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ŁOJASIEWICZ EXPONENT OF OVERDETERMINED SEMIALGEBRAIC MAPPINGS

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Abstract. We prove that both local and global Łojasiewicz exponent of a continuous overdetermined semialgebraic mapping $F : X \to \mathbb{R}^m$ on a closed semialgebraic set $X \subset \mathbb{R}^n$ (i.e. $m > \dim X$) are equal to the Łojasiewicz exponent of the composition $L \circ F : X \to \mathbb{R}^k$ for the generic linear mapping $L : \mathbb{R}^m \to \mathbb{R}^k$, where $k = \dim X$.

1. Introduction

Łojasiewicz inequalities are an important and useful tool in differential equations, singularity theory and optimization (see for instance [12, 13] in the local case and [18, 19] at infinity). In these considerations, estimations of the local and global Łojasiewicz exponents play a central role (see for instance [11, 14, 17, 18, 25] in the local case and [9, 16] at infinity). In the complex case, essential estimations of the Łojasiewicz exponent at infinity of a polynomial mapping $F = (f_1, \ldots, f_m) : \mathbb{C}^N \to \mathbb{C}^m$ (see Section 2.3) denoted by $\mathcal{L}^{\mathbb{C}}_{\infty}(F)$, was obtained by J. Chądzyński [4], J. Kollár [10], E. Cygan, T. Krasiński and P. Tworzewski [6] and E. Cygan [5].

We recall the estimation of Cygan, Krasiński and Tworzewski. Let $\deg f_j = d_j$, $j = 1, \ldots, m$, $d_1 \geq \ldots \geq d_m > 0$ and let

$$B(d_1, \ldots, d_m; k) = \begin{cases} d_1 \cdots d_m & \text{for } m \leqslant k, \\ d_1 \cdots d_{k-1} d_m & \text{for } m > k. \end{cases}$$

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Then for arbitrary $m \geq N$, under the assumption $\#F^{-1}(0) < \infty$, we have

$$\text{(CKT)} \qquad \mathcal{L}^{\mathbb{C}}_{\infty}(F) \geq d_m - B(d_1, \dots, d_m; N) + \sum_{b \in F^{-1}(0)} \mu_b(F),$$

where $\#A$ denotes the cardinality of a set A, and $\mu_b(F)$ is the intersection multiplicity (in general improper) in the sense of R. Achilles, P. Tworzewski and T. Winiarski of graph F and $\mathbb{C}^n \times \{0\}$ at the point $(b, 0)$ (see [1]). A generalization of (CKT) for regular mappings was obtained by Z. Jelonek [7, 8].

In the proof of (CKT) the following theorem was used (see [20, Corollary 1], in Polish).

Theorem 1.1. *Let $m > N > 1$, and let $\#F^{-1}(0) < \infty$. Then there exists a polynomial mapping $G = (g_1, \dots, g_N) : \mathbb{C}^N \to \mathbb{C}^N$ of the form*

$$g_i = f_i + \sum_{j=n}^{m-1} \alpha_{j,i} f_j \quad \text{for } i = 1, ..., N-1, \qquad g_N = f_m,$$

where $\alpha_{j,i} \in \mathbb{C}$, such that

$$\#G^{-1}(0) < \infty,$$

and

$$\mathcal{L}^{\mathbb{C}}_{\infty}(F) \geq \mathcal{L}^{\mathbb{C}}_{\infty}(G).$$

The above theorem has been generalized for complex polynomial mappings in [21, Theorem 2.1] and in the local case in [22, Theorem 2.1], and for real polynomial mappings in [24, Theorems 1–3] both at infinity and in the local case.

The purpose of the article is a generalization of the above fact to continuous semialgebraic mappings. More precisely, we prove that both: local and global Łojasiewicz exponent of an overdetermined semialgebraic mapping $F : X \to \mathbb{R}^m$ on a closed semialgebraic set $X \subset \mathbb{R}^N$ (i.e. $m > \dim X$) are equal to the Łojasiewicz exponent of the composition $L \circ F : X \to \mathbb{R}^k$ for the generic linear mapping $L : \mathbb{R}^m \to \mathbb{R}^k$, where $k = \dim X$ (see Theorems 2.2, and 2.3). For more detailed informations about semialgebraic sets and mappings, see for instance [2]. Moreover, we prove a version of the above fact for an analytic mapping with isolated zero (see Theorem 2.1).

A mapping $F : \mathbb{K}^N \to \mathbb{K}^m$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, is called *overdetermined* if $m > N$.

2. Results

2.1. **Notations.** Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By the dimension $\dim_{\mathbb{K},0} X$ at 0 of a set $X \subset \mathbb{K}^N$ we mean the infimum of the dimensions over $\mathbb{K}$ at 0 of local analytic sets $0 \in V \subset \mathbb{K}^N$ such that $X \cap U \subset V$ for some neighbourhood $U \subset \mathbb{K}^n$ of 0.

By the dimension $\dim_{\mathbb{R}} X$ of a set $X \subset \mathbb{R}^N$ we mean the infimum of dimensions of local analytic sets $V \subset \mathbb{R}^N$ such that $X \subset V$. In particular, if X is a semialgebraic set, $\dim_{\mathbb{R}} X$ is the infimum of dimensions of algebraic sets $V \subset \mathbb{R}^N$ such that $X \subset V$.

We will write "for the generic $x \in A$" instead of "there exists an algebraic set V such that $A \setminus V$ is a dense subset of A and for $x \in A \setminus V$".

By $\mathbf{L}^{\mathbb{K}}(m,k)$ we shall denote the set of all linear mappings $\mathbb{K}^m \to \mathbb{K}^k$ (we identify $\mathbb{K}^0$ with $\{0\}$). Let $m \geq k$. By $\Delta^{\mathbb{K}}(m,k)$ we denote the set of all linear mappings $L \in \mathbf{L}^{\mathbb{K}}(m,k)$ of the form $L = (L_1, ..., L_k)$,

$$L_i(y_1, ..., y_m) = y_i + \sum_{j=k+1}^{m} \alpha_{i,j} y_j, \qquad i = 1, ..., k,$$

where $\alpha_{i,j} \in \mathbb{K}$.

2.2. **The Łojasiewicz exponent at a point.** Let $X \subset \mathbb{K}^N$ be a closed subanalytic set. If $\mathbb{K} = \mathbb{C}$ we consider X as a subset of $\mathbb{R}^{2N}$. We will assume that the origin $0 \in \mathbb{K}^N$ belongs to X and it is an accumulation point of X. We denote by $F : (X, 0) \to (\mathbb{K}^m, 0)$ a mapping of a neighbourhood $U \subset X$ of the point $0 \in \mathbb{K}^N$ into $\mathbb{K}^m$ such that $F(0) = 0$, where the topology of X is induced from $\mathbb{K}^N$.

Let $F : (X, 0) \to (\mathbb{K}^m, 0)$ be a continuous subanalytic mapping, i.e. the graph of F is a closed subanalytic subset of $(X \cap U) \times \mathbb{K}^m$ for some neighbourhood $U \subset \mathbb{K}^N$ of the origin. If $\mathbb{K} = \mathbb{C}$, we consider $\mathbb{K}^N$ as $\mathbb{R}^{2N}$ and $\mathbb{K}^m$ as $\mathbb{R}^{2m}$. Then there are positive constants C, η, ε such that the following *Łojasiewicz inequality* holds:

$$(\text{Ł}_0) \qquad |F(x)| \geq C \operatorname{dist}(x, F^{-1}(0) \cap X)^{\eta} \quad \text{if} \quad x \in X, \quad |x| < \varepsilon,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{K}^n$, respectively in $\mathbb{K}^N$, and $\operatorname{dist}(x, V)$ is the distance of $x \in \mathbb{K}^N$ to the set $V \subset \mathbb{K}^N$ ($\operatorname{dist}(x, V) = 1$ if $V = \emptyset$). The smallest exponent η in (Ł$_0$) is called the *Łojasiewicz exponent* of F *on the set* X *at* 0 and is denoted by $\mathcal{L}_0^{\mathbb{K}}(F|X)$. If X contains a neighbourhood $U \subset \mathbb{K}^N$ of 0 we will call it the *Łojasiewicz exponent* of F *at* 0 and denote by $\mathcal{L}_0^{\mathbb{K}}(F)$. It is known that $\mathcal{L}_0^{\mathbb{K}}(F|X)$ is a rational number and (Ł$_0$) holds with any $\eta \geq \mathcal{L}_0^{\mathbb{K}}(F|S)$ and some positive constants C, ε, provided 0 is an accumulation point of $X \setminus F^{-1}(0)$ (see [3, 23]). If 0 is not an accumulation point of $X \setminus F^{-1}(0)$, we have $\mathcal{L}_0^{\mathbb{K}}(F|X) = 0$.

In Section 3 we will prove (cf [22, Theorem 2.1] and [24, Theorem 1])

Theorem 2.1. *Let $F = (f_1, \ldots, f_m) : (X, 0) \to (\mathbb{R}^m, 0)$ be an analytic mapping with isolated zero at the origin, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set and $0 \in X$. Let $\dim_{\mathbb{R},0} X = n$, and let $n \leq k \leq m$. Then for any $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ such that the origin is an isolated zero of $L \circ F|X$, we have*

$$(2.1) \qquad \mathcal{L}_0^{\mathbb{R}}(F|X) \leq \mathcal{L}_0^{\mathbb{R}}(L \circ F|X).$$

Moreover, for the generic $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ the origin is an isolated zero of $L \circ F|X$ and

$$(2.2) \qquad \mathcal{L}_0^{\mathbb{R}}(F|X) = \mathcal{L}_0^{\mathbb{R}}(L \circ F|X).$$

In particular, for the generic $L \in \Delta^{\mathbb{R}}(m,k)$ the origin is an isolated zero of $L \circ F|X$ and (2.2) holds.

The above theorem gives a method for reduction of the problem of calculating the Łojasiewicz exponent of overdetermined mappings to the case where the domain and codomain are equidimmensional. It is not clear to the authors whether the above statement is true if the origin is not isolated zero of f or the set X is subanalytic instead of semialgebraic.

If $F : X \to \mathbb{K}^m$ is a semialgebraic mapping then without any assumptions on the set of zeroes of F we will prove in Section 4 the following

Theorem 2.2. *Let $F : (X, 0) \to (\mathbb{K}^m, 0)$ be a continuous semialgebraic mapping, $X \subset \mathbb{K}^N$ be a closed semialgebraic set of dimension $\dim_{\mathbb{R},0} X = n$, and let $n \le k \le m$. Then for any $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ such that*

(2.3) $\quad F^{-1}(0) \cap U_L = (L \circ F)^{-1}(0) \cap U_L$ *for a neighbourhood* $U_L \subset X$ *of* 0

we have

$$\mathcal{L}_0^{\mathbb{K}}(F|X) \le \mathcal{L}_0^{\mathbb{K}}(L \circ F|X). \tag{2.4}$$

Moreover, for the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ the condition (2.3) *holds and*

$$\mathcal{L}_0^{\mathbb{K}}(F|X) = \mathcal{L}_0^{\mathbb{K}}(L \circ F|X). \tag{2.5}$$

In particular, for the generic $L \in \Delta^{\mathbb{K}}(m, k)$ the conditions (2.3) *and* (2.5) *hold.*

2.3. **The Łojasiewicz exponent at infinity.** The second aim of this article is to obtain a similar results as in the previous section but for the Łojasiewicz exponent at infinity.

By the *Łojasiewicz exponent at infinity of a mapping* $F : X \to \mathbb{K}^m$, where $X \subset \mathbb{K}^n$ is an unbounded set, we mean the supremum of the exponents ν in the following *Łojasiewicz inequality*:

$$|F(x)| \ge C|x|^\nu \quad \text{for} \quad x \in X, \quad |x| \ge R \tag{Ł$_\infty$}$$

for some positive constants C, R; we denote it by $\mathcal{L}_\infty^{\mathbb{K}}(F|X)$. If $X = \mathbb{K}^N$ we call the exponent $\mathcal{L}_\infty^{\mathbb{K}}(F|X)$ the *Łojasiewicz exponent at infinity* of F and denote by $\mathcal{L}_\infty^{\mathbb{K}}(F)$.

In Section 5 we will prove the following version of Theorem 2.1 for the Łojasiewicz exponent at infinity (cf [21, Theorem 2.1], [24, Theorem 3]).

Theorem 2.3. *Let $F = (f_1, \dots, f_m) : X \to \mathbb{R}^m$ be a continuous semialgebraic mapping having a compact set of zeros, where $X \subset \mathbb{R}^N$ is a closed semialgebraic set, $\dim X = n$, and let $n \le k \le m$. Then for any $L \in \mathbf{L}^{\mathbb{R}}(m, k)$ such that $(L \circ F)^{-1}(0) \cap X$ is compact, we have*

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) \ge \mathcal{L}_\infty^{\mathbb{R}}(L \circ F|X). \tag{2.6}$$

Moreover, for the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ the set $(L \circ F)^{-1}(0)$ is compact and

$$\mathcal{L}_\infty^{\mathbb{R}}(F|X) = \mathcal{L}_\infty^{\mathbb{R}}(L \circ F|X). \tag{2.7}$$

In particular, (2.7) *holds for the generic $L = (L_1, ..., L_k) \in \Delta^{\mathbb{R}}(m, k)$ and* $\deg f_j = \deg L_j \circ F$ *for $j = 1, \dots, k$, provided* $\deg f_1 \ge \dots \ge \deg f_m > 0$.

The above theorem gives a method of reduction of the problem of calculating the Łojasiewicz exponent at infinity of overdetermined semialgebraic mappings to the case where the dimensions of domain and codomain are equal.

3. Proof of Theorem 2.1

Let $k \in \mathbb{Z}$, $n \le k \le m$. Take a closed semialgebraic set $Z \subset \mathbb{R}^N$ of dimension $\dim_{\mathbb{R}} Z = n$, and let

$$\pi : Z \ni (x, y) \mapsto y \in \mathbb{R}^m.$$

Then the set $\pi(Z)$ is semialgebraic with $\dim_{\mathbb{R}} \pi(Z) \le n$. Denote by $Y \subset \mathbb{C}^m$ the complex Zariski closure of $\pi(Z)$. So, Y is an algebraic set of complex dimension $\dim_{\mathbb{C}} Y \le n$.

Assume that $0 \in Y$. Let $C_0(Y) \subset \mathbb{C}^m$ be the tangent cone to Y at 0 in the sense of Whitney [26, p. 510]. It is known that $C_0(Y)$ is an algebraic set and $\dim_{\mathbb{C}} C_0(Y) \le n$. So, we have

Lemma 3.1. *For the generic* $L \in \mathbf{L}^{\mathbb{K}}(m, k)$,

$$L^{-1}(0) \cap C_0(Y) \subset \{0\}.$$

In the proofs of Theorems 2.1, 2.2 and 2.3 we will need the following

Lemma 3.2. *If* $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ *satisfies* $L^{-1}(0) \cap C_0(Y) \subset \{0\}$, *then there exist* ε, C_1, $C_2 > 0$ *such that for* $z \in Z$, $|\pi(z)| < \varepsilon$ *we have*

$$C_1|\pi(z)| \le |L(\pi(z))| \le C_2|\pi(z)|. \tag{3.1}$$

Proof. It is obvious that for $C_2 = ||L||$ we obtain $|L(\pi(z))| \le C_2|\pi(z)|$ for $z \in Z$. This gives the right hand side inequality in (3.1).

Now, we show the left hand side inequality in (3.1). Assume to the contrary, that for any ε, $C_1 > 0$ there exists $z \in Z$ such that

$$C_1|\pi(z)| > |L(\pi(z))| \quad \text{and} \quad |\pi(z)| < \varepsilon.$$

In particular, for $\nu \in \mathbb{N}$, $C_1 = \frac{1}{\nu}$, $\varepsilon = \frac{1}{\nu}$ there exists $z_\nu \in Z$ such that

$$\frac{1}{\nu}|\pi(z_\nu)| > |L(\pi(z_\nu))| \quad \text{and} \quad |\pi(z_\nu)| < \frac{1}{\nu}.$$

Thus $|\pi(z_\nu)| > 0$ and

$$\frac{1}{\nu} > \frac{1}{|\pi(z_\nu)|}|L(\pi(z_\nu))| = \left| L\left(\frac{1}{|\pi(z_\nu)|}\pi(z_\nu)\right)\right|. \tag{3.2}$$

Let $\lambda_\nu = \frac{1}{|\pi(z_\nu)|}$ for $\nu \in \mathbb{N}$. Then $|\lambda_\nu \pi(z_\nu)| = 1$ so, by choosing subsequence, if necessary, we may assume that $\lambda_\nu \pi(z_\nu) \to v$ when $\nu \to \infty$, where $v \in \mathbb{C}^m$, $|v| = 1$ and $\pi(z_\nu) \to 0$ as $\nu \to \infty$, thus $v \in C_0(Y)$ and $v \ne 0$. Moreover, by (3.2), we have $L(v) = 0$. So $v \in L^{-1}(0) \cap C_0(Y) \subset \{0\}$. This contradicts the assumption and ends the proof. □

We will also need the following lemma (cf. [15, 22]). Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set such that $0 \in X$.

Lemma 3.3. *Let $F, G : (\mathbb{R}^N, 0) \to (\mathbb{R}^m, 0)$ be analytic mappings, such that $\mathrm{ord}_0(F - G) > \mathcal{L}_0^{\mathbb{R}}(F|X)$. If 0 is an isolated zero of $F|X$ then 0 is an isolated zero of $G|X$ and for some positive constants ε, C_1, C_2,*

$$C_1|F(x)| \le |G(x)| \le C_2|F(x)| \quad \text{for} \quad x \in X, \quad |x| < \varepsilon. \tag{3.3}$$

In particular, $\mathcal{L}_0^{\mathbb{R}}(F|X) = \mathcal{L}_0^{\mathbb{R}}(G|X)$.

Proof. Since F is a Lipschitz mapping in a neighbourhood of 0, then $1 \le \mathcal{L}_0^{\mathbb{R}}(F|X) < \infty$ and for some positive constants ε_0, C,

$$|F(x)| \ge C|x|^{\mathcal{L}_0^{\mathbb{R}}(F|X)} \quad \text{for} \quad x \in X, \quad |x| < \varepsilon_0. \tag{3.4}$$

From the assumption $\mathrm{ord}_0(F - G) > \mathcal{L}_0^{\mathbb{R}}(F|X)$ it follows that there exist $\eta \in \mathbb{R}$, $\eta > \mathcal{L}_0^{\mathbb{R}}(F|X)$ and $\varepsilon_1 > 0$ such that $||F(x)| - |G(x)|| \le |x|^\eta$ for $x \in X$, $|x| < \varepsilon_1$. Assume that (3.3) fails. Then for some sequence $x_\nu \in X$ such that $x_\nu \to 0$ as $\nu \to \infty$, we have

$$\frac{1}{\nu}|F(x_\nu)| > |G(x_\nu)| \quad \text{or} \quad \frac{1}{\nu}|G(x_\nu)| > |F(x_\nu)| \quad \text{for} \quad \nu \in \mathbb{N}.$$

So, in the both above cases, by (3.4) for $\nu \ge 2$, we have

$$\frac{C}{2}|x_\nu|^{\mathcal{L}_0^{\mathbb{R}}(F|X)} \le \frac{1}{2}|F(x_\nu)| < |F(x_\nu) - G(x_\nu)| \le |x_\nu|^\eta,$$

which is impossible. The last part of the assertion follows immediately from (3.3). □

Proof of Theorem 2.1. We prove the assertion (2.1) analogously as Theorem 2.1 in [22]. We will prove the second part of the assertion.

Let $G = (g_1, \ldots, g_m) : (\mathbb{R}^N, 0) \to (\mathbb{R}^m, 0)$ be a polynomial mapping such that $\mathrm{ord}_0^{\mathbb{R}}(F - G) > \mathcal{L}_0^{\mathbb{R}}(F|X)$. Obviously, such a mapping G does exist. By Lemma 3.3, $\mathcal{L}_0^{\mathbb{R}}(F|X) = \mathcal{L}_0^{\mathbb{R}}(G|X)$ and 0 is an isolated zero of $G|X$. Taking, if necessary, intersection of X with a ball B centered at zero, we may assume that $\dim_{\mathbb{R},0} X = \dim_{\mathbb{R}} X$. So, by Lemmas 3.1 and 3.2 for the generic $L \in \mathbf{L}^{\mathbb{R}}(m, k)$ we have that $L \circ G|X$ has an isolated zero at $0 \in \mathbb{R}^n$, $\mathcal{L}_0^{\mathbb{R}}(G|X) = \mathcal{L}_0^{\mathbb{R}}(L \circ G|X)$, and

$$\begin{aligned} \mathrm{ord}_0(L \circ G - L \circ F) &= \mathrm{ord}_0 L \circ (G - F) \ge \mathrm{ord}_0(G - F) \\ &> \mathcal{L}_0^{\mathbb{R}}(F|X) = \mathcal{L}_0^{\mathbb{R}}(G|X) = \mathcal{L}_0^{\mathbb{R}}(L \circ G|X), \end{aligned}$$

so, by Lemma 3.3, $\mathcal{L}_0^{\mathbb{R}}(L \circ F|X) = \mathcal{L}_0^{\mathbb{R}}(L \circ G|X) = \mathcal{L}_0^{\mathbb{R}}(F|X)$. This gives (2.2). The particular part of the assertion is proved analogously as in [22, Proposition 2.1]. □

4. Proof of Theorem 2.2

Let $X \subset \mathbb{R}^N$ be a closed semialgebraic set $\dim_{\mathbb{R}} X = n$, and let $0 \in X$. Taking, if necessary, intersection of X with a ball B centered at zero, we may assume that $\dim_{\mathbb{R},0} X = \dim_{\mathbb{R}} X$.

From [21, Proposition 1.1] we immediately obtain

Proposition 4.1. *Let $G = (g_1, ..., g_m) : X \to \mathbb{K}^m$ be a semialgebraic mapping, $g_j \neq 0$ for $j = 1, ..., m$, where $m \geq n \geq 1$, and let $k \in \mathbb{Z}$, $n \leq k \leq m$.*

(i) For the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$,

$$\#[(L \circ G)^{-1}(0) \setminus G^{-1}(0)] < \infty. \tag{4.1}$$

(ii) For the generic $L \in \Delta^{\mathbb{K}}(m, k)$,

$$\#[(L \circ G)^{-1}(0) \setminus G^{-1}(0)] < \infty. \tag{4.2}$$

Proof. Let $Y \subset \mathbb{C}^N \times \mathbb{C}^m$ be the Zariski closure of the graph of G, and let $\pi : Y \ni (x, y) \mapsto y \in \mathbb{C}^m$. Then for $(x, y) \in Y$ such that $x \in X$ and $y \in \mathbb{K}^m$ we have $y = G(x)$. Let us consider the case $n = k$. Let

$$U = \{L \in \mathbf{L}^{\mathbb{C}}(m, n) : \#[(L \circ \pi)^{-1}(0) \setminus \pi^{-1}(0)] < \infty\}.$$

By Proposition 1.1 in [21], U contains a non-empty Zariski open subset of $\mathbf{L}^{\mathbb{C}}(m, n)$. Then U contains a dense Zariski open subset W of $\mathbf{L}^{\mathbb{R}}(m, n)$. This gives the assertion (i) in the case $n = k$.

Let now $k > n$. Since for $L = (L_1, \ldots, L_k) \in \mathbf{L}^{\mathbb{K}}(m, k)$,

$$(L \circ \pi)^{-1}(0) \subset ((L_1, \ldots, L_n) \circ \pi)^{-1}(0),$$

then the assertion (i) follows from the previous case. We prove the assertion (ii) analogously as [21, Proposition 1.1]. □

Proof of Theorem 2.2. Without loss of generality we may assume that $F \neq 0$. By the definition, there exist $C, \varepsilon > 0$ such that for $x \in X$, $|x| < \varepsilon$ we have

$$|F(x)| \geq C \operatorname{dist}(x, F^{-1}(0))^{\mathcal{L}_0^{\mathbb{K}}(F|X)}, \tag{4.3}$$

and $\mathcal{L}_0^{\mathbb{K}}(F|X)$ is the smallest exponent for which the inequality holds. Let $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ be such that $F^{-1}(0) \cap U_L = (L \circ F)^{-1}(0) \cap U_L$ for some neighbourhood $U_L \subset \mathbb{K}^N$ of 0. Diminishing ε and the neighbourhood U_L, if necessary, we may assume that the equality $\operatorname{dist}(x, F^{-1}(0)) = \operatorname{dist}(x, F^{-1}(0) \cap U_L)$ holds for $x \in X$, $|x| < \varepsilon$. Obviously $L \neq 0$, so, $||L|| > 0$, and $|F(x)| \geq \frac{1}{||L||}|L(F(x))|$. Then by (4.3) we obtain $\mathcal{L}_0^{\mathbb{K}}(F|X) \leq \mathcal{L}_0^{\mathbb{K}}(L \circ F|X)$, and (2.4) is proved.

By Proposition 4.1 and Lemmas 3.1 and 3.2, for the generic $L \in \mathbf{L}^{\mathbb{K}}(m, k)$ we have that $F^{-1}(0) \cap U_L = (L \circ F)^{-1}(0) \cap U_L$ for some neighbourhood $U_L \subset \mathbb{K}^N$ of 0 and there exist ε, C_1, $C_2 > 0$ such that for $x \in X$, $|x| < \varepsilon$,

$$C_1|F(x)| \leq |L(F(x))| \leq C_2|F(x)|. \tag{4.4}$$

This and (4.3) gives (2.5) and ends the proof of Theorem 2.2. □

5. Proof of Theorem 2.3

The argument of Lemma 2.2 from [21] gives

Lemma 5.1. *Let $F : X \to \mathbb{R}^m$ with $m \geq n = \dim_{\mathbb{R}} X$ be a semialgebraic mapping, where $X \subset \mathbb{R}^N$, and let $n \leq k \leq m$. Then there exists a Zariski open and dense subset $U \subset \mathbf{L}^{\mathbb{R}}(m,k)$ such that for any $L \in U$ and any $\varepsilon > 0$ there exist $\delta > 0$ and $r > 0$ such that for any $x \in X$,*

$$|x| > r \ \wedge \ |L \circ F(x)| < \delta \ \Rightarrow \ |F(x)| < \varepsilon.$$

Proof. (cf. proof of Lemma 2.2 in [21]). Let us consider the case $k = n$. Let $W \subset \mathbb{C}^N$ be the Zariski closure of $F(X)$. Then $\dim_{\mathbb{C}} W \leq n$. In the case $\dim_{\mathbb{C}} W < n$, by Lemma 2.1 in [21] we easily obtain the assertion. Assume that $\dim W = n$. We easily see that for an algebraic set $V \subset W$, $\dim_{\mathbb{C}} V \leq n-1$, the mapping $F|_{X\setminus F^{-1}(V)} : X \setminus F^{-1}(V) \to W \setminus V$ is proper. By Lemma 2.1 in [21] there exists a Zariski open and dense subset $U_1 \subset \mathbf{L}^{\mathbb{R}}(m,k)$ such that for any $L \in U_1$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $z \in V$,

$$|L(z)| < \delta \Rightarrow |z| < \varepsilon. \tag{5.1}$$

Moreover, for $L \in U_1$,

$$W \subset \{z \in \mathbb{C}^m : |z| \leq C_L(1 + |L(z)|)\} \tag{5.2}$$

for some $C_L > 0$.

Let

$$U = \{L \in \mathbf{L}^{\mathbb{R}}(m,n) : L \in U_1\}.$$

Obviously, U is a dense and Zariski open subset of $\mathbf{L}^{\mathbb{R}}(m,n)$. Take $L \in U$ and $\varepsilon > 0$. Assume to the contrary that there exists a sequence $x_\nu \in X$ such that $|x_\nu| \to \infty$, $|L(f(x_\nu))| \to 0$ and $|f(x_\nu)| \geq \varepsilon$. By (5.2) we may assume that $f(x_\nu) \to y_0$ for some $y_0 \in W$. Since $F|_{X\setminus F^{-1}(V)} : X \setminus F^{-1}(V) \to W \setminus V$ is a proper mapping, we have $y_0 \in V$. So, $|y_0| \geq \varepsilon$ and $L(y_0) = 0$. This contradicts (5.1) and ends the proof in the case $n = k$.

Let now, $k > n$ and let

$$U = \{L = (L_1, \dots, L_k) \in \mathbf{L}^{\mathbb{R}}(m,k) : (L_1, \dots, L_n) \in U_1\}.$$

Then for any $L = (L_1, \dots, L_k) \in U$ and $x \in \mathbb{R}^n$ we have

$$|(L_1, \dots, L_n) \circ F(x)| \leq |L \circ F(x)|,$$

so, the assertion immediately follows from the previous case. □

Proof of Theorem 2.3 (cf. proof of Theorem 2.1 in [21]). Since for non-zero $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ we have $|L \circ F(x)| \leq ||L|||F(x)|$ and $||L|| > 0$, then by the definition of the Łojasiewicz exponent at infinity we obtain the first part of the assertion. We will prove the second part of the assertion.

Since $F^{-1}(0)$ is a compact set, by Proposition 4.1, there exists a dense Zariski open subset U of $\mathbf{L}^{\mathbb{R}}(m,k)$ such that

$$U \subset \{L \in \mathbf{L}^{\mathbb{R}}(m,k) : (L \circ F)^{-1}(0) \text{ is a compact set}\}.$$

So, for the generic $L \in \mathbf{L}^{\mathbb{R}}(m,k)$ the set $(L \circ F)^{-1}(0)$ is compact.

If $\mathcal{L}_{\infty}^{\mathbb{R}}(F|X) < 0$, the assertion (2.7) follows from Lemmas 3.1, 3.2 and 5.1.

Assume that $\mathcal{L}_{\infty}^{\mathbb{R}}(F|X) = 0$. Then there exist C, $R > 0$ such that $|F(x)| \geq C$ as $|x| \geq R$. Moreover, there exists a sequence $x_\nu \in X$ such that $|x_\nu| \to \infty$ as $\nu \to \infty$ and $|F(x_\nu)|$ is a bounded sequence. So, by Lemma 5.1 for the generic $L \in U$ and $\varepsilon = C$ there exist $r, \delta > 0$ such that $|L \circ F(x)| \geq \delta$ as $|x| > r$, so $\mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F|X) \geq 0$. Since $|L \circ F(x_\nu)|$ is a bounded sequence, we have $\mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F|X) \leq 0$. Summing up $\mathcal{L}_{\infty}^{\mathbb{R}}(L \circ F|X) = \mathcal{L}_{\infty}^{\mathbb{R}}(F|X)$ in the considered case.

In the case $\mathcal{L}_{\infty}^{\mathbb{R}}(F|X) > 0$, we obtain the assertion analogously as in the proof of Theorem 2.1 in [21]. □

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