Analytic and Algebraic Geometry
Łódź University Press 2013, 141 – 153
http://dx.doi.org/10.18778/7969-017-6.11

JUMPS OF MILNOR NUMBERS IN FAMILIES OF NON-DEGENERATE AND NON-CONVENIENT SINGULARITIES

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Abstract. The non-degenerate jump of the Milnor number of an isolated singularity f_0 is the minimal non–zero difference between the Milnor numbers of f_0 and one of its non-degenerate deformations (f_s). In the paper the results by Bodin and the author (concerning the non-degenerate jump) are generalized to non-convenient singularities.

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be *an isolated singularity*, i.e. f_0 is the germ of a holomorphic function having an isolated critical point at 0. In the sequel a *singularity* means an isolated singularity.

A deformation of f_0 is a family $(f_s)_{s\in U}$ of isolated singularities (or smooth germs) analytically dependent on the parameter s in an open neighborhood U of $0 \in \mathbb{C}$. Let $\mu(f_s)$ denote the Milnor number of f_s. By the upper semi-continuity of $\mu(f_s)$ with respect to the Zariski topology [see [4], Prop. 2.57] the difference

$$\mu(f_0) - \mu(f_s), \qquad s \neq 0,$$

is non-negative and independent of $s \neq 0$ in a sufficiently small neighborhood of $0 \in \mathbb{C}$. We call it *the jump of Milnor numbers of the deformation* $(f_s)_{s\in U}$ and denote $\lambda((f_s))$.

The jump $\lambda(f_0)$ (or *the first jump*) is the minimum of non-zero jumps over all deformations (f_s) of f_0. Gusein-Zade proved in [3] that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities it holds

2010 *Mathematics Subject Classification.* Primary 32S30, Secondary 14B07.

Key words and phrases. Deformation of singularity, Milnor number, Newton polygon, non-degenerate singularity.

$\lambda(f_0) = 1$. The paper concerns the non-degenerate jump of the Milnor number i.e. the case when deformations (f_s) consist of only non-degenerate singularities. First, we recall the needed notions.

Put $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let

$$f_0(x, y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij} x^i y^j \in \mathbb{C}\{x, y\}.$$

Put

$$\operatorname{supp}(f_0) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}.$$

The *Newton diagram* of f_0 is the convex hull of

$$\bigcup_{(i,j)\in\operatorname{supp}(f_0)} \left((i, j) + \mathbb{R}^2_+\right), \quad \text{where} \quad \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \wedge y \geq 0\}.$$

We will denote it by $\Gamma_+(f_0)$. The boundary of the Newton diagram $\Gamma_+(f_0)$ is the union of two semilines and a finite set (may be empty) of compact, non-parallel segments. These segments constitute *the Newton polygon of* f_0, which we will denote by $\Gamma(f_0)$. They can be ordered in a natural way from the highest segment (closest to the vertical axes) to the lowest one. Often we will identify pairs $(i, j) \in \mathbb{N}^2$ with monomials $x^i y^j$. The singularity f_0 is *convenient*, if $\Gamma(f_0)$ has common points with OX and OY axes.

For a segment $\gamma \in \Gamma(f_0)$ we define

$$(f_0)_\gamma := \sum_{(i,j)\in\gamma} a_{ij} x^i y^j.$$

A singularity f_0 is *non-degenerate on* $\gamma \in \Gamma(f_0)$ (in the Kouchnirenko sense), if the system of equations

$$\frac{\partial (f_0)_\gamma}{\partial x}(x, y) = 0, \ \frac{\partial (f_0)_\gamma}{\partial y}(x, y) = 0$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. We call a singularity f_0 *non-degenerate*, when f_0 is non-degenerate on every segment $\gamma \in \Gamma(f_0)$.

Let f_0 be a convenient singularity. By S we denote the area of the set bounded by OX and OY axes and the polygon $\Gamma(f_0)$. By a and b we denote the distances between the origin $(0, 0)$ and the common part of Newton polygon $\Gamma_+(f_0)$ with OX and OY axes, respectively.

We define the *Newton number* of f_0 by

$$\nu(f_0) := 2S - a - b + 1.$$

Let f_0 be a singularity. A deformation $(f_s)_{s\in U}$ of f_0 is called *non-degenerate* if f_s is non-degenerate for every $s \neq 0$ sufficielntly close to the origin. We will denote by $\mathcal{D}^{nd}(f_0)$ the set of all non-degenerate deformations of the singularity f_0. *The* *non-degenerate jump* $\lambda'(f_0)$ *of a singularity* f_0 is the minimum of non-zero jumps over all non-degenerate deformations (f_s) of f_0, i.e.

$$\lambda'(f_0) := \min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{nd}(f_0)$ we denote all the non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$.

Now, we recall some results on the jump of convenient and non-degenerate singularities, which we will generalize to the non-convenient case. First, we define specific deformations of a convenient non-degenerate singularity f_0. Let $J(f_0)$ be the set of integer points (monomials) lying under the Newton polygon of f_0 except $(0,0)$. For any $(p,q) \in J(f_0)$ we define a deformation

$$f_s(x,y) = f_0(x,y) + sx^p y^q, \qquad s \in \mathbb{C},$$

and denote it by $(f_s^{(p,q)})$.

Theorem 1 (Bodin [1], Walewska [10])**.** *If* f_0 *is a non-degenerate and convenient singularity, then*

$$\lambda'(f_0) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})),$$

where $J_0(f_0) \subset J(f_0)$ *is the set of points* $(p,q) \in J(f_0)$ *such that* $\lambda((f_s^{(p,q)})) \neq 0$.

Directly from the above theorem we have

Corollary 2. *If* f *and* $\tilde{f}$ *are two non-degenerate and convenient singularities, with the same Newton diagram, then* $\lambda'(f) = \lambda'(\tilde{f})$.

Using Theorem 1 Bodin gave the exact value of the non-degenerate jump of some singularities.

Theorem 3 (Bodin [1])**.** *Let* $f_0(x,y) = x^p - y^q$*, where* $p \geq q \geq 2$ *and let* d=GCD(p,q).

1. *If* $d < q$*, then* $\lambda'(f_0) = d$.
2. *If* $d = q$*, then* $\lambda'(f_0) = d - 1$.

In the first case the jump $\lambda'(f_0)$ *is realized by the deformation* $f_s^{(-b,q-a)}$*, where* $a, b \in \mathbb{Z}$ *are such that* $ap + bq = d$*, where* $0 < a < \dfrac{q}{d}$ *and* $b < 0$*. Moreover, the point* $(-b, q-a)$ *lies in an open triangle with vertices* $(0,q)$*,* $(0,0)$ *and* $(p,0)$.

In the second case the jump is realized by the deformation $f_s^{(p-1,0)}$.

Consider now a general case of a convenient and non-degenerate singularity f_0, whose Newton polygon consists of only one segment. Let $(p,0)$ and $(0,q)$ be the intersection points of the Newton polygon of f_0 with the axes OX and OY, respectively. From Corollary 2 and Theorem 3 we have the following

Theorem 4. *Let f_0 be a non-degenerate and convenient singularity, with the Newton polygon reduced to only one segment. Then this segment connects points $(p, 0)$ and $(0, q)$ for some $p, q \in \mathbb{N}$ such that $p, q \geq 2$. If $d := \mathrm{GCD}(p, q)$, then:*

1. *If $1 \leq d < \min(p, q)$, then $\lambda'(f_0) = d$,*
2. *If $d = \min(p, q)$, then $\lambda'(f_0) = d - 1$.*

Let f_0 be a non-degenerate and convenient singularity. Let

$$\Lambda'(f_0) = (\mu_0, \mu_1, \ldots, \mu_k)$$

be the strictly decreasing sequence of all possible Milnor numbers of all non-degenerate deformations (f_s) of f_0. In particular,

$$\mu_0 = \mu(f_0), \quad \mu_1 = \mu(f_0) - \lambda'(f_0), \quad \mu_k = 0.$$

From Theorem 4 we have a formula for μ_1 if f_0 is a singularity with one segment Newton polygon (in particular for irreducible f_0). The sequence $\Lambda'(f_0)$ may be strange. One can check that

1. for $f_0(x, y) = x^8 - y^5$, we have $\Lambda'(f_0) = (28, 27, \ldots, 0)$,
2. for $f_0(x, y) = x^8 - y^4$, we have $\Lambda'(f_0) = (21, 18, 17 \ldots, 0)$,
3. for $f_0(x, y) = x^7 - y^5$, we have $\Lambda'(f_0) = (24, 23, \ldots, 15, 13, 12, \ldots, 0)$.

Next theorem gives a formula for μ_2 for singularities with one segment Newton polygon.

Theorem 5 (Walewska [10])**.** *Let $f_0(x, y) = x^p - y^q$, $p \geq q \geq 2$, $p + q > 4$. Then $\mu_2 = \mu_1 - 1$, if μ_2 is defined.*

Consider now a general case of a singularity which Newton polygon consists of only one segment. From Corollary 2 and Theorem 5 we have the following

Theorem 6. *Let f_0 be a non-degenerate and convenient singularity whose Newton polygon consists of only one segment. If $\Lambda'(f_0) = (\mu_0, \mu_1, \ldots, \mu_k)$, $k \geq 2$, is the sequence of Milnor numbers associated to f_0, then $\mu_2 = \mu_1 - 1$.*

The main goal of this paper is to extend the above results to the case of non-convenient singularities.

2. Non-convenient singularities

A power series $f_0 \in \mathbb{C}\{x, y\}$ is *nearly convenient*, if the distance of the Newton diagram $\Gamma_+(f_0)$ to each axis of the coordinate system does not exceed 1. It is easy to notice that

Lemma 2.1. *If f_0 is a singularity, then f_0 is nearly convenient.*

Let f_0 be a singularity. Then f_0 is either convenient singularity or can be represented in one of the following forms

$$x\tilde{f}_1, \quad y\tilde{f}_2, \quad xy\tilde{f}_3, \tag{$\star$}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ can be smooth germs or a convenient singularity and $\tilde{f}_3$ can be an invertible or a smooth germ or a convenient singularity. First, we consider the simplest cases when $\tilde{f}_i$ is not a convenient singularity.

Lemma 2.2. *Let f_0 be a singularity of one of the form listed in ($\star$). Assume that $\tilde{f}_i$ is not a convenient singularity. Then $\lambda'(f_0) = 1$ and $\mu_2 = \mu_1 - 1$, when μ_2 is defined.*

Proof. Consider the possible cases:

1. $f_0 = x\tilde{f}_1$, where $\tilde{f}_1$ is a smooth germ and $y \nmid f_0$. Then

a) if $\operatorname{ord}\tilde{f}_1(0, y) = 1$, then we easily check that $\mu(f_0) = 1$. This means that $\lambda'(f_0) = 1$ and μ_2 is undefined.

b) if $\operatorname{ord}\tilde{f}_1(0, y) =: k > 1$, then $\mu(f_0) = 2k-1$ and for the deformations $f_s(x, y) = f_0(x, y) + sy^{2k-1}$ and $\tilde{f}_s(x, y) = f_0(x, y) + sy^{2k-1} + sxy^{k-1}$ we have $\mu(f_s) = 2k-2$ and $\mu(\tilde{f}_s) = 2k-3$ for $s \neq 0$. Hence $\lambda'(f_0) = 1$ and $\mu_2 = \mu_1 - 1$.

2. $f_0 = y\tilde{f}_2$, where $\tilde{f}_2$ is a smooth germ and $x \nmid f_0$. We proceed similarly to case 1.

3. $f_0 = xy\tilde{f}_3$. Then

a) if $\tilde{f}_3$ is an invertible series, then we easily check that $\mu(f_0) = 1$. This means that $\lambda'(f_0) = 1$ and μ_2 is undefined.

b) if $\tilde{f}_3$ is a smooth germ then we proceed similarly to case 1. ■

Let f_0 be a singularity. In the sequel we will assume that $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ in ($\star$) are convenient singularities. Denote by (a_i, b_i), $i = 0, \dots, k+1$ and γ_i, $i = 0, \dots, k$, the consecutive vertices and segments of the Newton polygon $\Gamma(f_0)$, respectively. Let L_{γ_0} and L_{γ_k} be the lines that include the segments $\gamma_0 = \overline{(a_0, b_0), (a_1, b_1)}$ and $\gamma_k = \overline{(a_k, b_k), (a_{k+1}, b_{k+1})}$, respectively. It may happen that $L_{\gamma_0} = L_{\gamma_k}$.

Denote by $(r, 0)$ and $(0, t)$ the points of intersection of the lines L_{γ_k} and L_{γ_0} with the axes OX and OY, respectively. Of course, the coordinates r and t do not have to be integers.

If $a_0 = 0$, then the point (a_0, b_0) will be denoted by $(0, b)$. Similarly, if $b_{k+1} = 0$, then the point (a_{k+1}, b_{k+1}) will be denoted by $(a, 0)$. We will denote by $J(f_0)$ the set of all monomials $x^p y^q$, where $p + q \geq 1$, lying in the closed domain bounded by the axes OX, OY and by the set

$$\operatorname{conv}\left\{\{(r, 0), (0, t), \operatorname{supp}(f_0)\} + \mathbb{R}_+^2\right\}.$$

Note that for a convenient singularity the definition of the set $J(f_0)$ agrees with the one given in Section 1.

We associate to a singularity f_0 a convenient one f_0^{con} defined by

$$f_0^{\text{con}} := \begin{cases} f_0, & \text{if } f_0 \text{ is a convenient singularity} \\ f_0 + x^m, & \text{if } f_0 \text{ is of the form } y\tilde{f}_1 \\ f_0 + y^n, & \text{if } f_0 \text{ is of the form } x\tilde{f}_2 \\ f_0 + x^m + y^n, & \text{if } f_0 \text{ is of the form } xy\tilde{f}_3 \end{cases}$$

where m and n are sufficiently large natural numbers.

It is easy to show that the Newton number of f_0^{con} does not depend on the choice of sufficiently large numbers m and n. So, we may define the Newton number of f_0 by

$$\nu(f_0) := \nu(f_0^{\text{con}}).$$

We have the following formulas for the Newton number (see [7]).

Property 7. *Let f_0 be a singularity.*

1. *If f_0 is a convenient singularity* (see Fig. 1a)), *then $\nu(f_0) = 2S - a - b + 1$.*
2. *If f_0 can be written as $x\tilde{f}_1$, where $\tilde{f}_1$ is a convenient singularity* (see Fig. 1b)), *then $\nu(f_0) = 2S - a + b_0 + 1$.*
3. *If f_0 can be written as $y\tilde{f}_2$, where $\tilde{f}_2$ is a convenient singularity* (see Fig. 1c)), *then $\nu(f_0) = 2S + a_{k+1} - b + 1$.*
4. *If f_0 can be written as $xy\tilde{f}_3$, where $\tilde{f}_3$ is a convenient singularity* (see Fig. 1d)), *then $\nu(f_0) = 2S + a_{k+1} + b_0 - 1$.*

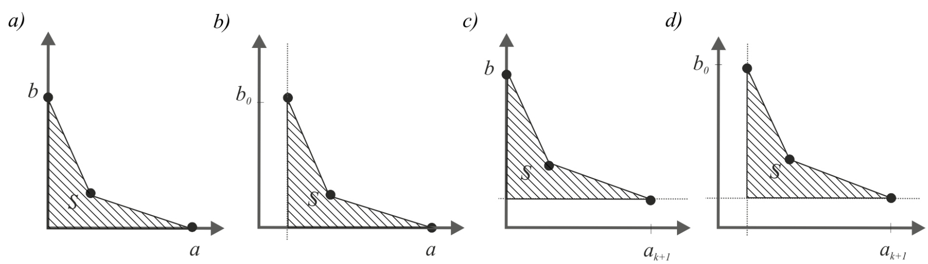

FIGURE 1. All possible variants of the Newton diagram of a nearly convenient singularity

From Kouchnirenko Theorem we have that if f_0 is a non-degenerate singularity, then $\mu(f_0) = \nu(f_0)$.

We prove that for any non-degenerate singularity f_0 there exists a deformation $(f_s^{(p,q)})$, where $(p,q) \in J(f_0)$, which realizes the jump $\lambda'(f_0)$.

Theorem 8. *If f_0 is non-degenerate, then*

$$\lambda'(f_0) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})),$$

where $J_0(f_0) \subset J(f_0)$ *is the set of points* (p, q) *such that* $\lambda((f_s^{(p,q)})) \neq 0$.

Proof. Let f_0 be a non-degenerate singularity. Then f_0 can be represented in one of the forms

$$\tilde{f}_0, x\tilde{f}_1, y\tilde{f}_2, xy\tilde{f}_3,$$

where $x \nmid \tilde{f}_0$, $y \nmid \tilde{f}_0$, $y \nmid \tilde{f}_1$, $x \nmid \tilde{f}_2$. Note that it suffices to consider the cases when $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ are convenient singularities because the other cases are included in the Lemma 2.2. We will consider cases:

1. $f_0 = \tilde{f}_0$. This means that the singularity is convenient and we may directly apply Theorem 1.

2. Suppose that $f_0 = x\tilde{f}_1$, where $\tilde{f}_1$ is a non-degenerate and convenient singularity. Denote by (a_i, b_i), $i = 0, \ldots, k+1$, the consecutive vertices of the Newton polygon $\Gamma(f_0)$. We have to prove

$$\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0) - \mu(f_s)) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})).$$

The inequality „$\leq$" is obvious. We will prove the opposite inequality. For sufficiently large n we have

$$\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0) - \mu(f_s)) = \min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0 + y^n) - \mu(f_s + y^n)).$$

Take any deformation $(f_s) \in \mathcal{D}_0^{nd}(f_0)$. Put $g_s := f_s + y^n$. Then g_s are convenient and $(g_s) \in \mathcal{D}_0^{nd}(f_0 + y^n)$ and $\mu(f_0 + y^n) - \mu(f_s + y^n) = \mu(f_0 + y^n) - \mu(g_s)$. We have

$$\begin{aligned}
\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0 + y^n) - \mu(f_s + y^n)) &\geq \min_{(h_s)\in\mathcal{D}_0^{nd}(f_0+y^n)} (\mu(f_0 + y^n) - \mu(h_s)) \overset{Th.1}{=} \\
&= \min_{(p,q)\in J_0(f_0+y^n)} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)\cup J_0'} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)),
\end{aligned}$$

where J_0' is the set of points $(0, l)$, where $l \in (t, n]$, for which $\lambda((f_s^{(p,q)})) \neq 0$. We claim that $J_0' = \emptyset$. Suppose to the contrary that $J_0' \neq \emptyset$. So there exists a point $(p, q) \in J_0'$. Then $(p, q) = (0, l)$, for some $l \in (t, n]$. It is easy to check $\mu(f_0 + y^n) = \mu(f_0 + y^n + sy^l)$, which contradicts the assumption that $(f_s^{(0,l)}) \in \mathcal{D}_0^{nd}(f_0)$. So

$$\begin{aligned}
&\min_{(p,q)\in J_0(f_0)\cup J_0'} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)} (\mu(f_0) - \mu(f_0 + sx^p y^q)).
\end{aligned}$$

3. In cases $f_0 = y\tilde{f}_2$ i $f_0 = xy\tilde{f}_3$ we proceed similarly to case 2. ■

3. The first jump of Milnor numbers

As for the non-degenerate and convenient singularities, we can give the exact value of the non-degenerate jump of some singularities. It happens that the Newton polygon of f_0 consists of only one segment. The following theorem extends Theorem 3 to the case of non-convenient singularities. It turns out that the formulas do not transfer automatically from convenient cases. There are new subcases.

Theorem 9. *Let* $f_0(x,y) = x^i y^j (x^p - y^q)$, *where* $i, j \in \{0,1\}$, $p \geq q \geq 2$, $p+q \geq 5$ *and let* $d = \mathrm{GCD}(p,q)$.

1. *If* $d < q$, *then* $\lambda'(f_0) = d$.
2. *If* $d = q$ *and* $i = 0$ *and* $j = 1$, *then* $\lambda'(f_0) = \begin{cases} d, & \text{for } q \neq p, \\ d-1, & \text{for } q = p. \end{cases}$
3. *If* $d = q$ *and* $i = 1$ *and* $j = 1$, *then* $\lambda'(f_0) = d$.
4. *If* $d = q$ *and* $j = 0$, *then* $\lambda'(f_0) = d - 1$.

Proof. Ad 1. Theorem 3, p. 1. implies that for the singularity $\tilde{f}_0(x,y) = x^p - y^q$ there exists a point P, which lies in the triangle with vertices $(0,q)$, $(0,0)$, $(p,0)$ and realizes the jump $\lambda'(\tilde{f}_0)$. According to the form of the singularity f_0 we consider the following cases.

a) $i = j = 0$. Then f_0 is a convenient singularity and from Theorem 3 we have $\lambda'(f_0) = d$.

b) $i = 1$ and $j = 0$. Translate the Newton diagram of $\tilde{f}_0$ together with the point P by the vector $[1,0]$. Using Property 7 p. 2. we easily check, that the point $P' := P + [1,0]$ realizes the jump equal to d.

Note that there exists no point P'' realizing a smaller jump than d. From Theorem 3, p. 1. we have that none of the points which lie on the axis OX realizes the jump smaller than d. We check, that for the points of the form $(0,k)$, where $k \in \mathbb{N}$ and $k \in (0,t)$ we have $\lambda((f_s^{(0,k)})) \geq d$. In fact, by assumption $p > q$ we have $|t - q| < 1$ (see Fig. 2). Moreover, Property 7, p. 2. implies that $\lambda((f_s^{(0,q)})) = q > d$ and $\lambda((f_s^{(0,q)})) < \lambda((f_s^{(0,k)}))$, where $k \in (0,q)$.

We check now that, for the points of the form $(1,m)$, where $m \in \mathbb{N}$ and $m \in (0,q)$ we get $\lambda((f_s^{(1,m)})) \geq d$. From Property 7, p. 2. $\lambda((f_s^{(1,q-1)})) = p + 1 > d$ and $\lambda((f_s^{(1,q-1)})) < \lambda((f_s^{(1,m)}))$, where $m \in (0, q-1)$ (see Fig. 2). This implies that $\lambda'(f_0) = d$ and this jump is realized by a point P'.

c) $i = 0$ and $j = 1$. Translate the Newton diagram of $\tilde{f}_0$ together with the point P by the vector $[0,1]$. From Property 7, p. 3. we have that the point $P' = P + [0,1]$ realizes the jump $\lambda'(f_0) = d$. Similarly to *b)* we easily check that, there exists no point which realizes the jump smaller than d.

d) $i = j = 1$. This follows from *b)* and *c)*.

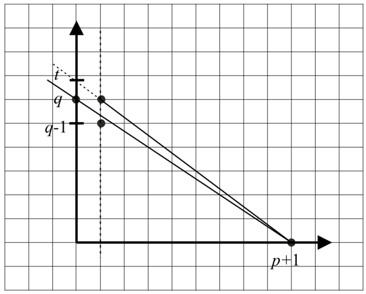

FIGURE 2. $f_0(x,y) = x(x^p - y^q)$

Ad 2. $d = q$, $i = 0$ and $j = 1$. In this case $r \in \mathbb{N}$ and $r = p + \dfrac{p}{q}$ (see Fig. 3). Consider the cases:

a) Let $q \neq p$. Note that $\lambda((f_s^{(r-1,0)})) = d$. It is sufficient to check that there exists no point realizing the jump smaller than d.

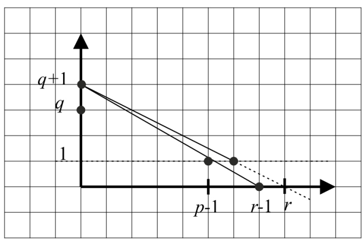

FIGURE 3. $f_0(x,y) = y(x^p - y^q)$

From Property 7, p. 3. $\lambda((f_s^{(p-1,1)})) = q+1 > d$ and $\lambda((f_s^{(0,q)})) = p-1 > d$ (see Fig. 3). Moreover $\lambda((f_s^{(k,0)})) > \lambda((f_s^{(r-1,0)}))$, if $k \in (0, r-1)$ and $\lambda((f_s^{(m,1)})) > \lambda((f_s^{(p-1,1)}))$, if $m \in (0, p-1)$ (see Fig. 3).

Moreover, Theorem 3, p. 2. implies that for the singularity $\tilde{f}_0(x,y) = x^p - y^q$ every point P which lies inside the triangle with vertices $(0,q)$, $(0,0)$, $(p,0)$ realizes the jump bigger or equal to d. If we translate the Newton diagram of $\tilde{f}_0$ by the vector $[0,1]$, then from Property 7, p. 3. we get, that every point P' lying inside the triangle with vertices $(0,q+1)$, $(0,1)$, $(p,1)$ realizes the jump bigger than d. So $\lambda'(f_0) = d$.

b) If $p = q$, then $\lambda((f_s^{(0,q)})) = d-1$. In this case $r = q+1$. Similarly to *a*) we check that there exists no point which realizes the jump smaller than $d-1$.

Ad 3. $d = q$, $i = 1$ and $j = 1$. Consider similarly to case 2.

Ad 4. Consider the cases:

a) $d = q$, $i = 0$ and $j = 0$. Then from Theorem 3 we have $\lambda'(f_0) = d - 1$.

b) $d = q$, $i = 1$ and $j = 0$. Note that $\lambda((f_s^{(p,0)})) = d-1$. It is sufficient to check that there exists no point realizing the jump better than $d - 1$. In fact, the assumption $p \geq q$ implies that $|t - q| \leq 1$ (see Fig. 4).

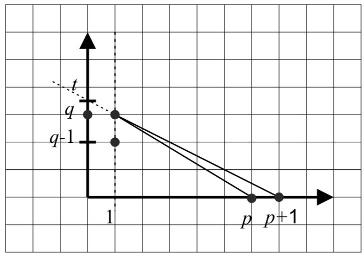

FIGURE 4. $f_0(x, y) = x(x^p - y^q)$

We have $\lambda((f_s^{(0,q)})) = q > d - 1$ and $\lambda((f_s^{(1,q-1)})) = p + 1 > d - 1$ (see Fig. 4). Property 7, p. 2. implies that $\lambda((f_s^{(0,k)})) > \lambda((f_s^{(0,q)}))$ for $k \in (0, q)$ and $\lambda((f_s^{(1,m)})) > \lambda((f_s^{(1,q-1)}))$ for $m \in (0, q - 1)$. Moreover, for singularity $\tilde{f}_0(x, y) = x^p - y^q$ each point P lying inside the triangle with vertices $(0, q)$, $(0, 0)$, $(p, 0)$ realizes the jump bigger than $d-1$. Hence and from Property 7, p. 2. we have that if we translate $\tilde{f}_0$ by the vector $[1, 0]$ then we get that each point P' lying inside the triangle with vertices $(1, q)$, $(1, 0)$, $(p+1, 0)$ realizes the jump bigger than $d-1$. Hence $\lambda'(f_0) = d - 1$. ■

From Lemma 2.2, Corollary 2 and Theorem 9 we have the following

Theorem 10. *Let f_0 be a non-degenerate singularity, with the Newton polygon reduced to at most one segment. Then $f_0(x, y) = x^i y^j \tilde{f}_0$, where $i, j \in \{0, 1\}$ and $\tilde{f}_0 \in \mathbb{C}\{x, y\}$ is a convenient power series. If $\tilde{f}_0$ smooth or invertible then $\lambda'(f_0) = 1$. If $\tilde{f}_0$ is a convenient singularity, which Newton polygon $\Gamma(\tilde{f}_0)$ has vertices at points $(p, 0)$ and $(0, q)$, $d := \mathrm{GCD}(p, q)$ and $p \geq q$, then*

1. *If $d < q$, then $\lambda'(f_0) = d$.*
2. *If $d = q$, $i = 0$ and $j = 1$, then $\lambda'(f_0) = \begin{cases} d, & \text{for } q < p, \\ d - 1, & \text{for } q = p. \end{cases}$*
3. *If $d = q$, $i = 1$ and $j = 1$, then $\lambda'(f_0) = d$.*
4. *If $d = q$ and $j = 0$, then $\lambda'(f_0) = d - 1$.*

4. THE SECOND JUMP OF MILNOR NUMBERS

Let f_0 be a non-degenerate singularity. Just as in the Introduction, we can consider the strictly decreasing sequence $(\mu_0, \mu_1, \ldots, \mu_k)$ of all possible Milnor numbers of all non-degenerate deformations (f_s) of f_0. In this case, we have results similar to the ones in the convenient case.

Theorem 11. *Let f_0 be a singularity of the form $f_0(x,y) = x^i y^j (x^p - y^q)$, $i, j \in \{0,1\}$, $p \geq q$. Then $\mu_2 = \mu_1 - 1$, if μ_2 is defined.*

Proof. For $i = 0$, $j = 0$ the assertion follows from Theorem 5. Note that if $x^p - y^q$ is not a singularity (i.e. $q = 1$) then the assertion follows from Lemma 2.2. If $x^p - y^q$ is a singularity we consider the case $i = 1$ or $j = 1$.

I. $q \nmid p$. Let us consider the subcases:

1. $i = 1$, $j = 0$. In this case we can repeat the argument of the proof of Theorem 5, p. 2. in [10] translating the whole configurations by the vector $[1, 0]$. Hence we get $\mu_2 = \mu_1 - 1$.

2. $i = 0$, $j = 1$. It suffices to consider only the case $q = 2$ because in the remaining cases we may repeat the argument from the proof of Theorem 5, p. 2 in [10]. Let $q = 2$. The fact $q \nmid p$ implies $\dfrac{3(p-1)}{2} \in \mathbb{N}$.

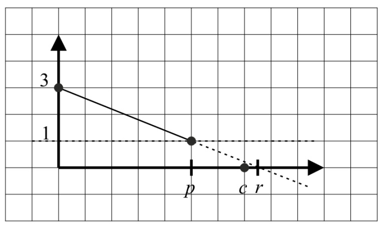

FIGURE 5. $f_0(x,y) = y(x^p - y^2)$

Moreover, for the point $(c, 0) := (\frac{3(p-1)}{2} + 1, 0)$ (see Fig. 5) we have $\lambda(f_s^{(c,0)}) = 1$. Of course $\mathrm{GCD}(c, 3) = 1$ hence from Theorem 3, p. 1. there exists a point lying inside the triangle with vertices $(0,3)$, $(0,0)$, $(c,0)$ realizing the jump $2 = \lambda'(f_0) + 1$. Hence $\mu_2 = \mu_1 - 1$.

3. $i = 1$, $j = 1$. It follows from 2.

II. $q \mid p$. Let us consider the subcases:

1. $i = 1$, $j = 0$. We have:

(i) $p = q = 2$. Then $f_0(x,y) = x(x^2 - y^2)$. It is easy to check that the point $(2, 0)$ realizes the jump equal to 1, while the deformation $f_s(x,y) = f_0(x,y) + sx^2 + sy^3$ realizes the jump equal to $2 = \lambda'(f_0) + 1$. Hence $\mu_2 = \mu_1 - 1$.

(ii) $p + q > 4$, $q \geq 2$. We repeat the argument from the proof of the Theorem 5, p. 1. in [10]. Hence and from Property 7 we have the assertion.

2. $i = 0$, $j = 1$. We have:

a) $q \neq p$. From Theorem 9 we have $\lambda'(f_0) = d$ and the deformation $f_s^{(r-1,0)}$ realizes this jump, where $r \in \mathbb{N}$, $r = p + \dfrac{p}{q}$ (see Fig. 6). Note that $\mathrm{GCD}(r-1, q+1) = 1$. From Theorem 3, p. 1., there exists a point (α, β) lying inside the triangle with vertices $(0, q+1)$, $(0,0)$, $(r-1, 0)$ realizing the jump equal to 1 for $f(x,y) = x^{r-1} - y^{q+1}$.

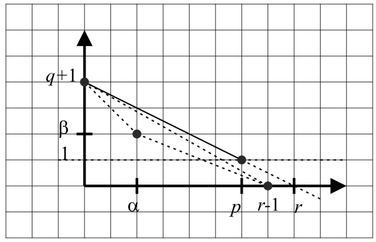

FIGURE 6. $f_0(x,y) = y(x^p - y^q)$

Therefore, the deformation $f_s(x,y) = f_0(x,y) + sx^{r-1} + sx^{\alpha}y^{\beta}$ realizes the jump $d+1 = \lambda'(f_0)+1$.

b) $q = p$. Let us consider the subcases:

i) $p = q = 2$. Then $f_0(x,y) = y(x^2 - y^2)$. It is easy to check that the point $(0,2)$ realizes the jump equal to 1, while the deformation $f_s(x,y) = f_0(x,y) + sx^3 + sy^2$ realizes the jump $2 = \lambda'(f_0)+1$. Hence $\mu_2 = \mu_1 - 1$.

ii) $p = q > 2$. From Theorem 9 $\lambda'(f_0) = d-1$ and this jump is realized by the point $(0,q)$. Note that $\mathrm{GCD}(q, q-1) = 1$. From Theorem 3, p. 1. there exists a point (α, β) lying inside the triangle with vertices $(0, q-1)$, $(0,0)$ and $(q,0)$ realizing the jump equal to 1 for $f(x,y) = x^q - y^{q-1}$.

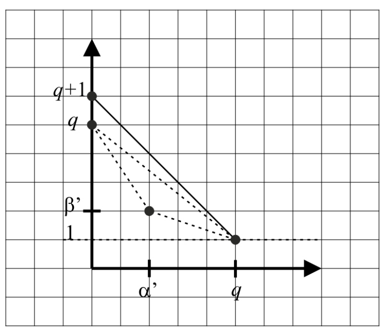

FIGURE 7. $f_0(x,y) = y(x^q - y^q)$

If we translate the diagram of $f(x,y) = x^q - y^{q-1}$ (with the point (α,β)) by the vector $[0,1]$ (see Fig. 7) we get the singularity $\tilde{f}(x,y) = x^q y - y^q$ and the point (α', β') such that the deformation $f_s(x,y) = f_0(x,y) + sy^q + sx^{\alpha'}y^{\beta'}$ realizes the jump $(d-1)+1 = d = \lambda'(f_0)+1$.

3. $i = j = 1$. Similarly to 2. ∎

From Lemma 2.2, Corollary 2 and Theorem 11 we have the following

Theorem 12. *Let f_0 be a non-degenerate singularity with the Newton polygon reduced to at most one segment. If $\Lambda'(f_0) = (\mu_0, \mu_1, \dots, \mu_k)$, $k \geq 2$, is the sequence of Milnor numbers associated to f_0, then*

$$\mu_2 = \mu_1 - 1,$$

provided μ_2 is defined.

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