ŁOJASIEWICZ EXPONENT OF SEMIALGEBRAIC SETS AND MAPPINGS

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SUMMARY OF THE DOCTORAL DISSERTATION

Łojasiewicz inequalities emerged in the late 1950s as the main tool in the division of distributions by real polynomials (L. Hörmander, 1958) and by real analytic functions (S. Łojasiewicz, 1958). Since then they have turned out to be of use in numerous branches of mathematics, including differential equations, dynamical systems and singularity theory (see for instance papers of K. Kurdyka, T. Mostowski, A. Parusiński and L. Simon). Quantitative versions of these inequalities, involving e.g. computing or estimating the relevant exponents, are of importance in real and complex algebraic geometry (see papers of B. Teissier in the complex case and S. Spodzieja in the real case). Recently a strong demand for explicit estimates of the Łojasiewicz exponent comes from optimization theory (see for instance papers of K. Kurdyka, S. Spodzieja and M. Schweighofer). Among others, S. Ji, J. Kollár, B. Shiffman, J. Kollár, J. Chądzyński, T. Krasiński, P. Tworzewski, A. Płoski, Z. Jelonek, E. Cygan also dealt with the Łojasiewicz exponent in the complex case and D. D'Acunto, K. Kurdyka, A. Gabrielov, J. Gwoździewicz in the real case.

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. Assume that $0 \in X$ is an accumulation point of X and $f, g : X \to \mathbb{R}$ are two continuous semialgebraic functions such that $0 \in g^{-1}(0) \subset f^{-1}(0) \neq X$. Then there are positive constants C, α, ε such that the following *Łojasiewicz inequality* holds:

$$|g(x)| \geq C|f(x)|^{\alpha} \quad \text{for} \quad x \in X,\ |x| < \varepsilon. \tag{1}$$

The infimum of the exponents α in (1) is called the *Łojasiewicz exponent of the pair* (f, g) *on the set* X *at* 0 and is denoted by $\mathcal{L}_0(f, g|X)$. It is known that $\mathcal{L}_0(f, g|X)$ is a rational number. Moreover, inequality (1) holds actually with $\alpha = \mathcal{L}_0(f, g|X)$ for some $\varepsilon, C > 0$.

An asymptotic estimate for $\mathcal{L}_0(f, g|X)$ was obtained by Solernó (1991). In general, his estimate is of the form

$$\mathcal{L}_0(f, g|X) \leq D^{M^{ca}}, \tag{S}$$

where D is a bound for the degrees of the polynomials involved in a description of f, g and X; M is the number of variables in these formulas (so in general $M \geq n$); a is the maximum number of alternating blocs of quantifiers in these formulas; and c is an (unspecified) universal constant. The estimate (S) was obtained from the effective Tarski-Seidenberg theorem.

The main result of the doctoral thesis is Theorem 4.2.1. In this theorem we give an estimate from above of the Łojasiewicz exponent for a pair of continuous semialgebraic functions f and g on a set $\Omega = \{x \in \mathbb{R}^n : |x| \leq 1\}$ at 0 in terms of degrees of polynomials describing these functions (P_k, Q_k, $k = 1, \ldots, l$) and some basic semialgebraic sets $X_1, \ldots, X_l \subset \mathbb{R}^n$ (described by polynomial equations $g_{k,i}$, $k = 1, \ldots, l$, $i = 1, \ldots, n-b_k$, where $b_k = \dim X_k$ and some polynomial inequalities) such that $\Omega = X_1 \cup \ldots \cup X_l$ and f, g are smooth on any X_i. More precisely, we have the following

Theorem 1. *Let* $d = \max\{\deg P_k, \deg Q_k, \deg g_{k,i} : k = 1, \ldots, l, i = 1, \ldots, n - b_k\}$ *and* $\alpha = d^{4n+1}$. *Then there exists a positive constant* C *such that*

$$|g(x)| \geq C\,|f(x)|^{\alpha} \quad \textit{for } x \in \Omega.$$

In particular,

$$\mathcal{L}_0(f, g|\Omega) \leq \alpha.$$

Asymptotically, Theorem 1 coincides with the Solernó estimate (S), however our estimate is explicit.

Chapters 1 and 2 have an auxiliary character. In Chapter 1 we collect definitions and known theorems used in the dissertation.

A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is called overdetermined if $m > n$. In the second chapter, we prove that the calculations of the both local and global Łojasiewicz exponent of a real overdetermined polynomial mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ can be reduced to the case $m = n$ by composing F with the generic linear mapping $\mathbb{R}^m \to \mathbb{R}^n$ (we write "the generic $x \in A$" instead of "there exists an algebraic set V such that $A \setminus V$ is a dense subset of A and $x \in A \setminus V$").

Let us recall the notions of Łojasiewicz exponent. By $F : (X, a) \to (\mathbb{R}^m, 0)$, where $a \in X \subset \mathbb{R}^n$, we denote a mapping from a neighbourhood $U \subset X$ of the point a to $\mathbb{R}^m$ such that $F(a) = 0$. If $F : (\mathbb{R}^n, a) \to (\mathbb{R}^m, 0)$ is an analytic or continous semialgebraic mapping, then there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds:

$$(\text{Ł}) \qquad |F(x)| \geq C \operatorname{dist}(x, F^{-1}(0))^{\eta} \quad \text{for} \quad x \in X,\ |x - a| < \varepsilon,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$ and $\operatorname{dist}(x, V)$ is the distance from $x \in \mathbb{R}^n$ to the set V ($\operatorname{dist}(x, V) = 1$ if $V = \emptyset$). The smallest exponent η in (Ł) is called the *Łojasiewicz exponent* of F *at* a and is denoted by $\mathcal{L}_a(F|X)$. It is known that $\mathcal{L}_a(F|X)$ is a rational number and (Ł) holds for any $\eta \geq \mathcal{L}_a(F|X)$ and some $C, \varepsilon > 0$. The exponent $\mathcal{L}_a(F|X)$ is an important invariant and tool in the singularity theory.

By the *Łojasiewicz exponent at infinity of a mapping* $F : X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$ is a closed unbounded set, we mean the supremum of exponents ν in the following Łojasiewicz inequality:

$$|F(x)| \geq C|x|^{\nu} \quad \text{for} \quad x \in X,\ |x| \geq R$$

for some positive constants C, R; we denote it by $\mathcal{L}_\infty(F|X)$. The Łojasiewicz exponent at infinity of a mapping has been considered by many authors in the contexts of effective Nullstellensatz and properness of mappings.

Let $X_1, \ldots, X_k \subset \mathbb{R}^n$ be closed semialgebraic sets such that $0 \in X_1 \cap \cdots \cap X_k$. Then there are a neighbourhood $U \subset \mathbb{R}^n$ of 0 and positive constants C, α such that the following inequality holds

$$(2) \qquad \operatorname{dist}(x, X_1) + \cdots + \operatorname{dist}(x, X_k) \geq C \operatorname{dist}(x, X_1 \cap \cdots \cap X_k)^{\alpha} \text{ for } x \in U.$$

The exponent α satisfying (2) for some U and $C > 0$ is called a *regular separation exponent of* $X_1, \ldots, X_k$ *at* 0. The infimum of all regular separation exponents of $X_1, \ldots, X_k$ at 0 is called the *Łojasiewicz exponent of* $X_1 \ldots, X_k$ *at* 0 and denoted by $\mathcal{L}_0(X_1, \ldots, X_k)$.

In Chapter 3 we give an estimate of the Łojasiewicz exponent for the regular separation of two closed semialgebraic sets and for continuous semialgebraic mappings on closed semialgebraic sets in local and global cases.

Let $X \subset \mathbb{R}^n$ be a closed semialgebraic set. It is known that X has a decomposition

$$X = X_1 \cup \cdots \cup X_k \tag{3}$$

into the union of closed basic semialgebraic sets

$$X_i = \{x \in \mathbb{R}^n : g_{i,1}(x) \geq 0, \ldots, g_{i,r_i}(x) \geq 0,\ h_{i,1}(x) = \cdots = h_{i,l_i}(x) = 0\}, \tag{4}$$

$i = 1, \ldots k$, where $g_{i,1}, \ldots, g_{i,r_i}, h_{i,1}, \ldots, h_{i,l_i} \in \mathbb{R}[x_1, \ldots, x_N]$. Assume that r_i is the smallest possible number of the inequalities $g_{i,j}(x) \geq 0$ in the definition of X_i, for $i = 1, \ldots, k$. Denote by $r(X)$ the minimum of $\max\{r_1, \ldots, r_k\}$ over all decompositions (3) into unions of sets of the form (4).

Denote by $\kappa(X)$ the minimum of the numbers

$$\max\{\deg g_{1,1}, \ldots, \deg g_{k,r_k}, \deg h_{1,1}, \ldots, \deg h_{k,l_k}\}$$

over all decompositions (3) of X into the union of sets of the form (4), provided $r_i \leq r(X)$.

The main results of Chapter 3 are the following theorems.

Theorem 2. *Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic sets such that $0 \in X \cap Y$. Put $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of 0 and a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \operatorname{dist}(x, X \cap Y)^{d(6d-3)^{n+r-1}} \quad \text{for } x \in U.$$

If additionally, 0 is an isolated point of $X \cap Y$, then for some neighbourhood $U \subset \mathbb{R}^N$ of 0 and some positive constant C,

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C|x|^{\frac{(2d-1)^{n+r}+1}{2}} \quad \text{for } x \in U.$$

Theorem 3. *Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic sets. Put $r = r(X) + r(Y)$ and $d = \max\{\kappa(X), \kappa(Y)\}$. Then there exists a positive constant C such that*

$$\operatorname{dist}(x, X) + \operatorname{dist}(x, Y) \geq C \left(\frac{\operatorname{dist}(x, X \cap Y)}{1 + |x|^d} \right)^{d(6d-3)^{n+r-1}} \quad \text{for } x \in \mathbb{R}^n.$$

From Theorem 2 we obtain an estimate of the Łojasiewicz exponent of semialgebraic mapping at zero. Namely, we have

Corollary 4. *Let $F : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set, and suppose $0 \in X$ and $F(0) = 0$. Set $r = r(X) + r(\operatorname{graph} F)$ and $d = \max\{\kappa(X), \kappa(\operatorname{graph} F)\}$. Then*

$$\mathcal{L}_0^{\mathbb{R}}(F|X) \leq d(6d-3)^{N+m+r-1}.$$

If additionally, 0 is an isolated zero of F, then

$$\mathcal{L}_0^{\mathbb{R}}(F|X) \leq \frac{(2d-1)^{N+m+r}+1}{2}.$$

Theorem 3 gives the following estimate of the Łojasiewicz exponent at infinity of semialgebraic mapping.

Corollary 5. *Let $F : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping, where $X \subset \mathbb{R}^n$ is a closed semialgebraic set. If $d = \max\{\kappa(X), \kappa(Y)\}$ and $r = r(X)+r(Y)$, where $Y = \operatorname{graph} F$, then there exists a positive constant C such that*

$$|F(x)| \geq C \left(\frac{\operatorname{dist}(x, F^{-1}(0) \cap X)}{1 + |x|^d} \right)^{d(6d-3)^{N+m+r-1}} \quad \text{for } x \in X.$$

In particular, if the set X is unbounded and $F^{-1}(0) \cap X$ is compact, then

$$\mathcal{L}_{\infty}^{\mathbb{R}}(F|X) \geq (1-d)d(6d-3)^{N+m+r-1}.$$

At the end of the thesis, in Chapter 5, we give some corollaries from the above results for regular separation exponent of a finite number of algebraic and semialgebraic sets at 0. Among other things, we prove the following

Theorem 6. *Let $X_1, \dots, X_k \subset \mathbb{R}^n$ be closed semialgebraic sets such that $0 \in X_1 \cap \dots \cap X_k$. Put $r = r(X_1) + \dots + r(X_k)$ and $d = \max\{\kappa(X_1), \dots, \kappa(X_k)\}$. Then there exist positive constants C, ε such that the following inequality holds*

$$\sum_{i=1}^{k} \operatorname{dist}(x, X_i) \geq C \operatorname{dist}(x, \bigcap_{i=1}^{k} X_i)^{d(6d-3)^{kn+r-1}} \quad \text{for} \quad x \in \mathbb{R}^n,\ |x| < \varepsilon.$$