Analytic and Algebraic Geometry
Łódź University Press 2013, 135 – 139

http://dx.doi.org/10.18778/7969-017-6.10

ABOUT CHOUIKHA'S ISOCHRONICITY CRITERION

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Abstract. Recently A.R.Chouikha gave a new characterization of isochronicity of center at the origin for the equation $x'' + g(x) = 0$, where g is a real smooth function defined in some neighborhood of $0 \in \mathbb{R}$. We present some new development of the subject. The present text is a short account of my paper "On Chouikha's isochronicity criterion", *arXiv:1201.6503*, where the proofs can be found. We correct the formulation of some results from the above paper.

Let us consider the second order differential equation

$$x'' + g(x) = 0 \tag{1}$$

where g is a real function defined in some neighborhood of $0 \in \mathbb{R}$ such that $g(0) = 0$, or equivalently the planar system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) \end{aligned}\right\} . \tag{2}$$

In what follows we shall exclusively concentrate on the system (2) with function g at least of class C^1.

As $g(0) = 0$, $0 \in \mathbb{R}^2$ is a singular point of the system (2). If in some neighborhood of a singular point all orbits of the system are closed and surround it, then the singular point is called a *center*.

A center is called *isochronous* if the periods of all orbits in some neighborhood of it are constant.

In future when speaking about isochronicity we always understand it with respect to $0 \in \mathbb{R}^2$ and the system (2).

The problem of characterization of isochronicity of the system (2) at $0 \in \mathbb{R}^2$ in term of function g is an old one.

2010 *Mathematics Subject Classification.* Primary 34C15, 34C25, 34C37.
Key words and phrases. Center, isochronicity, Urabe function.

To the best of our knowledge the first such characterization was done in 1937 by I.Kukles and N.Piskunov in [3], where even the case of continuous functions g is considered. The second one was described in 1962 by M.Urabe in [5] (see also [4]). Unfortunately these characterizations are not easy to handle and they are not really explicit.

We shall denote

$$G(x) = \int_0^x g(u)\,du. \tag{3}$$

Let us denote by X the continuous function defined in some neighborhood of $0 \in \mathbb{R}$ by

$$(X(x))^2 = 2\,G(x) \text{ and } xX(x) > 0 \; for \; x \neq 0. \tag{4}$$

Let us formulate now *Urabe Isochronicity Criterion.*

Theorem 1 ([5]). *Let g be a C^1 function defined in some neighborhood of $0 \in \mathbb{R}$. Let $g(0) = 0$ and $g'(0) = \lambda^2, \lambda > 0$. Then $0 \in \mathbb{R}^2$ is an isochronous center for the system* (2) *if and only if*

$$g(x) = \lambda \frac{X(x)}{1 + h(X(x))} \tag{5}$$

where the function X is defined by (4) *and where h is a continuous odd function defined in some neighborhood of $0 \in \mathbb{R}$.*

The function h is called *Urabe function* of the system (2).

Let us note that $\omega = \dfrac{2\pi}{\lambda}$ is the period of orbits of the above isochronous center.

Let us stress that from (3) and from assumptions on g in Urabe theorem it follows that $G(0) = 0$ and that in some punctured neighborhood of 0, $G(x) > 0$, G is of class C^2. Under our assumptions one proves that X is of class C^1. In fact, if $g \in C^k, k \geq 1$ (resp. g is real-analytic), then X is of class $C^k, X'(0) = \lambda > 0$ and h is of class C^{k-1} (resp. X and h are real-analytic).

From now on we shall always assume that $g \in C^1(]-\epsilon, \epsilon[))$ for some $\epsilon > 0$ and that

$$g'(0) = \lambda^2, \; \lambda > 0.$$

In September 2011 in a highly important paper [1], A.R.Chouikha published a completely new criterion of isochronicity ([1], Theorem B) which is much more direct and explicit that all previously known.

Theorem 2 ([1]). *Let $g \in C^1(]-\epsilon, \epsilon[)$ for some $\epsilon > 0$. Let $g(0) = 0$ and $g'(0) > 0$. If there exists δ, $0 < \delta \leq \epsilon$, such that for $|x| \leq \delta$ one has*

$$\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = f(G(x)) \tag{6}$$

where f is a continuous functions defined on some interval $[0,\eta]$, where $\eta > 0$, then $0 \in \mathbb{R}^2$ is an isochronous center for the system (2).

If $g \in C^2(]-\epsilon,\epsilon[)$ and $0 \in \mathbb{R}^2$ is an isochronous center for the system (2)*, then the condition* (6) *is satisfied.*

Consequently, if $g \in C^2(]-\epsilon,\epsilon[)$, then $0 \in \mathbb{R}^2$ is an isochronous center for the system (2) *if and only if the condition* (6) *is satisfied.*

We shall call the equation (6) the *Chouikha equation* and the function f is called *Chouikha function* of the system (2).

Let us pause now in the history of this theorem. In early February 2010, A.R. Chouikha communicated to me his *first* proof of his theorem valid only in real-analytic setting. Some time after he presented to me the *second* proof also valid only in real-analytic setting. The first proof was based on Urabe theorem, the second one on S.N. Chow and D. Wang [2] formula for the derivative of the first return map for the system (2). These proofs were not published at the time. At the beginning of July 2011, A.R. Chouikha and myself, simultaneously and independently obtained two different proofs of Chouikha theorem in smooth setting. Both proofs are the adaptation of the previous Chouikha's proofs in real-analytic setting. The Chouikha's proof published in [1] is the adaptation of his second proof. My proof is the adaptation of his first proof.

As a consequence of this last proof we obtain an unexpected closed relation between Urabe function h and Chouikha function f.

Theorem 3.

$$h(s) = \lambda \int_0^s f(\frac{q^2}{2})dq, \tag{7}$$

where $g'(0) = \lambda^2$, $\lambda > 0$. Thusf is real-analytic (resp. of class C^∞) if and only if h is real-analytic (resp. of class C^∞).

From now on we shall suppose that $f \in C^1([0,\epsilon]), \epsilon > 0$, where in 0 and in ϵ one considers the one-sided first derivatives. As before $g \in C^1(]-\delta,\delta[)$, $\delta > 0$.

Theorem 4. *Let $\epsilon > 0$ and $\lambda > 0$. Let $f \in C^1([0,\epsilon])$. There exists δ, $0 < \delta \le \epsilon$ and a unique function $g \in C^1(]-\delta,\delta[)$, $g'(0) = \lambda^2$ such that for every $|x| < \delta$ the Chouikha equation* (6)

$$\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = f(G(x))$$

is satisfied.

Let us stress that if $f_1, f_2 \in C^1([0,\epsilon]), \epsilon > 0$, and $f_1 \neq f_2$ on every interval $[0,\eta]$, $0 < \eta \le \epsilon$, then in any neighborhood of $0 \in \mathbb{R}$, $g_1 \neq g_2$, where g_1 and g_2 are the solutions of Chouikha equation that correspond to f_1 and to f_2 respectively.

Let us also note that if one supposes that $f \in C^k([0,\epsilon]), 1 \le k \le \infty$, or f is real-analytic, then the unique solution g of Chouikha equation is also of the same class. This gives a new light on the matter of Sec.4 of [1], proving the convergence of power series which appear there.

From now on we shall only consider the case of real-analytic or C^∞ functions g. Let us suppose that for function g, $0 \in \mathbb{R}^2$ is an isochronous center for the system (2).

In the real-analytic case there exists a natural bijective correspondence between the set of the couples of real-analytic functions f defined in some neighborhood of $0 \in \mathbb{R}$ and of real numbers $\lambda > 0$ with the set of the real-analytic functions g such that $0 \in \mathbb{R}^2$ is an isochronous center for the system (2). Indeed, to real-analytic function f defined in some neighborhood of $0 \in \mathbb{R}$ and to real number $\lambda > 0$ we associate the unique real-analytic function g such that $g(0) = 0,\ g'(0) = \lambda^2$ which is a solution of Chouikha equation, the existence of which is given by Theorem 4. Let us stress that the completely analogous statement is valid also in C^∞ framework.

As a consequence of Theorem 4 and of Theorem 3 we obtain a fact that seems to have been completely overlooked until now.

Theorem 5. *To every odd real-analytic (resp. of class C^∞) function h defined in some neighborhood of $0 \in \mathbb{R}$ and to every real number $\lambda > 0$ there corresponds a unique real-analytic (resp. of class C^∞) function g defined in some neighborhood of $0 \in \mathbb{R},\ g(0) = 0,\ g'(0) = \lambda^2$ such that $0 \in \mathbb{R}^2$ is an isochronous center for the system* (2) *and that h is its Urabe function.*

Let us denote by $Isochr(0, \omega)$ the germs of isochronous centers of the equation $x'' + g(x) = 0$ where g is a real-analytic function defined in some neighborhood of $0 \in \mathbb{R}$, $g(0) = 0$, $g'(0) > 0$. Let us denote by C_0^ω the germs of real-analytic functions defined in some neighborhood of $0 \in \mathbb{R}$. We can then state:

Theorem 6. *The Cartesian product $C_0^\omega \times \{x \in \mathbb{R}; x > 0\}$ and the set $Isochr(0, \omega)$ are in natural bijective correspondence. In other words the germs of real-analytic functions defined in some neighborhood of $0 \in \mathbb{R}$ and the strictly positive real numbers parametrize the germs of isochronous centers at 0 of equation $x'' + g(x) = 0$, with g a real-analytic function defined in some neighborhood of $0 \in \mathbb{R},\ g(0) = 0,\ g'(0) > 0$.*

Let us stress that the completely analogous statement to Theorem 6 is valid also in C^∞ framework.

Acknowledgment

I thank A.Raouf Chouikha (University Paris 13) for many years of helpful discussions on isochronous centers. I thank also Andrzej Maciejewski and Maria Przybylska (both from University of Zielona Gora, Poland) and Alain Albouy (Observatoire de Paris) for interesting discussions and important advices. Last but not least I thank Marie-Claude Werquin (University Paris 13) for linguistic corrections.

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