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Research Article

Best Proximity Point Theorem in Quasi-Pseudometric Spaces

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In quasi-pseudometric spaces (not necessarily sequentially complete), we continue the research on the quasi-generalized pseudodistances. We introduce the concepts of semiquasiclosed map and contraction of Nadler type with respect to generalized pseudodistances. Next, inspired by Abkar and Gabeleh we proved new best proximity point theorem in a quasi-pseudometric space. A best proximity point theorem furnishes sufficient conditions that ascertain the existence of an optimal solution to the problem of globally minimizing the error $\inf\{d(x,y) : y \in T(x)\}$, and hence the existence of a consummate approximate solution to the equation $T(X) = x$.

1. Preliminaries

Let A, B be nonempty subsets of a metric space (X, d). Then denote $\operatorname{dist}(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$; $D(x,B) = \inf\{d(x,y) : y \in B\}$ for $x \in X$; and

$$\begin{aligned} A_0 &= \{x \in A : d(x,y) = \operatorname{dist}(A,B) \text{ for some } y \in B\}; \\ B_0 &= \{y \in B : d(x,y) = \operatorname{dist}(A,B) \text{ for some } x \in A\}. \end{aligned} \tag{1}$$

We say that the pair (A, B) has the P-property if and only if

$$\begin{aligned} &\{d(x_1,y_1) = \operatorname{dist}(A,B) \wedge d(x_2,y_2) = \operatorname{dist}(A,B)\} \\ &\quad \Longrightarrow d(x_1,x_2) = d(y_1,y_2), \end{aligned} \tag{2}$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$. It is worth noticing that the concept of P-property was first introduced by Sankar Raj [1] (for details see also Abkar and Gabeleh [2]).

In 2013, Abkar and Gabeleh proved the following interesting results.

Theorem 1 (see [3]). *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that $A_0 \neq \emptyset$ and (A, B) has the P-property. We assume that $T : A \to 2^B$ is a multivalued non-self-contraction mapping; that is, $\{H(T(x),T(y)) \leqslant \lambda d(x,y)\}$ $\exists 0 \leqslant \lambda < 1$ $\forall x, y \in A$. If $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x_0) \subset B_0$ for each $x_0 \in A_0$, then T has a best proximity point in A.*

In this paper, inspired by Abkar and Gabeleh [3], we proved the best proximity point theorem in (not necessarily sequentially complete) quasi-pseudometric space. We introduced new class of multivalued contractions, which are generalization of classical contractions of Nadler type. For generality, this new class of narrower contractions is studied in quasi-pseudometric space. It is worth noticing that in the fixed point theory there exist many results in asymmetrics spaces (e.g., see Latif and Al-Mezel [4], Karuppiah and Marudai [5], Gaba [6, 7], and Otafudu [8]). The study in which conditions of contraction are defined by nonsymmetric distance is a new and extensive branch of metric fixed point theory. However, even in metric space, or b-metric space, these new contractions are extension of classical contractions of Nadler type. Furthermore, the concept of narrowing can be used not only for contractions of Nadler type, but also for Banach contraction (for single-valued map) and different generalizations of Banach and Nadler contractions.

The following terminologies from papers of Kelly [9], Reilly [10], and Reilly et al. [11] will be used in the sequel.

Definition 2. Let X be a nonempty set. A *quasi-pseudometric* on X is a map $p : X \times X \to [0, \infty)$ such that

($p1$) $\{p(x, x) = 0\}$ $\forall x \in X$; and

($p2$) $\{p(x, z) \leqslant p(x, y) + p(y, z)\}$ $\forall x, y, z \in X$.

For a given quasi-pseudometric p on X, a pair (X, p) is called quasi-pseudometric space. A quasi-pseudometric space (X, p) is called Hausdorff if

$$\{(x \neq y) \Longrightarrow (\max\{p(x, y), p(y, x)\} > 0)\} \quad \forall x, y \in X. \tag{3}$$

Definition 3. Let (X, p) be a quasi-pseudometric space. Then consider the following.

(i) ([10, Definition 5.1], [11, Definition 1(v) and p. 129]) One says that a sequence $(w_m : m \in \mathbb{N})$ in X is *left* (*right*) *Cauchy sequence in* X if

$$\begin{aligned} &\{p(w_m, w_n) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ k \leqslant m \leqslant n, \\ &(\{p(w_n, w_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ k \leqslant m \leqslant n). \end{aligned} \tag{4}$$

(ii) One says that a sequence $(w_m : m \in \mathbb{N})$ in X is *left (right) convergent in* X if

$$\begin{aligned} &\{p(w, w_m) < \varepsilon\} \quad \exists w \in X,\ \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m, \\ &(\{p(w_m, w) < \varepsilon\} \quad \exists w \in X,\ \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m), \end{aligned} \tag{5}$$

that is, if $\{\lim_{m\to\infty} p(w, w_m) = 0\}$ $\exists w \in X$ ($\exists w \in X$ $\{\lim_{m\to\infty} p(w, w_m) = 0\}$), for short.

(iii) ([10, Definition 5.3]) If every left (right) Cauchysequence in X is left (right) convergent to some point in X, then (X, p) is called left (right) sequentially complete quasi-pseudometric space.

Remark 4. Let (X, p) be a quasi-pseudometric space. Then (i) every left (right) convergent sequence in X is left (right) Cauchy sequence in X and the converse is false ([11, Example 2], [9, Example 5.8]); (ii) the limit of a left (right) convergent sequence is not unique. More precisely it is possible that if a sequence $(w_m : m \in \mathbb{N})$ in X is left (right) convergent in X then

$$\begin{aligned} &\{p(w, w_m) < \varepsilon\} \quad \exists W \subset X,\ \forall w \in W,\ \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m, \\ &(\{p(w_m, w) < \varepsilon\} \quad \exists W \subset X,\ \forall w \in W,\ \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m). \end{aligned} \tag{6}$$

Example 5 (see [12]). Let $X \subset \mathbb{R}$ be a nonempty set and let $p : X \times X \to [0, \infty)$ be given by the formula

$$p(x, y) = \begin{cases} 0 & \text{if } x \geqslant y, \\ 1 & \text{if } x < y. \end{cases} \tag{7}$$

The map p is a quasi-pseudometric on X and (X, p) is quasi-pseudometric space (for details see Reilly et al. [11]). Morever it is easy to verify that (X, p) is Hausdorff. Now, if $X = [0, 6]$ and we consider the sequence $(w_m = 1/m : m \in \mathbb{N})$ in X then we obtain that each point of the set $W = (0, 6]$ is a left limit of the sequence $(w_m : m \in \mathbb{N})$. Indeed, for each $w \in W$ there exists $k \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ such that $k \leqslant m$ we have $p(w, w_m) = 0$. Hence $\{p(w, w_m) < \varepsilon\}$ $\forall w \in W$ $\forall \varepsilon > 0$ $\exists k = k(\varepsilon) \in \mathbb{N}$ $\forall m \in \mathbb{N};\ k \leqslant m$.

Definition 6 (see [13, Section 3]). Let (X, p) be a quasi-pseudometric space. The map $J : X \times X \to [0, \infty)$ is said to be a *left* (*right*) *quasi-generalized pseudodistance on* X if the following two conditions hold:

($J1$) $\{J(u, w) \leqslant J(u, v) + J(v, w)\}$ $\forall u, v, w \in X$;

($J2$) for any sequences $(u_m : m \in \mathbb{N})$ and $(v_m : m \in \mathbb{N})$ in X satisfying

$$\begin{aligned} &\{J(u_m, u_n) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ k \leqslant m \leqslant n, \\ &(\{J(u_n, u_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ k \leqslant m \leqslant n), \\ &\{J(v_m, u_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m, \\ &(\{J(u_m, v_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m), \end{aligned} \tag{8}$$

the following holds

$$\{p(v_m, u_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m, \tag{9}$$

$$(\{p(u_m, v_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m \in \mathbb{N};\ k \leqslant m). \tag{10}$$

We observe that conditions (9) and (10) are equivalent to $\lim_{m\to\infty} p(v_m, u_m) = 0$ and $\lim_{m\to\infty} p(u_m, v_m) = 0$, respectively. In the following remark, we list some basic properties of left (right) generalized pseudodistance on (X, p).

Remark 7. Let (X, p) be a quasi-pseudometric space. The following hold: (a) quasi-pseudometric is left and right quasi-generalized pseudodistance on X; (b) let J be left (right) quasi-generalized pseudodistance on X. If $\forall u \in X$ $\{J(u, u) = 0\}$, then, J is quasi-pseudometric; (c) there are examples of left (right) generalized pseudodistance such that the map J is not quasi-pseudometrics (see Example 4.2 in [13]); (d) ([13, Proposition 3.1]) if (X, p) is a Hausdorff quasi-pseudometric space and J is a left (right) quasi-generalized pseudodistance, then $\{u \neq v \Rightarrow \{\max\{J(u,v), J(v,u)\} > 0\}\}$ $\forall u, v \in X$.

Definition 8 (see [13]). Let (X, p) be a quasi-pseudometric space and let $J : X \times X \to [0, \infty)$ be a left (right) quasi-generalized pseudodistance on X.

(i) One says that a sequence $(u_m : m \in \mathbb{N})$ in X is *left* (*right*) *J-Cauchy sequence in X* if

$$\begin{aligned}&\{J(u_m, u_n) < \varepsilon\}\\ &\qquad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ k \leqslant m \leqslant n,\\ &(\{J(u_n, u_m) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists k = k(\varepsilon) \in \mathbb{N},\ \forall m, n\\ &\quad \in \mathbb{N};\ k \leqslant m \leqslant n).\end{aligned} \tag{11}$$

(ii) Let $u \in X$ and let $(u_m : m \in \mathbb{N})$ be a sequence in X. One says that $(u_m : m \in \mathbb{N})$ is *left* (*right*) *J-convergent to u* if $\lim^{L-J}_{m\to\infty} u_m = u$; that is, $\lim_{m\to\infty} J(u, u_m) = 0$ ($\lim^{R-J}_{m\to\infty} u_m = u$); that is, $\lim_{m\to\infty} J(u_m, u) = 0$).

(iii) One says that a sequence $(u_m : m \in \mathbb{N})$ in X is *left* (*right*) *J-convergent in X* if $S^{L-J}_{(u_m:m\in\mathbb{N})} := \{u \in X : \lim^{L-J}_{m\to\infty} u_m = u\} \neq \varnothing$ ($S^{R-J}_{(u_m:m\in\mathbb{N})} := \{u \in X : \lim^{R-J}_{m\to\infty} u_m = u\} \neq \varnothing$).

(iv) If every left (right) J-Cauchy sequence $(u_m : m \in \mathbb{N})$ in X is left (right) J-convergent in X, that is, $S^{L-J}_{(u_m:m\in\mathbb{N})} \neq \varnothing$ ($S^{R-J}_{(u_m:m\in\mathbb{N})} \neq \varnothing$), then (X, p) is called *left* (*right*) *J-sequentially complete quasi-pseudometric space*.

(v) Let the class of all nonempty closed subsets of X be denoted by $\mathrm{Cl}(X)$. Let $\forall u \in X$ $\forall V \in \mathrm{Cl}(X)$ $\{J(u, V) = \inf_{v\in V} J(u, v)\}$. Define the distance of Hausdorff type, as the map $H_J : \mathrm{Cl}(X) \times \mathrm{Cl}(X) \to [0, \infty)$, where

$$\left\{H_J(A, B) = \max\left\{\sup_{u\in A} J(u, B), \sup_{v\in B} J(v, A)\right\}\right\} \qquad \forall A, B \in \mathrm{Cl}(X). \tag{12}$$

It is worth noticing that if (X, d) is a metric space and we put $J = d$, then we obtain the classical Hausdorff distance. Example of left J-sequentially complete quasi-pseudometric space which is not left sequentially complete is given in [12, Examples 6.1 and 6.2]. Now, we will present some indications that we will use later in the work.

Let (X, p) be a quasi-pseudometric space, and let $J : X \times X \to [0, \infty)$ be a left (right) quasi-generalized pseudodistance on X. Let $A \neq \emptyset$ and $B \neq \emptyset$ be subsets of X. We adopt the following notations and definitions: $\mathrm{dist}_p(A, B) = \inf\{p(x, y) : x \in A,\ y \in B\}$; $p(a, B) = \inf\{p(a, b) : b \in B\}$, where $a \in X$; and

$$\begin{aligned}A^J_0 &= \{x \in A : J(x, y) = \mathrm{dist}_p(A, B),\ \text{for some } y \in B\};\\ B^J_0 &= \{y \in B : J(x, y) = \mathrm{dist}_p(A, B),\ \text{for some } x \in A\}.\end{aligned} \tag{13}$$

Definition 9. Let (X, p) be a quasi-pseudometric space, and let $J : X \times X \to [0, \infty)$ be a left (right) quasi-generalized pseudodistance on X. Let (A, B) be a pair of nonempty subset of X with $A^J_0 \neq \emptyset$.

(i) The pair (A, B) is said to have the *WP^J-property* if and only if

$$\begin{matrix} J(x_1, y_1) = \mathrm{dist}_p(A, B) \\ J(x_2, y_2) = \mathrm{dist}_p(A, B) \end{matrix} \Longrightarrow (J(x_1, x_2) \leqslant J(y_1, y_2)), \tag{14}$$

where $x_1, x_2 \in A^J_0$ and $y_1, y_2 \in B^J_0$.

(ii) One says that a left (right) quasi-generalized pseudodistance on X *is associated with the pair* (A, B) if, for any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X such that $\exists x \in X$ $\{\lim_{m\to\infty} p(x, x_m) = 0\}$; $\exists y \in X$ $\{\lim_{m\to\infty} p(y, y_m) = 0\}$; and

$$\{J(x_m, y_{m-1}) = \mathrm{dist}_p(A, B)\} \quad \forall m \in \mathbb{N}, \tag{15}$$

one has $\max\{p(x, y), p(y, x)\} = \mathrm{dist}_p(A, B)$.

2. Best Proximity Point Theory in Quasi-Pseudometric Spaces

In this section we recall a definition of quasiclosed map and introduce the concepts of semiquasiclosed map and narrower J-contraction of Nadler type.

Definition 10. Let (X, p) be a quasi-pseudometric space and let A, B be a nonempty subsets of X.

(i) ([12, Definition 4.2(i)]) The set-valued non-self-mapping $T : A \to 2^B$ is called quasiclosed if whenever $(x_m : m \in \mathbb{N})$ is a sequence in A left converging to $W \subset A$ and $(y_m : m \in \mathbb{N})$ is a sequence in B satisfying the condition $\{y_m \in T(x_m)\}$ $\forall m \in \mathbb{N}$ and left converging to each point of the set $V \subset B$, then

$$\{v \in T(w)\} \quad \exists v \in V,\ \forall w \in W. \tag{16}$$

(ii) The set-valued non-self-mapping $T : A \to 2^B$ is called semiquasiclosed if whenever $(x_m : m \in \mathbb{N})$ is a sequence in A left converging to $W \subset A$ and $(y_m : m \in \mathbb{N})$ is a sequence in B satisfying the condition $\{y_m \in T(x_m)\}$ $\forall m \in \mathbb{N}$ and left converging to each point of the set $V \subset B$, then

$$\{v \in T(w)\} \quad \exists v \in V,\ \exists w \in W. \tag{17}$$

(iii) Let $J : X \times X \to [0, \infty)$ be a left (right) generalized pseudodistance on X. Let the map $T : A \to 2^B$ be such that $T(x) \in \mathrm{Cl}(X)$, for each $x \in X$. The map T is called a set-valued non-self-mapping J-contraction of Nadler type, if the following condition holds:

$$\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\} \quad \exists 0 \leqslant \lambda < 1 \ \forall x, y \in A. \tag{18}$$

(iv) The map T is called a set-valued non-self-mapping narrower J-contraction of Nadler type, if the following condition holds: $\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\}$ $\exists 0 \leqslant \lambda < 1$ $\forall x, y \in A_0^J$.

Theorem 11. *Let (X, p) be a Hausdorff left (right) J-sequentially complete quasi-pseudometric space, where $J : X \times X \to [0, \infty)$ is a left (right) quasi-generalized pseudodistance on X. Let (A, B) be a pair of nonempty subset of X with $A_0^J \neq \emptyset$ and such that (A, B) has the WP^J-property and J is associated with (A, B). Let $T : A \to 2^B$ be a semiquasiclosed set-valued non-self-mapping narrower contraction of Nadler type. Let $T(x)$ be bounded and closed in B for all $x \in A$, and $T(x) \subset B_0^J$ for each $x \in A_0^J$. Then T has a best proximity point in A.*

Proof. Part I. We assume that (X, p) is a quasi-pseudometric space and $J : X \times X \to [0, \infty)$ is a left generalized pseudodistance on X, such that (X, p) is a left J-sequentially complete quasi-pseudometric space. To begin, we observe that

$$\{J(w, v) \leqslant H_J(T(x), T(y)) + \gamma\} \quad \forall x, y \in A,\ \forall \gamma > 0,\ \forall w \in T(x),\ \exists v \in T(y). \tag{19}$$

Let $x, y \in A$, we know that $J(w, T(y)) \leqslant J(w, b)$ for all $b \in T(y)$ and for all $w \in T(x)$. Moreover for each $\gamma > 0$, using characterisation of infimum, there exists $v \in T(y)$ such that

$$J(w, T(y)) \leqslant J(w, v) \leqslant J(w, T(y)) + \gamma. \tag{20}$$

Property (20) implies that

$$\begin{aligned} J(w, T(y)) &\leqslant J(w, v) \leqslant J(w, T(y)) + \gamma \\ &\leqslant \sup_{w \in T(x)} J(w, T(y)) + \gamma. \end{aligned} \tag{21}$$

Since $\sup_{w \in T(x)} J(w, T(y)) \leqslant H_J(T(x), T(y))$, we conclude that

$$\begin{aligned} J(w, v) &\leqslant J(w, T(y)) + \gamma \leqslant \sup_{w \in T(x)} J(w, T(y)) + \gamma \\ &\leqslant H_J(T(x), T(y)) + \gamma. \end{aligned} \tag{22}$$

Hence property (19) holds. The proof will be broken into four steps.

Step 1. We can construct the sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ such that

$$\{w^m \in A_0^J \wedge v^m \in B_0^J\} \quad \forall m \in \{0\} \cup \mathbb{N}, \tag{23}$$

$$\{v^m \in T(w^m)\} \quad \forall m \in \{0\} \cup \mathbb{N}, \tag{24}$$

$$\{J(w^m, v^{m-1}) = \mathrm{dist}_p(A, B)\} \quad \forall m \in \mathbb{N}, \tag{25}$$

$$\{J(v^{m-1}, v^m) \leqslant H_J(T(w^{m-1}), T(w^m)) + \lambda^m\} \quad \forall m \in \mathbb{N}, \tag{26}$$

$$\{J(w^m, w^{m+1}) \leqslant J(v^{m-1}, v^m)\} \quad \forall m \in \mathbb{N}, \tag{27}$$

$$\{J(w^m, w^n) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ n_0 \leqslant m \leqslant n, \tag{28}$$

$$\{J(v^m, v^n) < \varepsilon\} \quad \forall \varepsilon > 0,\ \exists n_0 = n_0(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ n_0 \leqslant m \leqslant n. \tag{29}$$

Indeed, since $A_0^J \neq \emptyset$ and $T(x) \subseteq B_0^J$ for each $x \in A_0^J$, we may choose $w^0 \in A_0^J$ and next $v^0 \in T(w^0) \subseteq B_0^J$. By definition of B_0^J, there exists $w^1 \in A$ such that

$$J(w^1, v^0) = \mathrm{dist}_p(A, B). \tag{30}$$

Of course, since $v^0 \in B$, by (30), we have $w^1 \in A_0^J$. Next, since $T(x) \subseteq B_0^J$ for each $x \in A_0^J$, from (19) (for $x = w^0$, $y = w^1$, $\gamma = \lambda$, $w = v^0$) we conclude that there exists $v = v^1 \in T(w^1) \subseteq B_0^J$ (since $w^1 \in A_0^J$) such that

$$J(v^0, v^1) \leqslant H_J(T(w^0), T(w^1)) + \lambda. \tag{31}$$

Next, since $v^1 \in B_0^J$, by definition of B_0^J, there exists $w^2 \in A$ such that

$$J(w^2, v^1) = \mathrm{dist}_p(A, B). \tag{32}$$

Of course, since $v^1 \in B$, by (32), we have $w^2 \in A_0^J$. Since $T(x) \subseteq B_0^J$ for each $x \in A_0^J$, from (19) (for $x = w^1$, $y = w^2$, $\gamma = \lambda^2$, $w = v^1$) we conclude that there exists $v^2 \in T(w^2) \subseteq B_0^J$ (since $w^2 \in A_0^J$) such that

$$J(v^1, v^2) \leqslant H_J(T(w^1), T(w^2)) + \lambda^2. \tag{33}$$

By (30)–(33) and by the induction, we produce sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ such that $\{w^m \in A_0^J \wedge v^m \in B_0^J\}$ $\forall m \in \{0\} \cup \mathbb{N}$; $\{v^m \in T(w^m)\}$ $\forall m \in \{0\} \cup \mathbb{N}$; $\{J(w^m, v^{m-1}) = \mathrm{dist}_p(A, B)\}$ $\forall m \in \mathbb{N}$; and $\{J(v^{m-1}, v^m) \leqslant H_J(T(w^{m-1}), T(w^m)) + \lambda^m\}$ $\forall m \in \mathbb{N}$. Thus (23)–(26) hold. In particular (25) gives

$$\begin{aligned} \{J(w^m, v^{m-1}) &= \mathrm{dist}_p(A, B) \wedge J(w^{m+1}, v^m) \\ &= \mathrm{dist}_p(A, B)\} \quad \forall m \in \mathbb{N}. \end{aligned} \tag{34}$$

Now, since the pair (A, B) has the WP^J-property, from the above we conclude $\forall m \in \mathbb{N}\ \{J(w^m, w^{m+1}) \leqslant J(v^{m-1}, v^m)\}$. Consequently, property (27) holds.

We recall that the contractive condition is as follows:

$$\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\} \quad \exists 0 \leqslant \lambda < 1,\ \forall x, y \in A_0^J. \tag{35}$$

In particular, by (35) (for $x = w^m \in A_0^J$, $y = w^{m+1} \in A_0^J$, $m \in \{0\} \cup \mathbb{N}$) we obtain

$$\left\{H_J\left(T\left(w^m\right), T\left(w^{m+1}\right)\right) \leqslant \lambda J\left(w^m, w^{m+1}\right)\right\} \quad \exists 0 \leqslant \lambda < 1,\ \forall m \in \{0\} \cup \mathbb{N}. \tag{36}$$

Next, by (27), (26), and (36) we calculate

$$\begin{aligned}
&\left\{J\left(w^m, w^{m+1}\right) \leqslant J\left(v^{m-1}, v^m\right)\right.\\
&\quad \leqslant H_J\left(T\left(w^{m-1}\right), T\left(w^m\right)\right) + \lambda^m \leqslant \lambda J\left(w^{m-1}, w^m\right)\\
&\quad + \lambda^m \leqslant \lambda J\left(v^{m-2}, v^{m-1}\right) + \lambda^m\\
&\quad \leqslant \lambda\left[H_J\left(T\left(w^{m-2}\right), T\left(w^{m-1}\right)\right) + \lambda^{m-1}\right] + \lambda^m\\
&\quad = \lambda H_J\left(T\left(w^{m-2}\right), T\left(w^{m-1}\right)\right) + 2\lambda^m\\
&\quad \leqslant \lambda^2 J\left(w^{m-2}, w^{m-1}\right) + 2\lambda^m \leqslant \lambda^2 J\left(v^{m-3}, v^{m-2}\right)\\
&\quad + 2\lambda^m \leqslant \lambda^2\left[H_J\left(T\left(w^{m-3}\right), T\left(w^{m-2}\right)\right) + \lambda^{m-2}\right]\\
&\quad + 2\lambda^m = \lambda^2 H_J\left(T\left(w^{m-3}\right), T\left(w^{m-2}\right)\right) + 3\lambda^m\\
&\quad \leqslant \lambda^3 J\left(w^{m-3}, w^{m-2}\right) + 3\lambda^m \leqslant \cdots \leqslant \lambda^m J\left(w^0, w^1\right)\\
&\quad \left. + m\lambda^m\right\} \quad \exists 0 \leqslant \lambda < 1,\ \forall m \in \mathbb{N}.
\end{aligned} \tag{37}$$

Hence, $\exists 0 \leqslant \lambda < 1\ \forall m \in \mathbb{N}\ \{J(w^m, w^{m+1}) \leqslant \lambda^m J(w^0, w^1) + m\lambda^m\}$. This implies that $\sum_{m=0}^{\infty} J(w^m, w^{m+1}) < \infty$. Now, we have $\forall \varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N}\ \{\sum_{k=n_0}^{\infty} J(w^k, w^{k+1}) < \varepsilon\}$. Hence, by $(J1)$ we get

$$\begin{aligned}
&\left\{J\left(w^m, w^n\right) \leqslant \sup_{n \geqslant m} J\left(w^m, w^n\right) \leqslant \sum_{k=n_0}^{\infty} J\left(w^k, w^{k+1}\right)\right.\\
&\quad \left. < \varepsilon\right\}\\
&\quad \forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N},\ \forall m, n \in \mathbb{N};\ n_0 \leqslant m \leqslant n.
\end{aligned} \tag{38}$$

In consequence $\forall \varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N}\ \forall m, n \in \mathbb{N};\ n_0 \leqslant m \leqslant n\ \{J(w^m, w^n) < \varepsilon\}$. Similarly, by (27), (26), and (36) we obtain

$$\begin{aligned}
&\left\{J\left(v^{m-1}, v^m\right) \leqslant H_J\left(T\left(w^{m-1}\right), T\left(w^m\right)\right) + \lambda^m\right.\\
&\quad \leqslant \lambda J\left(w^{m-1}, w^m\right) + \lambda^m \leqslant \lambda J\left(v^{m-2}, v^{m-1}\right) + \lambda^m\\
&\quad \left. \leqslant \cdots \leqslant \lambda^m J\left(w^0, w^1\right) + m\lambda^m\right\}\\
&\quad \exists 0 \leqslant \lambda < 1,\ \forall m \in \mathbb{N}.
\end{aligned} \tag{39}$$

Using the analogous method as in the above we get $\forall \varepsilon > 0\ \exists n_0(\varepsilon) \in \mathbb{N}\ \forall m, n \in \mathbb{N};\ n_0 \leqslant m \leqslant n\ \{J(w^m, w^n) < \varepsilon\}$. Then properties (23)–(29) hold.

Step 2. We can show that the sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ *and* $(v^m : m \in \{0\} \cup \mathbb{N})$ *are left* J*-Cauchy sequences in* X. Indeed, it is an easy consequence of (28) and (29).

Step 3. We can show that the sets $S^{L-J}_{(w^m : m \in \mathbb{N})}$ *and* $S^{L-J}_{(v^m : m \in \mathbb{N})}$ *are nonempty*. Indeed, by Step 2, the sequences $(w^m : m \in \{0\} \cup \mathbb{N})$ and $(v^m : m \in \{0\} \cup \mathbb{N})$ are left J-Cauchy. By left J-sequentially completeness, both sequences are left J-convergent in X; that is, $S^{L-J}_{(w^m : m \in \mathbb{N})} \neq \varnothing$ and $S^{L-J}_{(v^m : m \in \mathbb{N})} \neq \varnothing$.

Step 4. We can show that

$$\begin{aligned}
&\left\{\lim_{m \to \infty} p\left(w, w_m\right) = 0\right\} \quad \forall w \in S^{L-J}_{(w^m : m \in \mathbb{N})},\\
&\left\{\lim_{m \to \infty} p\left(v, v_m\right) = 0\right\} \quad \forall v \in S^{L-J}_{(v^m : m \in \mathbb{N})}.
\end{aligned} \tag{40}$$

Indeed, by Step 3, $S^{L-J}_{(w^m : m \in \mathbb{N})} \neq \varnothing$ and $S^{L-J}_{(v^m : m \in \mathbb{N})} \neq \varnothing$. Let $w \in S^{L-J}_{(w^m : m \in \mathbb{N})}$ and $v \in S^{L-J}_{(v^m : m \in \mathbb{N})}$ be arbitrary and fixed. From Definition 8(iii), $\lim^{L-J}_{m \to \infty} w^m = w$ and $\lim^{L-J}_{m \to \infty} v^m = v$, which by Definition 8(ii) gives $\lim_{m \to \infty} J(w, w^m) = 0$, and $\lim_{m \to \infty} J(v, v^m) = 0$. Hence, if we define the sequences $(z^m = w : m \in \{0\} \cup \mathbb{N})$ and $(y^m = v : m \in \{0\} \cup \mathbb{N})$, we obtain

$$\lim_{m \to \infty} J\left(z^m, w^m\right) = 0, \tag{41}$$

$$\lim_{m \to \infty} J\left(y^m, v^m\right) = 0. \tag{42}$$

In consequence, by (28) and (41) we have that (8) hold. Next by $(J2)$ we obtain that

$$\lim_{m \to \infty} p\left(z^m, w^m\right) = 0. \tag{43}$$

Similarly, by (29) and (42) and $(J2)$ we obtain that

$$\lim_{m \to \infty} p\left(y^m, v^m\right) = 0. \tag{44}$$

Next, by (43), (44), and definition of sequences $(z^m = w : m \in \{0\} \cup \mathbb{N})$ and $(y^m = v : m \in \{0\} \cup \mathbb{N})$ and from arbitrariness $w \in S^{L-J}_{(w^m : m \in \mathbb{N})}$ and $v \in S^{L-J}_{(v^m : m \in \mathbb{N})}$ we obtain that (40) hold.

Step 5. We can show that the there exists a best proximity point; that is, there exists $w_0 \in A$ such that $\inf\{p(w_0, z) : z \in T(w_0)\} = \operatorname{dist}_p(A, B)$. Indeed, if we denote $W_0 = S^{L-J}_{(w^m : m \in \mathbb{N})}$ and $V_0 = S^{L-J}_{(v^m : m \in \mathbb{N})}$, then, by Step 4, $\{\lim_{m \to \infty} p(w, w^m) = 0\}\ \forall w \in W_0$ and $\{\lim_{m \to \infty} p(v, v^m) = 0\}\ \forall v \in V_0$. Now, since A and B are left quasiclosed (we recall that $\{w^m \in A \wedge v^m \in B\}\ \forall m \in \{0\} \cup \mathbb{N}$), thus $W_0 \subset A$ and $V_0 \subset B$. Finally, since by (24) we have $\{v^m \in T(w^m)\}\ \forall m \in \{0\} \cup \mathbb{N}$, and since T is left semiquasiclosed, we have

$$\{v \in T(w)\} \quad \exists v \in V_0,\ \exists w \in W_0. \tag{45}$$

Next, since $W_0 \subset A$, $V_0 \subset B$ and $T(A) \subset B$, by (45) we have $T(w) \subset B$ and

$$\begin{aligned}\operatorname{dist}_p(A,B) &= \inf\{p(a,b) : a \in A \wedge b \in B\} \\ &\leqslant p(w,B) \leqslant p(w,T(w)) \\ &= \inf\{p(w,z) : z \in T(w)\} \leqslant p(w,v).\end{aligned} \tag{46}$$

We know that $\lim_{m\to\infty} p(w,w^m) = 0$ and $\lim_{m\to\infty} p(v,v^m) = 0$. Moreover by (25) we get $\{J(w^m, v^{m-1}) = \operatorname{dist}_p(A,B)\}$ $\forall m \in \mathbb{N}$. Thus, since the map J is associated with the pair (A,B), then by Definition 9(ii), we conclude that

$$\max\{p(w,v), p(v,w)\} = \operatorname{dist}_p(A,B). \tag{47}$$

Finally, (46) and (47), we obtain

$$\begin{aligned}\operatorname{dist}_p(A,B) &\leqslant \inf\{p(w,z) : z \in T(w)\} \leqslant p(w,v) \\ &\leqslant \max\{p(w,v), p(v,w)\} = \operatorname{dist}_p(A,B),\end{aligned} \tag{48}$$

and hence

$$\begin{aligned}\inf\{p(w,z) : z \in T(w)\} &= \max\{p(w,v), p(v,w)\} \\ &= \operatorname{dist}_p(A,B);\end{aligned} \tag{49}$$

that is, w is a best proximity point of the mapping T.

Part II. We assume that (X,p) is a quasi-pseudometric space and $J : X\times X \to [0,\infty)$ is a right generalized pseudodistance on X, such that (X,p) is a right J-sequentially complete quasi-pseudometric space. Then proof is analogous as in Part I. □

Remark 12. It is worth noticing that, (a) in assumption of Theorem 11, the space (X,p) does not need to be left (right) sequentially complete. Consequently if, in particular, we put $p = d$ and we consider usual metric space, then in Theorem 11 the assumption about sequential completeness will be not necessary. (b) The class of set-valued non-self-mapping J-contractions of Nadler type is wider than the class of set-valued non-self-mapping contractions of Nadler type. (c) The class of set-valued non-self-mapping narrower J-contractions of Nadler type is wider than the class of set-valued non-self-mapping J-contractions of Nadler type.

Remark 13. It is worth noticing that, in a metric space X, a point $x \in A$ is said to be a best proximity point of a mapping $T : A \to B$ if $d(x,Tx) = \operatorname{dist}(A,B)$, where A, B are nonempty subsets of X. If $A = B$, then $\operatorname{dist}(A,B) = 0$ and a best proximity point reduces to a fixed point of a self-mapping. In our theorem, let (X,p) be a Hausdorff left (right) sequentially complete quasi-pseudometric space, and let $J : X\times X \to [0,\infty)$ be a left (right) quasi-generalized pseudodistance on X. Let (A,B) be a pair of nonempty subset of X with $A_0^J \neq \emptyset$ and such that (A,B) has the WP^J-property, J is associated with (A,B), and $A = B$. Then if $T : A \to 2^A$ is a semiquasiclosed set-valued non-self-mapping narrower contraction of Nadler type and $T(x)$ is bounded and closed in $B = A$ for all $x \in A$, and $T(x) \subset B_0^J$ for each $x \in A_0^J$, then we have that T has a fixed point in A. Indeed, it is consequence of the proof of Theorem 11. More precisely, by (45) we have $v \in T(w)$. Moreover by (49) we have $p(v,w) = 0$ and $p(w,v) = 0$. Since (X,p) is a Hausdorff space, we conclude that $v = w$, so $v \in T(v)$, and consequently v is a fixed point of T.

Next results are straightforward consequences of Theorem 11.

Corollary 14. *Let (X,p) be a Hausdorff left (right) J-sequentially complete quasi-pseudometric space, where $J : X\times X \to [0,\infty)$ is a left (right) quasi-generalized pseudodistance on X. Let (A,B) be a pair of nonempty subset of X with $A_0^J \neq \emptyset$ and such that (A,B) has the WP^J-property and J is associated with (A,B). Let $T : A \to B$ be a continuous single-valued narrower contraction of Banach type; that is,*

$$\begin{aligned}&\{J(T(x),T(y)) \leqslant \lambda J(x,y)\} \\ &\qquad \exists 0 \leqslant \lambda < 1,\ \forall x,y \in A_0^J.\end{aligned} \tag{50}$$

If $T(A_0^J) \subset B_0^J$, then T has a best proximity point in A.

Now we give some examples which illustrate the main results of the paper.

Example 15. Let (X,d) be a metric space, where $X = \mathbb{R}$, $d(x,y) = |x-y|$, $x,y \in X$. Let (A,B) be a pair of subset X, where $A = [4,5]$ and $B = [2,3] \cup [6,7]$. Let $E = [2,3] \cup [4, 4(1/2)]$ and let $J : X\times X \to [0,\infty)$ be defined by the formula

$$J(x,y) = \begin{cases} d(x,y) & \text{if } E\cap\{x,y\} = \{x,y\}, \\ 9 & \text{if } E\cap\{x,y\} \neq \{x,y\}, \end{cases} \qquad x,y \in X. \tag{51}$$

The map J is a generalized pseudodistance on X (see Example 4.2 in [13]). It is clear that J is associated with the pair (A,B). Assume that $T : A \to 2^B$ is of the form

$$T(x) = \begin{cases} [-2x+11, 3] \cup [6, 2x-2] & \text{if } x \in \left[4, 4\frac{1}{2}\right), \\ [2x-7, 3] \cup [6, -2x+16] & \text{if } x \in \left[4\frac{1}{2}, 5\right]. \end{cases} \tag{52}$$

(I) We show that the pair (A,B) has the WP^J-property.

Indeed, we observe that $\operatorname{dist}(A,B) = 1$ and

$$\begin{aligned}A_0^J &= \{x \in A : \text{there exists } u \in B \text{ such that } J(x,u) \\ &\quad = \operatorname{dist}(A,B)\} = \{4,5\}, \\ B_0^J &= \{u \in B : \text{there exists } x \in B \text{ such that } J(x,u) \\ &\quad = \operatorname{dist}(A,B)\} = \{3,6\}.\end{aligned} \tag{53}$$

Hence, it is easy to verify that the pair (A,B) has the weak WP^J-property.

(II) We see that A is complete and by (52) we have $T(A_0^J) = \{3, 6\} \subset B_0^J$.

(III) We see that T is a set-valued non-self-mapping narrower J-contraction of Nadler type; that is,

$$\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\} \quad \exists 0 \leqslant \lambda < 1, \ \forall x, y \in A_0^J. \tag{54}$$

Indeed, let $x, y \in A_0^J$ be arbitrary and fixed. Then by (52), $T(x) = T(y) = \{5, 6\} \subset E$, which, by (51), gives

$$H_J(T(x), T(y)) = H_J(\{5, 6\}, \{5, 6\}) = 0 \leqslant \lambda d(x, y) = \lambda J(x, y). \tag{55}$$

In consequence the map T is a set-valued non-self-mapping narrower J-contraction of Nadler type.

(V) We see that there exists a best proximity point of T.

Indeed, for $z = 4$ we have $d(z, T(z)) = d(4, \{3, 6\}) = 1 = \operatorname{dist}(A, B)$ and for $z = 5$ we have $d(z, T(z)) = d(5, \{3, 6\}) = 1 = \operatorname{dist}(A, B)$.

Now, we will compare our result with another result for J-generalized pseudodistance in b-metric space (with $s \leqslant 1$) [14]. For the reader's convenience, we formulate this result in metric spaces (with $s = 1$).

Theorem 16 (see [14]). *Let X be a complete metric space and let the map $J : X \times X \to [0, \infty)$ be a b-generalized pseudodistance on X. Let (A, B) be a pair of nonempty closed subsets of X with $A_0^J \neq \emptyset$ and such that (A, B) has the P^J-property and J is associated with (A, B). Let $T : A \to 2^B$ be a closed set-valued non-self-mapping J-contraction of Nadler type. That is,*

$$\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\} \quad \exists_{0 \leqslant \lambda < 1}, \ \forall_{x, y \in A}. \tag{56}$$

If $T(x)$ is bounded and closed in B for all $x \in A$, and $T(x) \subset B_0$ for each $x \in A_0$, then T has a best proximity point in A.

Remark 17. Let X, A, B, T, E, and J be as in Example 15.

(I) We see that the map T is not a set-valued non-self-mapping J-contraction of Nadler type.

Indeed, we suppose that for T the following condition holds:

$$\{H_J(T(x), T(y)) \leqslant \lambda_0 J(x, y)\} \quad \exists_{0 \leqslant \lambda_0 < 1}, \ \forall_{x, y \in A}. \tag{57}$$

In particular, for $x_0 = 4$ and $y_0 = 4$, by (51) we have $J(x_0, y_0) = 9$ and $H_J(T(x_0), T(y_0)) = 9$ (since $7 \in T(y_0)$ and $7 \notin E$). Hence, by (57) we get

$$9 = H_J(T(x_0), T(y_0)) \leqslant \lambda_0 J(x_0, y_0) = \lambda_0 \cdot 9 < 9, \tag{58}$$

which is absurd.

(II) We show that the pair (A, B) does not have the P^J-property.

Indeed, we observe that $\operatorname{dist}(A, B) = 1$ and

$$\begin{aligned} A_0^J &= \{x \in A : \text{there exists } u \in B \text{ such that } J(x, u) \\ &= d(A, B)\} = \{4, 5\}, \\ B_0^J &= \{u \in B : \text{there exists } x \in B \text{ such that } J(x, u) \\ &= d(A, B)\} = \{3, 6\}. \end{aligned} \tag{59}$$

Hence, it is easy to verify that, for the pairs $(x_1, y_1) = (4, 3)$ and $(x_2, y_2) = (5, 6)$, by (51) we have $J(x_1, y_1) = d(x_1, y_1) = 1$ and $J(x_2, y_2) = d(x_2, y_2) = 1$, but $J(x_1, x_2) \neq J(y_1, y_2)$. Therefore, the pair (A, B) does not have the P^J-property.

Now we give the examples which illustrate the main results of the paper in case when X is quasi-pseudometric space.

Example 18. Let $X = [0, 1] \subset \mathbb{R}$, $A = [0, 1/8] \cup [5/8, 6/8] \subset X$, $B = [2/8, 3/8] \cup [7/8, 1] \subset X$ and let $F = \{1/2^n : n \in \mathbb{N}\}$. Let $p : X \times X \to [0, \infty)$ be given by the formula

$$p(x, y) = \begin{cases} |x - y| + 1 & \text{if } x \notin F \wedge y \in F, \\ |x - y| & \text{if } x \in F \vee y \notin F, \end{cases} \quad x, y \in X. \tag{60}$$

Then (X, p) is a noncomplete quasi-pseudometric space (for details see Examples 6.1–6.3 in [12]). Let $E = [1/8, 1]$ and $J : X \times X \to [0, \infty)$ be defined by the formula

$$J(x, y) = \begin{cases} p(x, y) & \text{if } E \cap \{x, y\} = \{x, y\}, \\ 4 & \text{if } E \cap \{x, y\} \neq \{x, y\}, \end{cases} \quad x, y \in X. \tag{61}$$

The map J is a generalized pseudodistance and X is a Hausdorff left (right) J-sequentially complete quasi-pseudometric space (see Examples 6.1–6.4 in [12]). It is clear that J is also associated with the pair (A, B). Assume that $T : A \to 2^B$ is of the form

$$T(x) = \begin{cases} \left\{-8x^2 + \frac{3}{8}, -8x^2 + 1\right\} & \text{if } x \in \left[0, \frac{1}{8}\right] \\ \left\{-8x^2 + 10x - \frac{22}{8}, -8x^2 + 10x - \frac{17}{8}\right\} & \text{if } x \in \left[\frac{5}{8}, \frac{6}{8}\right]. \end{cases} \tag{62}$$

(I) We show that the pair (A, B) has the WP^J-property.

Indeed, we observe that $\operatorname{dist}_p(A, B) = 1/8$ and

$$\begin{aligned} A_0^J &= \{x \in A : \text{there exists } u \in B \text{ such that } J(x, u) \\ &= \operatorname{dist}(A, B)\} = \left\{\frac{1}{8}, \frac{6}{8}\right\}, \\ B_0^J &= \{u \in B : \text{there exists } x \in B \text{ such that } J(x, u) \\ &= \operatorname{dist}(A, B)\} = \left\{\frac{2}{8}, \frac{7}{8}\right\}. \end{aligned} \tag{63}$$

Hence, it is easy to verify that the pair (A, B) has the weak WP^J-property. Indeed, the assumption of definition of WP^J-property is satisfied only in the two following cases:

(1) if $x_1 = 1/8$, $y_1 = 2/8$, $x_2 = 6/8$, and $y_2 = 7/8$, and then, by (61) and (60) we obtain

$$\begin{aligned} J(x_1, x_2) &= J\left(\frac{1}{8}, \frac{6}{8}\right) = \frac{5}{8} \leqslant \frac{5}{8} = J\left(\frac{2}{8}, \frac{7}{8}\right) \\ &= J(y_1, y_2); \end{aligned} \tag{64}$$

(2) if $x_1 = 6/8$, $y_1 = 7/8$, $x_2 = 1/8$ and $y_2 = 2/8$, and then, by (61) and (60) we obtain

$$\begin{aligned} J(x_1, x_2) &= J\left(\frac{6}{8}, \frac{1}{8}\right) = \left|\frac{6}{8} - \frac{1}{8}\right| + 1 = 1 + \frac{5}{8} \leqslant 1 + \frac{5}{8} \\ &= \left|\frac{7}{8} - \frac{2}{8}\right| + 1 = J\left(\frac{7}{8}, \frac{2}{8}\right) = J(y_1, y_2). \end{aligned} \tag{65}$$

(II) We see that A is complete and by (62) we have $T(A_0^J) \subset B_0^J$.

(III) We see that T is a set-valued non-self-mapping narrower J-contraction of Nadler type; that is,

$$\{H_J(T(x), T(y)) \leqslant \lambda J(x, y)\} \quad \exists_{0 \leqslant \lambda < 1}, \ \forall_{x, y \in A_0^J}. \tag{66}$$

Indeed, let $x, y \in A_0^J$ be arbitrary and fixed. Then by (52), $T(x) = T(y) = \{2/8, 7/8\} \subset E$, which, by (61), gives

$$\begin{aligned} H_J(T(x), T(y)) &= H_J\left(\left\{\frac{2}{8}, \frac{7}{8}\right\}, \left\{\frac{2}{8}, \frac{7}{8}\right\}\right) = 0 \\ &\leqslant \lambda p(x, y) = \lambda J(x, y). \end{aligned} \tag{67}$$

In consequence the map T is a set-valued non-self-mapping narrower J-contraction of Nadler type. Moreover, by (60), (61), and Definition 10(ii), we obtain that T is semi-quasiclosed.

(V) We see that there exists a best proximity point of T.

Indeed, by (61), (60), and (62), for $z = 1/8$ we have $p(z, T(z)) = p(1/8, \{2/8, 7/8\}) = p(1/8, 2/8) = 1/8 = \text{dist}_p(A, B)$ and for $z = 6/8$ we have $p(z, T(z)) = p(6/8, \{2/8, 7/8\}) = p(6/8, 7/8) = 1/8 = \text{dist}_p(A, B)$.

Conflict of Interests

The author declares that they have no conflict of interests.

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