m(I)\to 0} \frac{m(E' \cap I)}{m(I)\psi(m(I))} = \infty$$

for all $x \in E$.

Following Taylor we introduce a notion of ψ-density (compare with [22]). Let $\mathscr{C}$ be the family of all nondecreasing continuous functions $\psi\colon (0,\infty) \to (0,\infty)$ such that $\lim_{t\to 0^+} \psi(t) = 0$.

We say that $x \in \mathbb{R}$ is a ψ-density point of $A \in \mathscr{L}$ (we will write $x \in \Phi_\psi(A)$) if and only if

$$\lim_{h\to 0^+} \frac{m(A' \cap [x-h, x+h])}{2h\psi(2h)} = 0.$$

From Theorem 4.2 we obtain that the operator Φ_ψ is not a lower density operator. However, this operator is an almost lower density operator (see [23] and [15]) and the family

$$\mathscr{T}_\psi = \{A \in \mathscr{L} : A \subset \Phi_\psi(A)\}$$

forms a topology called ψ-density topology. Clearly, $\mathscr{T}_o \subsetneq \mathscr{T}_\psi \subsetneq \mathscr{T}_d$. Let us notice that, despite of topologies generated by lower density operators, for any ψ-density topology there is a set of positive measure and empty $\mathscr{T}_\psi$-interior. On the other hand, $\mathscr{T}_\psi$ has a lot of properties similar to properties generated by lower density operators. For any $\psi \in \mathscr{C}$ null sets are $\mathscr{T}_\psi$-closed and consequently the space $(\mathbb{R}, \mathscr{T}_\psi)$ is neither first countable, nor second countable, nor Lindelöf, nor separable. A set is compact in $\mathscr{T}_\psi$ if and only if it is finite; a set is connected if and only if it is connected in $\mathscr{T}_o$; a set is $\mathscr{T}_\psi$-Borel if and only if it is measurable (for details see [11]).

For any topologies $\mathscr{T}_a$, $\mathscr{T}_b \subset 2^{\mathbb{R}}$ we will denote by $\mathscr{C}_{ab}$ be the family of all continuous functions $f\colon (\mathbb{R}, \mathscr{T}_a) \to (\mathbb{R}, \mathscr{T}_b)$. It is easy to observe that

(P1) for any topologies $\mathscr{T}_a$, $\mathscr{T}_b$ and $\mathscr{T}_c$, if $\mathscr{T}_b \subset \mathscr{T}_c$ then $\mathscr{C}_{ab} \supset \mathscr{C}_{ac}$ and $\mathscr{C}_{ba} \subset \mathscr{C}_{ca}$;

(P2) for any pair of topologies $\mathscr{T}_a$, $\mathscr{T}_b$, if $\mathscr{T}_a \subset \mathscr{T}_b$ then $\mathscr{C}_{ab} \subset \mathscr{C}_{aa} \cap \mathscr{C}_{bb}$ and $\mathscr{C}_{ba} \supset \mathscr{C}_{aa} \cup \mathscr{C}_{bb}$.

If we start with topologies considered above: natural topology $\mathscr{T}_o$, density topology $\mathscr{T}_d$ and ψ-density topology $\mathscr{T}_\psi$, we may obtain nine classes of continuous functions: $\mathscr{C}_{oo}$, $\mathscr{C}_{od}$, $\mathscr{C}_{o\psi}$, $\mathscr{C}_{do}$, $\mathscr{C}_{dd}$, $\mathscr{C}_{d\psi}$, $\mathscr{C}_{\psi o}$, $\mathscr{C}_{\psi d}$ and $\mathscr{C}_{\psi\psi}$.

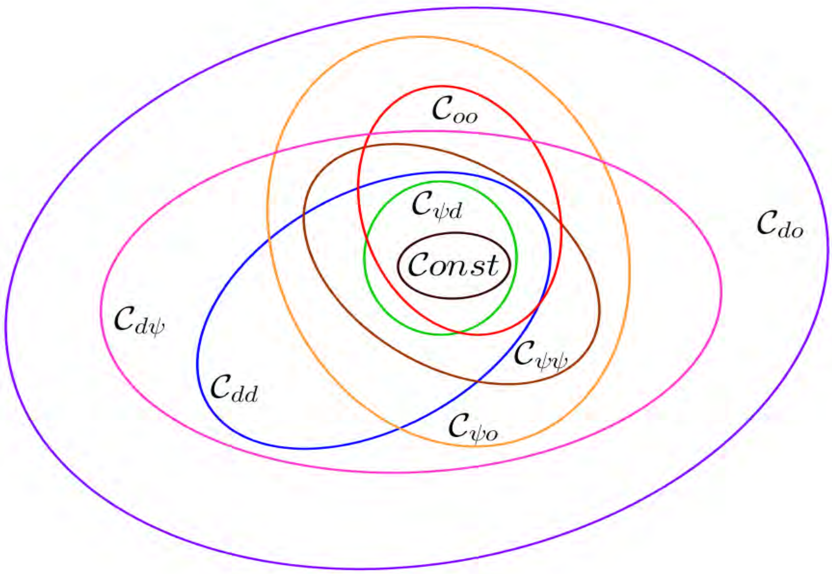

Fig. 1. Relations between the classes of continuous functions connected with $\mathscr{T}_o$, $\mathscr{T}_d$, $\mathscr{T}_\psi$

Denote by $\mathscr{F}$ the set of all these nine families. Some of them have been examined:

1. $\mathscr{C}_{oo}$ is the class of all ordinary continuous functions.
2. $\mathscr{C}_{do}$ is the class of approximately continuous functions (see [4], [12] and [23]). Let us remind that such functions are Darboux Baire 1 (a function f is Baire 1 if for any perfect set P the restriction $f|_P$ has a point of continuity). Moreover, each approximately continuous and bounded function is a derivative.
3. $\mathscr{C}_{dd}$ is the class of density continuous functions (for the results see [2], [4] and [20]). This class is not additive. All density continuous functions are approximately continuous and they belong to Darboux Baire*1 class. Remind that f is Baire*1 if for each perfect set P there is a portion $Q \subset P$ such that $f|_Q$ is continuous.

We will consider properties of functions from classes connected with ψ-density topology. We present selected results of published articles and some open problems. For the convenience of the reader we quote examples and sketches of the proofs.

Observe that all the families from $\mathscr{F}$ have some common properties. Assume that $\mathscr{C}_{ab} \in \mathscr{F}$. It is evident that

1. $\mathscr{C}onst \subset \mathscr{C}_{ab}$,
2. if $f \in \mathscr{C}_{ab}$, then $-f \in \mathscr{C}_{ab}$,
3. if $f \in \mathscr{C}_{ab}$ and $k \in \mathbb{R}$, then $f+k \in \mathscr{C}_{ab}$.

Moreover, $\mathscr{C}_{ab}$ forms a lattice.

Remark 4.1. If $f,g \in \mathscr{C}_{ab}$ then $\max\{f,g\} \in \mathscr{C}_{ab}$ and $\min\{f,g\} \in \mathscr{C}_{ab}$.

Proof. Let $h = \max\{f,g\}$ and $x_0 \in \mathbb{R}$. We will show that the function $h\colon (\mathbb{R},\mathscr{T}_a) \to (\mathbb{R},\mathscr{T}_b)$ is continuous at x_0. If $f(x_0) > g(x_0)$ (or $g(x_0) > f(x_0)$) then h is equal to f (g) on some neighbourhood of x_0 and the thesis is obvious. Assume then that $h(x_0) = f(x_0) = g(x_0)$ and $G \in \mathscr{T}_b$ is an open neighbourhood of the point $h(x_0)$. The sets $f^{-1}(G)$ and $g^{-1}(G)$ are open and

$$x_0 \in H = f^{-1}(G) \cap g^{-1}(G) \in \mathscr{T}_a.$$

As $\min\{f,g\} = \max\{-f,-g\}$ we obtain that $\min\{f,g\} \in \mathscr{C}_{ab}$. □

If $f,g \in \mathscr{C}_{ab}$ then $f+g$ and kf for a real k may not be in $\mathscr{C}_{ab}$. Also the limit of uniformly convergent sequence of functions from $\mathscr{C}_{ab}$ may not belong to $\mathscr{C}_{ab}$.

4.2 Continuity related to natural topology

It appears that classes $\mathscr{C}_{o\psi}$ and $\mathscr{C}_{od}$ are surprisingly small. Following [4, Theorem 3.1] and [8, Theorem 3] we can check that:

Theorem 4.3. *$\mathscr{C}_{o\psi} = \mathscr{C}onst$ for any function $\psi \in \mathscr{C}$.*

Indeed, suppose that $f \in \mathscr{C}_{o\psi}$ and $a < b$. Then $f([a,b])$ is compact and connected in $\mathscr{T}_\psi$. Therefore, $f([a,b])$ is a singleton and f has to be constant. Another proof can be found in [8].
By (P1) we obtain that $\mathscr{C}_{od} \subset \mathscr{C}_{o\psi}$, hence $\mathscr{C}_{od} = \mathscr{C}onst$.

Let us examine classes $\mathscr{C}_{\psi o}$. First we introduce the notion of the inner ψ-density point of a set $A \subset \mathbb{R}$: x is said the inner ψ-density point of $A \subset \mathbb{R}$ if and only if there exists a set $B \in \mathscr{L}$ such that $B \subset A$ and x is a ψ-density point of B. Clearly, for a measurable set the notions of ψ-density point and inner ψ-density point are equivalent. Now we can define ψ-approximately continuous function.

Definition 4.1 ([22]). We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is ψ-approximately continuous at x_0 if and only if x_0 is the inner ψ-density point of $f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ for each $\varepsilon > 0$.

It is evident, that f is ψ-approximately continuous at x_0 if and only if x_0 is the inner ψ-density point of $f^{-1}((a,b))$ for each interval (a,b) such that $f(x_0) \in (a,b)$. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *ψ-approximately continuous* if and only if f is ψ-approximately continuous at each point. Note that a set A is open in the topology $\mathscr{T}_\psi$ if and only if each point of A is the inner ψ-density point of A ([22, Theorem 3.3]). Therefore, a function $f\colon \mathbb{R} \to \mathbb{R}$ is ψ-approximately continuous if and only if for each interval (a,b) the set $f^{-1}((a,b))$ belongs to $\mathscr{T}_\psi$ ([22, Theorem 3.6]), so $f \in \mathscr{C}_{\psi o}$.

As $\mathscr{T}_o \subset \mathscr{T}_\psi \subset \mathscr{T}_d$ we obtain

$$\mathscr{C}_{oo} \subset \mathscr{C}_{\psi o} \subset \mathscr{C}_{do}.$$

From [7, Property 5] it is known that for any function $\psi \in \mathscr{C}$ there are functions $\psi_1,\ \psi_2 \in \mathscr{C}$ such that

$$\mathscr{C}_{\psi_1 o} \subsetneq \mathscr{C}_{\psi o} \subsetneq \mathscr{C}_{\psi_2 o}.$$

Hence for any function $\psi \in \mathscr{C}$

$$\mathscr{C}_{oo} \subsetneq \mathscr{C}_{\psi o} \subsetneq \mathscr{C}_{do} \subsetneq \mathscr{D}\mathscr{B}_1.$$

Moreover, in [24, Theorem 9] it is proved that the family $\mathscr{C}_{\psi o}$ is not contained in the class of Baire*1 functions.

It is easy to check that

Theorem 4.4. *Let $\psi \in \mathscr{C}$.*

1. *If $f,\ g$ are ψ-approximately continuous functions, then $f+g,\ f-g$, $f\cdot g$, $\max\{f,g\}$ and $\min\{f,g\}$ are ψ-approximately continuous. If f is a ψ-approximately continuous function and $f(x) \neq 0$ for $x \in X$, then $\frac{1}{f}$ is ψ-approximately continuous.*
2. *If $f_n \in \mathscr{C}_{\psi o}$ for any $n \in \mathbb{N}$ and a sequence $\{f_n\}_{n\in\mathbb{N}}$ uniformly converges to f, then $f \in \mathscr{C}_{\psi o}$.*
3. *If $f \in \mathscr{C}_{\psi o}$ and $g \in \mathscr{C}_{oo}$, then $g \circ f \in \mathscr{C}_{\psi o}$.*

Remind that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called measurable if $f^{-1}(U) \in \mathscr{L}$ for any $U \in \mathscr{T}_o$. Obviously, any approximately continuous and any ψ-approximately continuous function are measurable. The famous theorem of Denjoy ([5]) states that f is measurable if and only if it is approximately continuous almost everywhere. The similar theorem holds for ψ-approximate continuity.

Theorem 4.5 ([22, Theorem 3.7]). *A function $f\colon \mathbb{R} \to \mathbb{R}$ is measurable if and only if there exists a function $\psi \in \mathscr{C}$ such that f is ψ-approximately continuous almost everywhere.*

Proof. In the face of the theorem of Denjoy it is sufficient to prove that if f is measurable, then there exists $\psi \in \mathscr{C}$ such that f is ψ-approximately continuous almost everywhere. Suppose that f is measurable. Let (a_n, b_n), $n \in \mathbb{N}$, be a basis of Euclidean topology on the real line. From [22, Theorem 2.12] there exists a function $\psi \in \mathscr{C}$ such that for any $n \in \mathbb{N}$ almost all points of $f^{-1}((a_n, b_n))$ are its ψ-density points. Denote by $N(f)$ the set of all points in which f is not ψ-approximately continuous. Then

$$N(f) \subset \bigcup_{n\in\mathbb{N}} [f^{-1}((a_n,b_n)) \setminus \Phi_\psi(f^{-1}((a_n,b_n)))]$$

and consequently $m(N(f)) = 0$. □

It is known that if $\mathscr{T}_{\psi_1} \setminus \mathscr{T}_{\psi_2} \neq \emptyset$, then $\mathscr{C}_{\psi_1 o} \setminus \mathscr{C}_{\psi_2 o} \neq \emptyset$ ([7, Theorem 2]). From [7, Proposition 4] it follows that there exist continuum different topologies $\mathscr{T}_{\psi_b} \subsetneq \mathscr{T}_{\psi_a}$, where $0 < a < b < 1$, and continuum different classes of continuous functions such that

$$\mathscr{C}_{oo} \subsetneq \mathscr{C}_{\psi_b o} \subsetneq \mathscr{C}_{\psi_a o} \subsetneq \mathscr{C}_{do}$$

for any $0 < a < b < 1$. Moreover, if $\mathscr{C}_{\psi_b o} \subsetneq \mathscr{C}_{\psi_a o}$, then there exists a $\mathfrak{c}$-generated algebra $\mathscr{F}$ of functions which is contained in the difference $\mathscr{C}_{\psi_a o} \subsetneq \mathscr{C}_{\psi_b o}$ (compare [17]).

4.2.1 Classes $\mathscr{C}_{\psi\psi}$

Let us fix a function ψ from the family $\mathscr{C}$. We will consider continuous functions $f\colon (\mathbb{R}, \mathscr{T}_\psi) \to (\mathbb{R}, \mathscr{T}_\psi)$. Such functions are called ψ-continuous. Evidently,

$$\mathscr{C}onst \subsetneq \mathscr{C}_{\psi\psi} \subset \mathscr{C}_{\psi o} \subsetneq \mathscr{C}_{do}.$$

Since $\mathscr{C}_{do} \subsetneq \mathscr{D}\mathscr{B}_1$ ([4, Theorem 1.3.1]) any function from $\mathscr{C}_{\psi\psi}$ is measurable and $\mathscr{D}\mathscr{B}_1$.

Remind some additional information about ψ-density topologies. All of them are invariant under translation, but they may not be invariant under multiplication. More precisely, if $|\alpha| \geq 1$ and 0 is a ψ-density point of measurable set A, then 0 is a ψ-density point of αA. Indeed, it follows from monotonicity of $\psi \in \mathscr{C}$ and inequality

$$\frac{m(\alpha A' \cap [-h,h])}{2h\psi(2h)} = \frac{\alpha m(A' \cap [-\frac{h}{\alpha}, \frac{h}{\alpha}])}{\frac{2h}{\alpha}\psi(2h)} \leq \frac{\alpha m(A' \cap [-\frac{h}{\alpha}, \frac{h}{\alpha}])}{\frac{2h}{\alpha}\psi(\frac{2h}{\alpha})}.$$

On the other hand, if $\limsup_{x\to 0^+} \frac{\psi(\alpha x)}{\psi(x)} = \infty$, then there exists a set $A \in \mathscr{T}_\psi$ such that $\frac{1}{\alpha}A \notin \mathscr{T}_\psi$ (compare with [22, Theorem 2.8]). It is not difficult to check (compare [6] and [22]), that the topology $\mathscr{T}_\psi$ is invariant under multiplication by a nonzero number if and only if ψ fulfills the condition

$$\limsup_{x\to 0^+} \frac{\psi(2x)}{\psi(x)} < \infty. \qquad (\Delta_2)$$

We will write then $\psi \in \Delta_2$.

Proposition 4.1 ([8, Remark 9]). *Assume that* $\psi \in \mathscr{C}$.

(1) If $\psi \in \Delta_2$ *and* $f \in \mathscr{C}_{\psi\psi}$, *then* $kf \in \mathscr{C}_{\psi\psi}$ *for any number* $k \in \mathbb{R}$.
(2) If $\psi \in \Delta_2$, *then any piecewise linear function is* ψ*-continuous.*
(3) If $\psi \notin \Delta_2$, *then no linear function* $f(x) = kx$ *with* $|k| > 1$ *is* ψ*-continuous.*

For $\psi = id$ the topology $\mathscr{T}_\psi$ coincides with superdensity topology ([18]). This function evidently fulfills (Δ_2). The functions $\psi(x) = x^\alpha$ for $\alpha \geq 1$ are the most useful for obtaining topologies $\mathscr{T}_\psi$ satisfying (Δ_2). Note that, there are functions which do not satisfy (Δ_2) but are in $\mathscr{C}$, for instance $\psi(x) = e^{-\ln^2 x}$ for $x \in (0,1)$ and linear for $x \geq 1$ (one can find another example in [6]).

Let us remind that if $\psi \notin \Delta_2$ then even linear functions may not be continuous. In [10] it is proved that if $\psi \in \Delta_2$ and there exist numbers $\alpha, \beta > 0$ such that

$$0 < \alpha < \frac{f(x) - f(y)}{x - y} < \beta < \infty$$

for any $x \neq y$, then f is ψ-continuous.

In [2] it is shown that the sum of two density continuous functions need not be density continuous. We will show a similar result for ψ-continuous functions. Moreover, we observe that for any $\psi \in \mathscr{C}$ there exists a function f such that $f + id$ is not ψ-continuous.

Theorem 4.6 ([9, Theorem 2]). *For any $\psi \in \mathscr{C}$ there exists a ψ-continuous function f such that $f + id$ is not ψ-continuous.*

Proof. If ψ does not satisfy the condition (Δ_2) then we can put $f(x) = \frac{1}{2}x$. Then, from Proposition 4.1 (3), the function $g(x) = f(x) + x$ is not ψ-continuous. Assume then that $\psi \in \Delta_2$. Let $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$ be an interval set such that 0 is its right-hand ψ-dispersion point, $0 < a_n < b_n < a_{n-1}$, $\lim_{n\to\infty} b_n = 0$ and $b_{n+1} - a_{n+1} \le b_n - a_n$ for any natural number n. Put $c_0 = b_1$, $c_n = a_n + \frac{b_n - a_n}{4}$, $d_n = b_n - \frac{b_n - a_n}{4}$ for $n \in \mathbb{N}$ and $C = \bigcup_{n=1}^{\infty}[c_n, d_n]$. We define a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ in the following way:

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty, 0], \\ x + \frac{b_n - a_n}{4} & \text{for } x \in [d_n, c_{n-1}), n \in \mathbb{N}, \\ \text{linear} & \text{for } x \in [c_n, d_n], n \in \mathbb{N}, \\ c_1 & \text{for } x \ge c_1. \end{cases}$$

The function f is ψ-continuous at any point $x \neq 0$ (as piecewice linear). It is also ψ-continuous at $x = 0$, because for any measurable set $V \in \mathscr{T}_\psi$ such that $0 \in V$ there exists a set $U \in \mathscr{T}_\psi$ such that $0 \in U$ and $f(U) \subset V$ (for details look [9, Theorem 2]).

The function

$$g(x) = -f(x) + x$$

is constant on each interval $[d_n, c_{n-1}]$ for all natural numbers $n \ge 2$ and $g(d_n) = \frac{b_n - a_n}{4}$. The set

$$B = \left\{ \frac{b_1 - a_1}{4}, \frac{b_2 - a_2}{4}, \dots \right\}$$

is denumerable so it is closed in topology $\mathscr{T}_\psi$. But its preimage $g^{-1}(B) = \bigcup_{n=2}^{\infty}[d_n, c_{n-1}]$ is not closed in $\mathscr{T}_\psi$. Hence g is not ψ-continuous. □

Theorem 4.7. *For any $\psi \in \mathscr{C}$*

(1) $\mathscr{C}_{oo} \setminus \mathscr{C}_{\psi\psi} \neq \emptyset$,
(2) $\mathscr{C}_{\psi o} \setminus (\mathscr{C}_{oo} \cup \mathscr{C}_{\psi\psi}) \neq \emptyset$,
(3) $\mathscr{C}_{dd} \setminus \mathscr{C}_{\psi\psi} \neq \emptyset$.

Proof. (1). Let $A = \bigcup_{n=1}^{\infty}[a_n, b_n]$ be a set satisfying the conditions: $b_{n+1} < a_n < b_n$, $\lim_{n\to\infty} b_n = 0$, such that $\lim_{h\to 0^+} \frac{m(A' \cap (0,h))}{2h\psi(2h)} = 0$ and let $\{y_n\}_{n\in\mathbb{N}}$ be a sequence strictly decreasing to 0. The function

$$f(x) = \begin{cases} y_n & \text{for } x \in [a_n, b_n], n \in \mathbb{N} \\ 0 & \text{for } x \le 0, \\ \text{linear} & \text{for } x \in [b_{n+1}, a_n], n \in \mathbb{N} \end{cases}$$

is continuous at every point $x \neq 0$ as piecewise linear function. At $x = 0$ f is continuous, because $\lim_{n\to\infty} f(y_n) = 0 = f(0)$.

From the construction of A we have that $x = 0$ is its ψ-density point and $0 \notin A$, so A is not a $\mathscr{T}_\psi$-closed set. The set $\{y_n\}_{n\in\mathbb{N}}$ is $\mathscr{T}_\psi$-closed and $f^{-1}(\{y_n\}) = A$. Therefore $f \notin \mathscr{C}_{\psi\psi}$ ([8, Example 7]).

(2). Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers strictly decreasing to 0. Take an interval set $A = \bigcup_{n=1}^{\infty} (a_n, b_n)$ such that $\lim_{h\to 0^+} \frac{m(A'\cap[-h,h])}{2h\psi(2h)} = 0$ and $[a_n, b_n] \subset (t_{n+1}, t_n)$. Let $B = \bigcup_{n=1}^{\infty} (c_n, d_n)$ be a set such that $\lim_{h\to 0^+} \frac{m(B\cap[-h,h])}{2h\psi(2h)} = 0$ and $[c_n, d_n] \subset (a_n, b_n)$ (compare [7, Proposition 1]). We define

$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \text{ and } x = a_n,\ n \in \mathbb{N} \\ \frac{1}{n} & \text{for } x \in (a_n, b_n) \setminus (c_n, d_n), n \in \mathbb{N} \\ 1 & \text{for } x = \frac{c_n + d_n}{2}, n \in \mathbb{N} \\ \text{linear} & \text{for the remaining } x. \end{cases}$$

The function f is ψ-approximately continuous at any point. It is neither continuous nor ψ-continuous at $x = 0$ (for details see [8, Example 12]).

(3). There exist sets $C = \bigcup_{n=1}^{\infty} [c_n, d_n]$ and $E = \bigcup_{n=1}^{\infty} [e_n, f_n]$ with $0 < d_{n+1} < c_n < d_n < e_n < f_n < d_n$, $\lim_{n\to\infty} d_n = 0$ such that

$$\lim_{h\to 0+} \frac{m(C \cap (0,h))}{2h} = 0 \text{ and } \frac{m(C \cap (0, d_n))}{2d_n \psi(2d_n)} > \frac{1}{4}.$$

The function

$$g(x) = \begin{cases} 0 & \text{for } x \notin C, \\ 1 & \text{for } x \in [e_n, f_n],\ n \in \mathbb{N}, \\ \text{linear} & \text{on } [c_n, e_n] \text{ and } [f_n, d_n],\ n \in \mathbb{N}. \end{cases}$$

has required properties ([9, Proposition 5]). □

Theorem 4.8 ([8, Example 11]). *If $\psi \in \Delta_2$ then $\mathscr{C}_{\psi\psi} \setminus \mathscr{C}_{oo} \neq \emptyset$.*

Proof. Let $\psi \in \mathscr{C}$ be a function fulfilling the condition (Δ_2) and $A = \bigcup_{n=1}^{\infty} (a_n, b_n)$ be an interval set such that $\lim_{h\to 0^+} \frac{m(A\cap(0,h))}{2h\psi(2h)} = 0$. The function

$$f(x)=\begin{cases}0 & \text{for } x\notin A,\\ 1 & \text{for } x=\frac{a_n+b_n}{2}, n\in\mathbb{N}\\ \text{linear} & \text{for } x\in\left[a_n,\frac{a_n+b_n}{2}\right]\cup\left[\frac{a_n+b_n}{2},b_n\right], n\in\mathbb{N}.\end{cases}$$

has required properties. □

Proposition 4.2 ([9, Proposition 4]). *If $\psi\in\Delta_2$, then the family of ψ-continuous functions is not closed under the uniform convergence.*

The proof of this property is not complicated and it is based on the facts that there is a continuous function which is not ψ-continuous and each continuous function is a limit of uniformly convergent sequence of piecewise linear continuous functions.

The following problems are still open: Is the family $\mathscr{C}_{\psi\psi}\setminus\mathscr{C}_{oo}$ nonempty if $\psi\notin\Delta_2$? Is the class $\mathscr{C}_{\psi\psi}$ closed under uniform convergence if we do not assume condition (Δ_2)?

4.2.2 Classes $\mathscr{C}_{\psi d}$ and $\mathscr{C}_{d\psi}$

Let $\psi\in\mathscr{C}$. By property (P2) $\mathscr{C}_{\psi d}\subset\mathscr{C}_{dd}\cap\mathscr{C}_{\psi\psi}$. Hence every function from the family $\mathscr{C}_{\psi d}$ is measurable and $\mathscr{D}\mathscr{B}_1^*$ because functions from $\mathscr{C}_{dd}$ are measurable and $\mathscr{D}\mathscr{B}_1^*$ ([4, Theorem 4.1]).

The class $\mathscr{C}_{\psi d}$ is relatively small. It is easy to check that linear functions do not belong to it. Moreover, what is surprising, no bijection belongs to $\mathscr{C}_{\psi d}$.

Theorem 4.9. *If $f\in\mathscr{C}_{\psi d}$ then f is not a bijection.*

Proof. Let $f\colon(\mathbb{R},\mathscr{T}_\psi)\to(\mathbb{R},\mathscr{T}_d)$ be a continuous function. Since f is a Darboux function, $f(I)$ is an interval for any interval I. Consequently, for any $U\in\mathscr{T}_o$, $f(U)$ is measurable.

Suppose that there exists the inverse function $f^{-1}\colon\mathbb{R}\to\mathbb{R}$. By Theorem 4.2 there is a nowhere dense and perfect set A such that $m(A)>0$ and $\mathrm{I}nt_\psi(A)=\emptyset$. A is closed in $\mathscr{T}_o$, so it is closed and nowhere dense in $\mathscr{T}_\psi$. Hence A' is dense in $\mathscr{T}_\psi$. Moreover $A'\in\mathscr{T}_o$, so $f(A')$ is a measurable set. Hence the set $\mathbb{R}\setminus f(A')=f(A)$ is measurable.

Suppose that $m(f(A))>0$. Then there is a nonempty set $U\subset f(A)$ open in topology $\mathscr{T}_d$. Since f is continuous, $f^{-1}(U)\in\mathscr{T}_\psi$. But A' is dense in $\mathscr{T}_\psi$, then $f^{-1}(U)\cap A'\neq\emptyset$ and, consequently, $U\cap f(A')\neq\emptyset$. It is a contradiction, because $U\subset f(A)$. This proves that $m(f(A))=0$.

Consequently, the set $f(A)$ and any subset B of $f(A)$ is closed in $\mathscr{T}_d$. Since $f \in \mathscr{C}_{\psi d}$, $f^{-1}(B)$ is closed in $\mathscr{T}_\psi$ for any $B \subset f(A)$. Any $\mathscr{T}_\psi$-closed set is measurable. Obviously, for any $C \subset A$ there exists $B \subset f(A)$ such that $f^{-1}(B) = C$. It gives a contradiction, because the set A (of positive measure) contains a nonmeasurable set. □

Modifying the last proof we can prove the following

Theorem 4.10 (compare [8]). *If $f \in \mathscr{C}_{\psi d}$ and $a < b$ then $f|_{(a,b)}$ is not an injection.*

We do not know if there exists nonconstant function in $\mathscr{C}_{\psi d}$.

From (P2) it follows that $\mathscr{C}_{d\psi} \supset \mathscr{C}_{dd} \cup \mathscr{C}_{\psi\psi}$ and $\mathscr{C}_{d\psi} \subset \mathscr{C}_{do}$, therefore functions from $\mathscr{C}_{d\psi}$ are $\mathscr{D}\mathscr{B}_1$ and may not be $\mathscr{D}\mathscr{B}_1^*$. The inclusion $\mathscr{C}_{d\psi} \subset \mathscr{C}_{do}$ is proper. Indeed, take the interval set $A = \bigcup_{n=1}^{\infty} [a_n, b_n]$ such that $0 \in \Phi^+(A)$ and put

$$f(x) = \begin{cases} \frac{1}{n} & \text{for } x \in [a_n, b_n],\ n \in \mathbb{N}, \\ \text{linear} & \text{for } x \in [b_n, a_{n-1}], n \in \mathbb{N} \\ 0 & \text{for } x \le 0. \end{cases}$$

Then $Y = (0,\infty) \setminus \{\frac{1}{n}; n \in \mathbb{N}\} \in \mathscr{T}_\psi$, but $f^{-1}(Y) = (0,\infty) \setminus A \notin \mathscr{T}_d$. Hence $f \notin \mathscr{C}_{\psi d}$. Simultaneously, f is continuous, therefore approximately continuous, so $\mathscr{C}_{d\psi} \subsetneq \mathscr{C}_{do}$ and, additionally, we obtained that $\mathscr{C}_{oo} \not\subset \mathscr{C}_{\psi d}$.

Moreover $f \in \mathscr{C}_{do} \setminus \mathscr{C}_{d\psi}$ for any $\psi \in \mathscr{C}$. Therefore, $\mathscr{C}_{do} \setminus \bigcup_{\psi \in \mathscr{C}} \mathscr{C}_{d\psi} \neq \emptyset$. However we do not know how to describe the union $\bigcup_{\psi \in \mathscr{C}} \mathscr{C}_{d\psi}$. Obviously, if $\mathscr{T}_{\psi_1} \subsetneq \mathscr{T}_{\psi_2}$ then $\mathscr{C}_{d\psi_2} \subset \mathscr{C}_{d\psi_1}$, but we even can not say whether this inclusion is proper.

4.3 Functions preserving ψ-density points

In [1] there was introduced the concept of homeomorphism preserving density points. This notion was examined also in [19]. We will adopt this idea to the theory of ψ-density continuous functions.

Fix a function $\psi \in \mathscr{C}$ and let introduce the notion of a function preserving ψ-density points.

Definition 4.2. We will say that a homeomorphism h preserves ψ-density points if for any measurable set $S \subset \mathbb{R}$ and any $x_0 \in \Phi_\psi(S)$

$$\lim_{t\to 0^+} \frac{m^*((h(S))' \cap [h(x_0)-t, h(x_0)+t])}{2t\psi(2t)} = 0$$

(m^* stands for the outer Lebesgue measure).

Observe that if a homeomorphism h preserves ψ-density points, then it also preserves ψ-dispersion points.

Proposition 4.3. *If h is a homeomorphism preserving ψ-density points, then h satisfies Lusin's condition (N).*

Proof. Let Z be a set of Lebesgue measure zero. There exists a G_δ-set $A \supset Z$ of measure zero. Then $h(A)$ is also a G_δ-set, so it is measurable. Suppose that Lebesgue measure of $h(A)$ is positive. Hence $h(A)$ has density 1 at a certain point $y_0 \in h(A)$:

$$\lim_{t\to 0^+} \frac{m(h(A) \cap [y_0-t, y_0+t])}{2t} = 1.$$

Observe that for $S = A'$ and any $t > 0$ such that $\psi(2t) \le 1$, we have

$$\begin{aligned}\frac{m^*((h(S))' \cap [y_0-t, y_0+t])}{2t\psi(2t)} &= \frac{m^*(h(S') \cap [y_0-t, y_0+t])}{2t\psi(2t)} \ge \\ &\ge \frac{m(h(A) \cap [y_0-t, y_0+t])}{2t},\end{aligned}$$

therefore h does not preserve ψ-density points. □

Corollary 4.1. *If a homeomorphism $h\colon [0,1] \to [0,1]$ preserves ψ-density points then it is an absolutely continuous function.*

From Proposition 4.3 it follows that if the homeomorphism h preserves ψ-density points then, for any measurable set $S \subset \mathbb{R}$, $h(S)$ is a measurable set and we need not use the outer measure in Definition 4.2.

Theorem 4.11. *A homeomorphism $h\colon \mathbb{R} \to \mathbb{R}$ preserves ψ-density points if and only if h^{-1} is a ψ-continuous function.*

Proof. First we assume that h preserves ψ-density points. We will show that h^{-1} is a ψ-continuous function at any point. Fix a point y_0 and a set $V \in \mathscr{T}_\psi$ such that $x_0 = h^{-1}(y_0) \in V$. We will show that there exists a set $U \in \mathscr{T}_\psi$ such that $y_0 \in U$ and $h^{-1}(U) \subset V$. Since V is open in $\mathscr{T}_\psi$, for any $x \in V$ we have $x \in \Phi_\psi(V)$. The homeomorphism preserves ψ-density points, so $h(x) \in \Phi_\psi(h(V))$. Hence $h(V)$ is open in $\mathscr{T}_\psi$ and putting $U = h(V)$, we complete the proof of this implication.

Suppose now that h does not preserve ψ-density points. Set $x_0 \in \mathbb{R}$ and $S \in \mathscr{L}$ such that x_0 is a ψ-dispersion point of S and

$$\limsup_{t\to 0^+} \frac{m^*(h(S)\cap[h(x_0)-t,h(x_0)+t])}{2t\psi(2t)} > 0.$$

Take a G_δ-set $A \supset S$ such that $m(A\setminus S)=0$, a sequence $(a_n)_{n\in\mathbb{N}}$ decreasing to 0 and a number $\alpha > 0$ for which

$$\frac{m\,(h(A)\cap[h(x_0)-a_n,h(x_0)+a_n])}{2a_n\psi(2a_n)} > \alpha$$

for all $n \in \mathbb{N}$. We can assume that for $n \in \mathbb{N}$

$$\frac{m\,(h(A)\cap[h(x_0),h(x_0)+a_n])}{2a_n\psi(2a_n)} > \frac{\alpha}{2} \tag{4.3}$$

For any natural n there exists a closed set $B_n \subset h(A)\cap[h(x_0)+a_{n+1},h(x_0)+a_n]$ such that

$$m(B_n) > m\Big(h(A)\cap[h(x_0)+a_{n+1},h(x_0)+a_n]\Big) - \frac{\alpha}{4}\cdot\frac{1}{2^n}\cdot 2a_n\psi(2a_n). \tag{4.4}$$

The set $B = \bigcup_{n=1}^{\infty} B_n \cup \{h(x_0)\}$ is closed in natural topology and from (4.3) and (4.4) we obtain

$$\frac{m\,(B\cap[h(x_0),h(x_0)+a_n])}{2a_n\psi(2a_n)} \geq \frac{\alpha}{4} > 0$$

for any n. Hence $h(x_0)$ is not a ψ-dispersion point of the set B.

On the other hand, x_0 is a ψ-dispersion point of $h^{-1}(B)$ and the set $C = \mathbb{R}\setminus h^{-1}(B)\cup\{x_0\} \in \mathscr{T}_\psi$, but $h(C) = \mathbb{R}\setminus B\cup\{h(x_0)\} \notin \mathscr{T}_\psi$, so the function h^{-1} is not ψ-continuous. $\square$

This survey still leaves numerous questions without answers. In particular we do not know if:

1. Is the difference $\mathscr{C}_{\psi\psi}\setminus\mathscr{C}_{oo}$ nonempty and is the class $\mathscr{C}_{\psi\psi}$ closed under uniform convergence if $\psi \notin \Delta_2$?
2. Does there exist a nonconstant function $f \in \mathscr{C}_{\psi d}$?
3. Is the inclusion $\mathscr{C}_{d\psi_2} \subset \mathscr{C}_{d\psi_1}$ proper, if $\mathscr{T}_{\psi_1} \subsetneq \mathscr{T}_{\psi_2}$?
4. What is the union $\bigcup_{\psi\in\mathscr{C}} \mathscr{C}_{d\psi}$ and intersection $\bigcap_{\psi\in\mathscr{C}} \mathscr{C}_{d\psi}$?
5. What is the relation between classes $\mathscr{C}_{\psi_1\psi_1}$ and $\mathscr{C}_{\psi_2\psi_2}$ for different functions $\psi_1,\psi_2 \in \mathscr{C}$ such that $\mathscr{T}_{\psi_1} \neq \mathscr{T}_{\psi_2}$?

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