VECTOR BUNDLES AND BLOWUPS

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Abstract. Let $X$ be a nonsingular quasi-projective complex algebraic variety and let $E$ be an algebraic vector bundle on $X$ of rank $r \geq 2$. The pullback of $E$ by the blowup of $X$ at a suitably chosen nonsingular subvariety of $X$ of codimension $r$ contains a line subbundle that can be explicitly described.

1. Introduction

Kleiman [2, Problem 1] considers the problem of splitting vector bundles on a nonsingular quasi-projective variety $V$ over an infinite field $k$: For any vector bundle $G$ on $V$ of rank at least 2, Kleiman [2, Theorem 4.7] proves that the pullback of $G$ by the blowup of a suitably chosen nonsingular subvariety contains a line bundle. Henceforth we assume that $k = \mathbb{C}$ and obtain Kleiman’s theorem as Corollary 1.3, which is a special case of Corollary 1.2 derived from Theorem 1.1. It does not seem possible to deduce Theorem 1.1 and Corollary 1.2 directly from [2]. Furthermore, the proof of Theorem 1.1 is short and very simple. In fact the main virtues of our note are its simplicity and brevity.

Let $X$ be a nonsingular quasi-projective complex algebraic variety. For any closed nonsingular (not necessarily irreducible) subvariety $Z$ of $X$, let

$$\pi(X, Z) : B(X, Z) \to X$$

denote the blowup of $X$ at $Z$. As usual, the line bundle determined by the exceptional divisor $D := \pi(X, Z)^{-1}(Z)$ will be denoted by $\mathcal{O}(D)$. If $Z$ is empty, then $B(X, Z) = X$ and $\pi(X, Z)$ is the identity map, $D = 0$ and $\mathcal{O}(D)$ is the standard

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trivial line bundle on $X$. For an algebraic vector bundle $E$ on $X$ and a section $u$ of $E$, the zero locus of $u$ will be denoted by $Z(u)$,

$$Z(u) := \{ x \in X : u(x) = 0 \}.$$

If $u$ is transverse to the zero section, then $Z(u)$ is a closed nonsingular subvariety of $X$ which is either empty or of codimension equal to the rank of $E$.

The main result, whose proof is postponed until Section 2, is the following:

**Theorem 1.1.** Let $E$ be an algebraic vector bundle on $X$ of rank $r \geq 2$. If $s$ is a section of $E$ which is transverse to the zero section and $Z := Z(s)$, then the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle isomorphic to $O(D)$, where $D$ is the exceptional divisor of the blowup $\pi(X,Z) : B(X,Z) \to X$.

Of course, $E$ may not have a section that is transverse to the zero section. However, if $E$ is generated by global sections $s_1, \ldots, s_k$, then for a general point $(t_1, \ldots, t_k) \in \mathbb{C}^k$, the section

$$s = t_1s_1 + \cdots + t_ks_k$$

is transverse to the zero section. There is always an algebraic line bundle $L$ on $X$ such that the vector bundle $E \otimes L$ is generated by global sections. It suffices to take as $L$ a high tensor power of an ample line bundle on $X$, cf. [1].

**Corollary 1.2.** Let $E$ be an algebraic vector bundle on $X$ of rank $r \geq 2$. Let $L$ be an algebraic line bundle on $X$ such that the vector bundle $E \otimes L$ admits a section $v$ transverse to the zero section, and let $Z := Z(v)$. Then the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle isomorphic to $O(D) \otimes \pi(X,Z)^*L^\vee$, where $D$ is the exceptional divisor of the blowup $\pi(X,Z) : B(X,Z) \to X$ and $L^\vee$ stands for the dual line bundle to $L$.

**Proof.** According to Theorem 1.1, the pullback vector bundle $\pi(X,Z)^*(E \otimes L)$ on $B(X,Z)$ contains an algebraic subbundle isomorphic to $O(D)$. The vector bundle $\pi(X,Z)^*E$ is isomorphic to

$$\pi(X,Z)^*(E \otimes L) \otimes \pi(X,Z)^*L^\vee,$$

and hence it contains a line subbundle isomorphic to $O(D) \otimes \pi(X,Z)^*L^\vee$. $\square$

Since for a suitably chosen line bundle $L$, the vector bundle $E \otimes L$ admits a section transverse to the zero section, the next result follows immediately.

**Corollary 1.3.** Let $E$ be an algebraic vector bundle on $X$ of rank $r \geq 2$. Then there exists a closed nonsingular subvariety $Z$ of $X$, either empty or of codimension $r$, such that the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle.
Corollary 1.3 is not a new result. It is proved (for varieties over an arbitrary infinite field) in Kleiman’s paper [2].

2. Proof of Theorem 1.1

For any nonsingular complex algebraic variety $Y$, denote by $T_Y$ its tangent bundle. Let $X$ be a nonsingular quasi-projective complex algebraic variety and let $Z$ be a closed nonsingular subvariety of $X$ with $\dim Z < \dim X - 1$. Consider the blowup

$$\pi(X, Z) : B(X, Z) \to X$$

of $X$ at $Z$. As a point set $B(X, Z)$ is the union of $X \setminus Z$ and the projective bundle $\mathbb{P}(N_Z X)$ on $Z$ associated with the normal bundle $N_Z X := (T_X | Z) / T_Z$ to $Z$ in $X$. The map $\pi(X, Z)$ is the identity on $X \setminus Z$ and the bundle projection $\mathbb{P}(N_Z X) \to Z$ on $\mathbb{P}(N_Z X)$.

Proof of Theorem 1.1. By abuse of notation, the total space of the vector bundle $E$ will also be denoted by $E$. Regard $X$ as subvariety of $E$, identifying it with its image by the zero section. Furthermore, identify the normal bundle to $X$ in $E$ with the vector bundle $E$. Thus as a point set the space $B(E, X)$ is the union of $E \setminus X$ and the projective bundle $\mathbb{P}(E)$ on $X$ that contains an algebraic line subbundle $L$ defined as follows. The fiber of $L$ over a point $e \in (E \setminus X)$ is the line $\{e\} \times C e$, and the restriction $L|\mathbb{P}(E)$ is the tautological line bundle on $\mathbb{P}(E)$. Note that $u : B(E, X) \to L$, defined by $u(e) = (e, e)$ for $e \in (E \setminus X)$ and $u|\mathbb{P}(E) = 0$, is a section of $L$, transverse to the zero section and satisfying $Z(u) = \mathbb{P}(E)$.

Since the section $s$ is transverse to $X$ in $E$, for each point $z$ in $Z$, the differential $ds_z : T_{X, z} \to T_{E, z}$ induces a linear isomorphism

$$\bar{d}s_z : (N_Z X)_z \to (N_X E)_z = E_z$$

between the fibers over $z$ of the normal bundle to $Z$ in $X$ and the normal bundle to $X$ in $E$. Define $\bar{s} : B(X, Z) \to B(E, X)$ by $\bar{s}(x) = s(x)$ for $x \in X \setminus Z$ and $\bar{s}(l) = \bar{d}s_z(l)$ for $l \in \mathbb{P}(N_Z X)_z$ with $z \in Z$. Thus $\bar{s}(l)$ is in $\mathbb{P}(E_z)$. By construction, $\bar{s}$ is an algebraic morphism satisfying

$$p \circ \pi(E, X) \circ \bar{s} = \pi(X, Z).$$
Hence the pullback $s^*L$ is an algebraic line subbundle of
\[
s^*((p \circ \pi(E, X))^* E) = (p \circ \pi(E, X) \circ \bar{s})^* E = \pi(X, Z)^* E.
\]

It remains to prove that the line bundles $s^*L$ and $O(D)$ are isomorphic. By construction, $\bar{s}$ is transverse to $\mathbb{P}(E)$ in $B(E, X)$ and $\bar{s}^{-1}(\mathbb{P}(E)) = \pi(X, Z)^{-1}(Z)$. Since the section $u : B(E, X) \to L$ is transverse to the zero section and $Z(u) = \mathbb{P}(E)$, the pullback section $s^*u : B, (X, Z) \to \bar{s}^*L$ is also transverse to the zero section and $Z(s^*u) = \pi(X, Z)^{-1}(Z) = D$. Consequently, the vector bundle $s^*L$ is isomorphic to $O(D)$, as required.

References