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VECTOR BUNDLES AND BLOWUPS

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Abstract. Let X be a nonsingular quasi-projective complex algebraic variety and let E be an algebraic vector bundle on X of rank $r \geq 2$. The pullback of E by the blowup of X at a suitably chosen nonsingular subvariety of X of codimension r contains a line subbundle that can be explicitly described.

1. Introduction

Kleiman [2, Problem 1] considers the problem of splitting vector bundles on a nonsingular quasi-projective variety V over an infinite field k: For any vector bundle G on V of rank at least 2, Kleiman [2, Theorem 4.7] proves that the pullback of G by the blowup of a suitably chosen nonsingular subvariety contains a line bundle. Henceforth we assume that $k = \mathbb{C}$ and obtain Kleiman's theorem as Corollary 1.3, which is a special case of Corollary 1.2 derived from Theorem 1.1. It does not seem possible to deduce Theorem 1.1 and Corollary 1.2 directly from [2]. Furthermore, the proof of Theorem 1.1 is short and very simple. In fact the main virtues of our note are its simplicity and brevity.

Let X be a nonsingular quasi-projective complex algebraic variety. For any closed nonsingular (not necessarily irreducible) subvariety Z of X, let

$$\pi(X,Z) : B(X,Z) \to X$$

denote the blowup of X at Z. As usual, the line bundle determined by the exceptional divisor $D := \pi(X,Z)^{-1}(Z)$ will be denoted by $\mathcal{O}(D)$. If Z is empty, then $B(X,Z) = X$ and $\pi(X,Z)$ is the identity map, $D = 0$ and $\mathcal{O}(D)$ is the standard

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trivial line bundle on X. For an algebraic vector bundle E on X and a section u of E, the zero locus of u will be denoted by $Z(u)$,

$$Z(u) := \{x \in X : u(x) = 0\}.$$

If u is transverse to the zero section, then $Z(u)$ is a closed nonsingular subvariety of X which is either empty or of codimension equal to the rank of E.

The main result, whose proof is postponed until Section 2, is the following:

Theorem 1.1. *Let E be an algebraic vector bundle on X of rank $r \geq 2$. If s is a section of E which is transverse to the zero section and $Z := Z(s)$, then the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle isomorphic to $\mathcal{O}(D)$, where D is the exceptional divisor of the blowup $\pi(X,Z) : B(X,Z) \to X$.*

Of course, E may not have a section that is transverse to the zero section. However, if E is generated by global sections $s_1, \dots, s_k$, then for a general point $(t_1, \dots, t_k) \in \mathbb{C}^k$, the section

$$s = t_1 s_1 + \cdots + t_k s_k$$

is transverse to the zero section. There is always an algebraic line bundle L on X such that the vector bundle $E \otimes L$ is generated by global sections. It suffices to take as L a high tensor power of an ample line bundle on X, cf. [1].

Corollary 1.2. *Let E be an algebraic vector bundle on X of rank $r \geq 2$. Let L be an algebraic line bundle on X such that the vector bundle $E \otimes L$ admits a section v transverse to the zero section, and let $Z := Z(v)$. Then the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle isomorphic to $\mathcal{O}(D) \otimes \pi(X,Z)^* L^\vee$, where D is the exceptional divisor of the blowup $\pi(X,Z) : B(X,Z) \to X$ and $L^\vee$ stands for the dual line bundle to L.*

Proof. According to Theorem 1.1, the pullback vector bundle $\pi(X,Z)^*(E \otimes L)$ on $B(X,Z)$ contains an algebraic subbundle isomorphic to $\mathcal{O}(D)$. The vector bundle $\pi(X,Z)^*E$ is isomorphic to

$$\pi(X,Z)^*(E \otimes L) \otimes \pi(X,Z)^* L^\vee,$$

and hence it contains a line subbundle isomorphic to $\mathcal{O}(D) \otimes \pi(X,Z)^* L^\vee$. □

Since for a suitably chosen line bundle L, the vector bundle $E \otimes L$ admits a section transverse to the zero section, the next result follows immediatly.

Corollary 1.3. *Let E be an algebraic vector bundle on X of rank $r \geq 2$. Then there exists a closed nonsingular subvariety Z of X, either empty or of codimesion r, such that the pullback vector bundle $\pi(X,Z)^*E$ on $B(X,Z)$ contains an algebraic line subbundle.*

Corollary 1.3 is not a new result. It is proved (for varieties over an arbitrary infinite field) in Kleiman's paper [2].

2. Proof of Theorem 1.1

For any nonsingular complex algebraic variety Y, denote by T_Y its tangent bundle. Let X be a nonsingular quasi-projective complex algebraic variety and let Z be a closed nonsingular subvariety of X with $\dim Z < \dim X - 1$. Consider the blowup

$$\pi(X, Z) : B(X, Z) \to X$$

of X at Z. As a point set $B(X, Z)$ is the union of $X \setminus Z$ and the projective bundle $\mathbb{P}(N_Z X)$ on Z associated with the normal bundle

$$N_Z X := (T_X|Z)/T_Z$$

to Z in X. The map $\pi(X, Z)$ is the identity on $X \setminus Z$ and the bundle projection $\mathbb{P}(N_Z X) \to Z$ on $\mathbb{P}(N_Z X)$.

Proof of Theorem 1.1. By abuse of notation, the total space of the vector bundle E will also be denoted by E. Regard X as subvariety of E, identifying it with its image by the zero section. Furthermore, identify the normal bundle to X in E with the vector bundle E. Thus as a point set the space $B(E, X)$ is the union of $E \setminus X$ and the projective bundle $\mathbb{P}(E)$ associated with E, while $\pi(E, X) : B(E, X) \to E$ is the identity on $E \setminus X$ and the bundle projection $\mathbb{P}(E) \to X$ on $\mathbb{P}(E)$. If $p : E \to X$ is the bundle projection, then the pullback vector bundle $(p \circ \pi(E, X))^* E$ on $B(E, X)$ contains an algebraic line subbundle L defined as follows. The fiber of L over a point $e \in (E \setminus X)$ is the line $\{e\} \times \mathbb{C}e$, and the restriction $L|\mathbb{P}(E)$ is the tautological line bundle on $\mathbb{P}(E)$. Note that $u : B(E, X) \to L$, defined by $u(e) = (e, e)$ for $e \in (E \setminus X)$ and $u|\mathbb{P}(E) = 0$, is a section of L, transverse to the zero section and satisfying $Z(u) = \mathbb{P}(E)$.

Since the section s is transverse to X in E, for each point z in Z, the differential $ds_z : T_{X,z} \to T_{E,z}$ induces a linear isomorphism

$$\bar{d}s_z : (N_Z X)_z \to (N_X E)_z = E_z$$

between the fibers over z of the normal bundle to Z in X and the normal bundle to X in E. Define $\bar{s} : B(X, Z) \to B(E, X)$ by $\bar{s}(x) = s(x)$ for $x \in X \setminus Z$ and $\bar{s}(l) = \bar{d}s_z(l)$ for $l \in \mathbb{P}(N_Z X)_z$ with $z \in Z$. Thus $\bar{s}(l)$ is in $\mathbb{P}(E_z)$. By construction, $\bar{s}$ is an algebraic morphism satisfying

$$p \circ \pi(E, X) \circ \bar{s} = \pi(X, Z).$$

Hence the pullback $\bar{s}^*L$ is an algebraic line subbundle of

$$\bar{s}^*((p \circ \pi(E,X))^*E) = (p \circ \pi(E,X) \circ \bar{s})^*E = \pi(X,Z)^*E.$$

It remains to prove that the line bundles $\bar{s}^*L$ and $\mathcal{O}(D)$ are isomorphic. By construction, $\bar{s}$ is transverse to $\mathbb{P}(E)$ in $B(E,X)$ and $\bar{s}^{-1}(\mathbb{P}(E)) = \pi(X,Z)^{-1}(Z)$. Since the section $u : B(E,X) \to L$ is transverse to the zero section and $Z(u) = \mathbb{P}(E)$, the pullback section $\bar{s}^*u : B,(X,Z) \to \bar{s}^*L$ is also transverse to the zero section and $Z(\bar{s}^*u) = \pi(X,Z)^{-1}(Z) = D$. Consequently, the vector bundle $\bar{s}^*L$ is isomorphic to $\mathcal{O}(D)$, as required.

References

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