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ON $\mathcal{C}^0$-SUFFICIENCY OF JETS

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Abstract. The paper presents some details of the proofs by Kuiper and Kuo, and Bochnak and Łojasiewicz that refer to the impact of the Łojasiewicz exponent of gradient mappings on $\mathcal{C}^0$-sufficiency of jets.

Introduction

Let $\omega : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a k-jet and $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ - one of its $\mathcal{C}^k$-realizations. We say that f is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^k$ class if, for any other $\mathcal{C}^k$-realization $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of ω there exist homeomorphisms $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that

$$g \circ \varphi = \psi \circ f \quad \text{in a neighbourhood of the origin.}$$

If this is the case, we say that f and g are $\mathcal{C}^0$-right-left equivalent, and if $\psi = \mathrm{id}_{\mathbb{R}}$ we say that f and g are $\mathcal{C}^0$-right equivalent. We say that f and g are V-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic as germs at 0.

The sufficiency of jets was studied by many authors, among them: Kuiper, Kuo, Bochnak and Łojasiewicz. In their, nowadays considered classical papers, the sufficiency of k-jets with respect to $\mathcal{C}^0$-right equivalence and the sufficiency of k-jets with respect to V-equivalence were studied, and necessary and sufficient conditions for sufficiency were given. In these cases the necessary and sufficient condition was formulated in terms of the Łojasiewicz inequality.

The present article presents some details of the proofs by Kuiper and Kuo and Bochnak and Łojasiewicz.

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1. $\mathcal{C}^r$-equivalence of functions

One of the major problems of catastrophe theory proposed by René Thom [30] is the classification of singularities of mappings and smooth functions at a point. If $f : (\mathbb{R}^n, a) \to (\mathbb{R}^s, b)$ will stand for the mapping f defined in a neighbourhood of the point $a \in \mathbb{R}^n$ with values in $\mathbb{R}^s$ such that $f(a) = b$, this problem can be formulated as follows:

Problem 1. *What conditions must be satisfied by smooth mappings $f, g : (\mathbb{R}^n, a) \to (\mathbb{R}^s, b)$ (of class $\mathcal{C}^k$; analytic), for the existence of diffeomorphisms $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$, $\psi : (\mathbb{R}^s, b) \to (\mathbb{R}^s, b)$ (of class $\mathcal{C}^r$; analytic isomorphisms) such that*

$$(1) \qquad g \circ \varphi = \psi \circ f \quad \textit{in a neighbourhood of the point } a.$$

The mappings f, g satisfying (1) are called *equivalent at the point* a (respectively $\mathcal{C}^r$*-equivalent; analytically equivalent*), if φ, ψ are smooth diffeomorphisms (respectively of class $\mathcal{C}^r$; analytic isomorphisms).

We will illustrate the above problem by the following examples.

Example 1. *Let $k \in \mathbb{Z}$, $k > 0$. All functions $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, defined by the formula*

$$f(x) = a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots, \qquad a_k \neq 0,$$

are analytically equivalent at zero. Indeed, it is sufficient to show that any such function is analytically equivalent at zero to the function $g(x) = x^k$, $x \in \mathbb{R}$. Taking $\psi(t) = t \operatorname{sgn} a_k$, $t \in \mathbb{R}$, and

$$\varphi(x) = x \sqrt[k]{|a_k + a_{k+1} x + a_{k+2} x^2 + \cdots|} \quad \textit{in a neighbourhood of zero,}$$

we see that φ and ψ are analytic isomorphisms and $\psi \circ f = g \circ \varphi$ in a neighbourhood of zero.

For the functions of several variables, Problem 1 is not so simple as in Example 1 for one variable.

Example 2. *Let*

$$f(x_1, x_2) = x_1^2 x_2 + a x_2^5, \qquad g(x_1, x_2) = x_1^2 x_2 + x_2^5, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

where $a \in \mathbb{R}$ is a parameter. Then the polynomials f and g have the same Taylor polynomial of order 3 at zero, equals to $x_1^2 x_2$, however

- *For $a > 0$, the functions f and g are analytically equivalent at zero, because for the analytic isomorphism*

  $$\varphi(x_1, x_2) = \left(\frac{1}{\sqrt[10]{a}} \cdot x_1,\ \sqrt[5]{a} \cdot x_2 \right), \qquad (x_1, x_2) \in \mathbb{R}^2,$$

  we have $f = g \circ \varphi$ in $\mathbb{R}^2$.

- *For $a \leqslant 0$, the functions f and g are not even $\mathcal{C}^0$-equivalent at zero, because by simple calculation we check that their sets of zeros have different numbers of topological components in each neighbourhood of the point $(0,0) \in \mathbb{R}^2$. Thus they can not be homeomorphic in any neighbourhood of the point $(0,0)$.*

In Examples 1, 2 we received analytic equivalence of analytic functions. There are analytic functions which are $\mathcal{C}^0$-equivalent at a point but are not analytically equivalent, as the following example shows.

Example 3. (Whitney). *Let*

$$f(x_1,x_2) = x_1x_2(x_1+x_2)(x_1-ax_2), \quad g(x_1,x_2) = x_1x_2(x_1+x_2)(x_1-bx_2),$$

where $a,b>0$ are parameters. According to Corollary 1 in Section 2, for every $a,b>0$ functions f and g are $\mathcal{C}^0$-equivalent at zero. For $a \neq b$, the functions f and g are not even $\mathcal{C}^1$-equivalent. If there were diffeomorphisms $\varphi : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$, $\psi : (\mathbb{R},0) \to (\mathbb{R},0)$ of class $\mathcal{C}^1$ such that $\psi \circ f = g \circ \varphi$ in a neighbourhood of zero, then the differential $d_0\varphi$ at zero would transform the tangent spaces at zero of the components of $f^{-1}(0)$ to the corresponding tangent spaces of the components of $g^{-1}(0)$. Then identify the tangent spaces to $\mathbb{R}^2$ at 0 with $\mathbb{R}^2$ we would get $d_0\varphi(f^{-1}(0)) = g^{-1}(0)$, which is impossible.

In view of this example, we see that the analytic classification of functions leads to a very rich family of different classes. This redirected the study of equivalence of functions to the study of $\mathcal{C}^r$-equivalence, especially to study of $\mathcal{C}^0$-equivalence at a point. In this paper we concentrate on study the $\mathcal{C}^0$-equivalence of $\mathcal{C}^k$ functions.

2. $\mathcal{C}^0$-SUFFICIENCY OF JETS

Examples 1 and 2 impose the following particularly important case of the Problem 1.

Problem 2. *What conditions should be imposed on the Taylor polynomials of functions f and g such that these functions were $\mathcal{C}^0$-equivalent at zero?*

This problem leads to the notion of $\mathcal{C}^0$-sufficiency of jets.

By a *k-jet of $\mathcal{C}^k$ function* $f : (\mathbb{R}^n,0) \to (\mathbb{R},0)$ we mean a family v of all functions $g : (\mathbb{R}^n,0) \to (\mathbb{R},0)$ of class $\mathcal{C}^k$ with the same k-th Taylor polynomial centered at zero as a Taylor polynomial of function f:

$$\sum_{j=1}^{k} \frac{1}{j!} \sum_{i_1,\dots,i_j=1}^{n} \frac{\partial^j f}{\partial x_{i_1} \cdots \partial x_{i_j}}(0) x_{i_1} \cdots x_{i_j}.$$

The function f is called then *$\mathcal{C}^k$-realization* of the jet v. By $J^k(n)$ we denote the set of all k-jets of $\mathcal{C}^k$ functions in n variables. The k-jet of a function f can be identified with the k-th Taylor polynomial of the function. So $J^k(n)$ is isomorphic to $\mathbb{R}^N$, where $N = \binom{n+k}{n} - 1$.

A k-jet is called *$\mathcal{C}^0$-sufficient in* the $\mathcal{C}^k$ class, if any two of its $\mathcal{C}^k$-realizations are $\mathcal{C}^0$-equivalent at zero.

R. Thom [30] (see also [13]) proved that by adding to any polynomial a "generic" form of "high degrees" we get a $\mathcal{C}^0$-sufficient k-jet in an appropriate class (the same is also true for the k-jets of mappings). Precisely, we have

Theorem 1. (R. Thom). *Let us denote by $\pi_s : J^{k+s}(n) \to J^k(n)$ the natural projection. Let $v \in J^k(n)$. Then there is an integer $s > 0$ and there is a proper algebraic subset $\Sigma \subset \pi_s^{-1}(v)$ such that every $(k+s)$-jet $w \in \pi_s^{-1}(v) \setminus \Sigma$ is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^{k+s}$ class.*

Bochnak and Łojasiewicz generalized this theorem (see [1]) showing that $s = 1$ (see Proposition 1 in Section 3).

In the language of k-jets Problem 2 can be written as follows.

Problem 3. *What conditions should be imposed on the k-jet to make it $\mathcal{C}^0$-sufficient in the $\mathcal{C}^k$ class?*

The $\mathcal{C}^0$-sufficiency of jet implies a topological equivalence (in a neighbourhood of zero) of sets of zeros of its realizations. This leads to the following definition:

A k-jet is called *V-sufficient in* $\mathcal{C}^k$ class, if for any two its $\mathcal{C}^k$-realizations f and g, the sets $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic in a neighbourhood of zero.

The following beautiful theorem is a solution of Problem 3.

Theorem 2. (Kuiper-Kuo, Bochnak-Łojasiewicz). *Let v be a k-jet with f as its $\mathcal{C}^k$-realization, where $k \in \mathbb{Z}$, $k > 0$. The following conditions are equivalent:*

(a) *v is $\mathcal{C}^0$-sufficient in $\mathcal{C}^k$ class,*
(b) *v is V-sufficient in $\mathcal{C}^k$ class,*
(c) *$|\nabla f(x)| \geqslant C|x|^{k-1}$ in a neighbourhood of the point $0 \in \mathbb{R}^n$ for some constant $C > 0$, where ∇f is the gradient of the function f.*

In the above theorem the implication (a)⇒(b) is obvious; the implication (b)⇒(c) was proved by Bochnak and Łojasiewicz [1]; the implication (c)⇒(a) was proved by Kuiper [11] and Kuo [12]. The proof of Bochnak and Łojasiewicz (by contradiction) is based on the construction of an appropriate $\mathcal{C}^k$-realization of the jet, whose set of zeros is not a topological manifold in any neighbourhood of the point 0. It is known that for every $\mathcal{C}^k$-realization f of V-sufficient k-jet, the set of zeros $f^{-1}(0)$ is a topological manifold in some neighbourhood of zero or an empty set (see Lemma 2 in Section 4). The proofs of Kuiper and Kuo are based on the construction of a homeomorphism φ (see definition of $\mathcal{C}^0$-equivalence) using the general solution of an appropriate system of ordinary differential equations. The proof of Theorem 2 is discussed further in Section 4.

In Section 4, as the implication (c)⇒(a) of Theorem 2, we similarly prove the following

Corollary 1. *Let $f, g \in \mathbb{R}[x_1, x_2]$ be homogeneous forms that are decomposed in the products of linear forms without multiple factors. If* $\deg f = \deg g$, *then f and g are $\mathcal{C}^0$-equivalent at zero.*

Of course, the implication (a)⇒(b) in Theorem 2 holds also in the complex domain, where instead of the $\mathcal{C}^k$ functions it should be considered the class of holomorphic functions. It is easy to check that the proof of the implication (c)⇒(a) is transferred without any changes to the complex case. Unfortunately, the Bochnak and Łojasiewicz proof of the implication (b)⇒(c) is typically real and cannot be transferred to the case of holomorphic function. This implication over $\mathbb{C}$ was generalized by Teissier [29], who showed that for the holomorphic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, the smallest integer k such that k-jet of function f is $\mathcal{C}^0$-sufficient in the class of holomorphic functions, satisfies the inequality $k \geqslant \lceil \mathcal{L}_0(\nabla f) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer $k \geq x$ and $\mathcal{L}_0(\nabla f)$ – the Łojasiewicz exponent of ∇f at zero (see Section 3). The inequality $k \leqslant \lceil \mathcal{L}_0(\nabla f) \rceil + 1$ was proved by Chang and Lu [3], who based on the article of Kuo [12].

The problem of sufficiency of jets is of interest to many mathematicians, besides the mentioned above, inter alia: Kirschenbaum and Lu [8]; Koike [9]; Kucharz [10]; Kuo [13]; Kuo and Lu [15]; Lu [17]; Pelczar [21], [22]; Płoski [24]; Randall [25]; Takens [28]; Trotman [32].

3. The Łojasiewicz exponent

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function of class $\mathcal{C}^k$. In view of Theorem 2, a special importance is imposed on the optimal (i. e. the smallest) exponent α in *the Łojasiewicz inequality* [20]

(Ł) $$|\nabla f(x)| \geqslant C|x|^\alpha \quad \text{in a neighbourhood of zero for some } C > 0.$$

This exponent is called *the Łojasiewicz exponent of gradient* ∇f at a zero and is denoted by $\mathcal{L}_0(\nabla f)$. This is obviously an invariant of singularities, that is, it stays invariant under a diffeomorphic change of variables. The knowledge of the exponent and its connections to other invariants of singularities helps in a more accurate characterization of different classes of singularities. This fact caused a great interest and an intense study of the exponent $\mathcal{L}_0(\nabla f)$. It was of interest to many scientists, among others: Chądzyński [4], Chądzyński and Krasiński [6]; Khadiri and Tougeron [7]; Kuo and Lu [14]; Lejeune-Jalabert and Teissier [19]; Płoski [23]; Teissier [29]; Tougeron [31].

Bochnak and Łojasiewicz generalized Theorem 1 (see [1], page 259) showing that $s = 1$. In the proof of this generalization they use Theorem 2 (c)⇒(a) to the following fact.

Proposition 1. *For a polynomial $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ of degree at most k there is a proper algebraic subset $\Sigma\subset\mathbb{R}^N$, where $N=\binom{n+k}{n-1}$, such that for every polynomial*

$$H_c(x)=\sum_{i_1+\cdots+i_n=k+1} c_{i_1,\ldots,i_n}x_1^{i_1}\cdots x_n^{i_n},$$

where $c=(c_{i_1,\ldots,i_n};i_1+\cdots+i_n=k+1)\in\mathbb{R}^N\setminus\Sigma$, we have

$$\mathcal{L}_0\left(\nabla(f+H_c)\right)\leqslant k, \tag{2}$$

so then $|\nabla(f+H_c)(x)|\geqslant C|x|^k$ in a neighbourhood of the point $0\in\mathbb{R}^n$ for some constant $C>0$ (that is $f+H_c$ satisfies the condition (c) *of Theorem 2 for $k+1$).*

Proof. Since for every proper algebraic subset $V\subset\mathbb{C}^N$, a set $V\cap\mathbb{R}^N$ is a proper algebraic subset of $\mathbb{R}^N$, then it suffices to prove the proposition over $\mathbb{C}$. Let

$$\begin{aligned}\Omega=&\{c\in\mathbb{C}^N:\ \exists_{r>0}\ \ \nabla(f+H_c)(x)\neq 0\ \text{ for }\ 0<|x|<r\},\\ \Delta=&\{c\in\mathbb{C}^N:\ \exists_{r>0}\ \ \nabla(f+H_b)(x)\neq 0\ \text{ for }\ 0<|x|<r,\ |b-c|<r\}.\\ G=&\{c\in\mathbb{C}^N:\ \exists_{C,r>0}\ \ |\nabla(f+H_c)(x)|\geqslant C|x|^k\ \text{ for }\ |x|<r\}.\end{aligned}$$

Note first that the set Ω has a nonempty interior. Indeed, let us consider an algebraic set:

$$\Gamma=\{(c,x)\in\mathbb{C}^N\times\mathbb{C}^n:\nabla H_c(x)=0\}.$$

Let $\Gamma=\Gamma_1\cup\ldots\cup\Gamma_l$ be a decomposiotion of Γ into irreducible components. Of course, $\mathbb{C}^N\times\{0\}\subset\Gamma$. Take any component Γ_{i_0} of the set Γ such that $\mathbb{C}^N\times\{0\}\subset\Gamma_{i_0}$. We will show that $\mathbb{C}^N\times\{0\}=\Gamma_{i_0}$. Suppose to the contrary, that $\mathbb{C}^N\times\{0\}\subsetneq\Gamma_{i_0}$, then $\dim_{\mathbb{C}}\Gamma_{i_0}>N$. Since $\nabla H_c(x)=0$ is a system of homogenous equations, it is easy to check that for each $c\in\mathbb{C}^N$ there is $x\neq 0$, such that $(c,x)\in\Gamma_{i_0}$. However, it is impossible, because for $c\in\mathbb{C}^N$ such that $H_c(x)=x_1^{k+1}+\cdots+x_n^{k+1}$ there is no $x\neq 0$ satisfying $\nabla H_c(x)=0$. Summing up $\Gamma_{i_0}=\mathbb{C}^N\times\{0\}$. Denoting by A the set $\bigcup_{i\neq i_0}\{c\in\mathbb{C}^N:(c,0)\in\Gamma_i\}$, we see that this is a proper algebraic subset of $\mathbb{C}^N$. Moreover, for $c\in\mathbb{C}^N\setminus A$ the gradient $\nabla(f+H_c)$ has no zeros at infinity. Thus, the set of zeros of $\nabla(f+H_c)$ is finite. This gives that $\mathbb{C}^N\setminus\Omega\subset A$ and prove the announced remark.

Taking into account the above remark we will prove that $\mathbb{C}^N\setminus\Delta$ is contained in a proper algebraic subset Σ of space $\mathbb{C}^N$. In fact, let

$$\Omega_j=\{c\in\mathbb{C}^N:\nabla(f+H_c)(x)\neq 0\ \text{ for }\ 0<|x|<\frac{1}{j}\},\quad j\in\mathbb{N}.$$

Then $\Omega=\bigcup_{j=1}^{\infty}\Omega_j$. From the previous observation $\operatorname{Int}\Omega\neq\emptyset$, so from the Baire theorem, there is $j_0\in\mathbb{N}$ such that $\operatorname{Int}\Omega_{j_0}\neq\emptyset$. Let

$$T=\{(c,x)\in\mathbb{C}^N\times\mathbb{C}^n:\ \nabla(f+H_c)(x)=0\}$$

and let $T=T_1\cup\ldots\cup T_m$ be a decomposiotion of T into irreducible components. If $\mathbb{C}^N\times\{0\}\not\subset T$, then by setting $\Sigma=\{c\in\mathbb{C}^N:(c,0)\in T\}$ we get the mentioned remark in this case. So, assume that $\mathbb{C}^N\times\{0\}\subset T$. Then there is i_0 such that $\mathbb{C}^N\times\{0\}\subset T_{i_0}$. We will show that $\mathbb{C}^N\times\{0\}=T_{i_0}$. Assuming the contrary, we get $\dim_{\mathbb{C}} T_{i_0} > N$. Thus, each point $(c,0)$ is an accumulation point of the set $T_{i_0} \setminus [\mathbb{C}^N \times \{0\}]$. In particular, each point $(c,0)$, where $c \in \Omega_{j_0}$ is an accumulation point of the set $T_{i_0} \setminus [\mathbb{C}^N \times \{0\}]$. This is impossible, because Ω_{j_0} has nonempty interior. As a consequence $\mathbb{C}^N \times \{0\} = T_{i_0}$. Now, setting $\Sigma = \bigcup_{i \neq i_0} \{c \in \mathbb{C}^N : (c,0) \in T_i\}$ we get the mentioned remark, too.

Finally we will show that $\mathbb{C}^N \setminus \Sigma \subset G$, which finishes the proof of the proposition. We will base on the original Bochnak and Łojasiewicz proof [1], p. 259. Suppose to the contrary, that there exists $c \in \mathbb{C}^N \setminus \Sigma$ such that $c \notin G$. Then there exists a sequence $(a_\nu) \subset \mathbb{C}^n \setminus \{0\}$, $a_\nu \to 0$ such that

$$\frac{|\nabla(f + H_c)(a_\nu)|}{|a_\nu|^k} \to 0 \quad \text{as} \quad \nu \to \infty. \tag{3}$$

We will prove that there exists a sequence $b_\nu \in \mathbb{C}^N$ such that

$$\nabla(f + H_c)(a_\nu) = \nabla H_{b_\nu}(a_\nu) \quad \text{and} \quad b_\nu \to 0. \tag{4}$$

Indeed, let $\delta_\nu = \nabla(f + H_c)(a_\nu)$ and $L_\nu : \mathbb{C}^n \to \mathbb{C}^n$ be an isometry such that $L_\nu(\frac{a_\nu}{|a_\nu|}) = (1, 0, \ldots, 0)$ and $L_\nu(0) = 0$. Denote by M_ν the matrix of mapping L_ν. Then all the coefficients of the matrices M_ν and M_ν^{-1} are bounded by 1. Let $\delta_\nu \cdot M_\nu^{-1} = (\theta_{\nu,1}, \ldots, \theta_{\nu,n})$. Then from (3) we have

$$\frac{\theta_{\nu,i}}{|a_\nu|^k} \to 0 \quad \text{as} \quad \nu \to \infty \quad \text{for} \quad i = 1, \ldots, n. \tag{5}$$

Take polynomials

$$G_\nu(x) = \frac{\theta_{\nu,1}}{k+1} x_1^{k+1} + \sum_{i=2}^{n} \theta_{\nu,i} x_1^k x_i$$

and

$$H_{b_\nu} = \frac{1}{|a_\nu|^k} G_\nu \circ L_\nu.$$

Then

$$\nabla G_\nu(x) = \left(x_1^{k-1} \left(\theta_{\nu,1} x_1 + k\theta_{\nu,2} x_2 + \cdots + k\theta_{\nu,n} x_n\right), \theta_{\nu,2} x_1^k, \ldots, \theta_{\nu,n} x_1^k\right),$$

so $\nabla G_\nu(1, 0, \ldots, 0) = (\theta_{\nu,1}, \ldots, \theta_{\nu,n})$. Hence

$$\nabla H_{b_\nu}(a_\nu) = \frac{1}{|a_\nu|^k} \nabla G_\nu(L_\nu(\frac{a_\nu}{|a_\nu|} |a_\nu|)) \cdot M_\nu = (\theta_{\nu,1}, \ldots, \theta_{\nu,n}) \cdot M_\nu = \delta_\nu.$$

Moreover, (5) implies that $b_\nu \to 0$ as $\nu \to \infty$, because b_ν are made of points $\left(\frac{\theta_{\nu,1}}{|a_\nu|^k (k+1)}, \frac{\theta_{\nu,2}}{|a_\nu|^k}, \ldots, \frac{\theta_{\nu,n}}{|a_\nu|^k}\right)$ by the linear transformations with the uniformly bounded coefficients. As a result, (4) has been proved. In summary, from (4) and the definition of sequence δ_ν we get

$$\nabla(f + H_{c-b_\nu})(a_\nu) = \nabla(f + H_c)(a_\nu) - \nabla H_{b_\nu}(a_\nu) = 0$$

and $c - b_\nu \in \mathbb{C}^N \setminus \Sigma \subset \Delta$ for sufficiently large ν (because $c - b_\nu \to c$ as $\nu \to \infty$). This contradicts the definition of set Δ and completes the proof. □

From the Proposition 1 we deduce immediately its generalization.

Corollary 2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function and $k \in \mathbb{Z}$, $k > 0$. Then there is a proper algebraic subset $\Sigma \subset \mathbb{R}^N$, where $N = \binom{n+k}{n-1}$, such that for each $c = (c_{i_1,\dots,i_n}; i_1 + \cdots + i_n = k+1) \in \mathbb{R}^N \setminus \Sigma$ we have $\mathcal{L}_0(\nabla(f + H_c)) \leqslant k$, where $H_c(x) = \sum_{i_1+\cdots+i_n=k+1} c_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}$.*

Proof. Let $f = g + h + u$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the polynomial of degree at most k, $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the homogeneous polynomial of degree $k+1$ and $u : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the analytic function such that $\operatorname{ord}_0 u > k+1$. According to Proposition 1, there exists a proper algebraic subset $\Sigma_1 \subset \mathbb{R}^N$ such that the inequality $\mathcal{L}_0(\nabla(g + H_c)) \leqslant k$ holds for every $c \in \mathbb{R}^N \setminus \Sigma_1$. If $c_0 \in \mathbb{R}^N$ is a system of coefficients of h, then $\Sigma_2 = \{c - c_0 : c \in \Sigma_1\}$ is a proper algebraic subset of $\mathbb{R}^N$ and $\mathcal{L}_0(\nabla(g + h + H_c)) \leqslant k$ for every $c \in \mathbb{R}^N \setminus \Sigma_2$. Since $\operatorname{ord}_0 u > k+1$, we obtain $|\nabla u(x)| \leqslant C|x|^{k+1}$ in a neighbourhood of zero, for some $C > 0$. This and the previous one implies the inequality $\mathcal{L}_0(\nabla(f + H_c)) \leqslant k$. $\square$

The example which follows will illustrate the preceding results: Theorem 1 and Proposition 1 .

Example 4. *Let $f \in J^2(2)$ be of the form $f(x_1, x_2) = x_1^2$.*

Then the 2-jet f is not $\mathcal{C}^0$-sufficient in $\mathcal{C}^2$ class, because, for example, a set of zeros of its $\mathcal{C}^2$-realization $g(x_1, x_2) = x_1^2 - x_2^4$ is not homeomorphic to $f^{-1}(0)$ in any neighbourhood of zero.

Let $\Sigma = \mathbb{R}^3 \times \{0\}$, for every $c = (c_1, c_2, c_3, c_4) \in \mathbb{R}^4 \setminus \Sigma$ and let $H_c(x) = c_1 x_1^3 + c_2 x_1^2 x_2 + c_3 x_1 x_2^2 + c_4 x_2^3$. Then the sets of zeros of $\frac{\partial(f+H_c)}{\partial x_1}$ and $\frac{\partial(f+H_c)}{\partial x_2}$ have no common tangents at a point zero. Thus $\mathcal{L}_0(\nabla(f + H_c)) \leqslant 2$ and according to the Theorem 2, the 3-jet $f + H_c$, $c \in \mathbb{R}^4 \setminus \Sigma$, is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^3$ class.

Remark 1. *It is worth going back for a moment to the polynomial $g(x_1, x_2) = x_1^2 x_2 + x_2^5$ in Example 2. We will calculate $\mathcal{L}_0(\nabla g)$. In these calculations, it is convenient to pass to the complex case. In this case, the Łojasiewicz exponent of gradient ∇g is defined in the same way as above and denoted by $\mathcal{L}_0^{\mathbb{C}}(\nabla g)$. Using the results of Chądzyński and Krasiński (Theorem 1 in [6], see also [5]) we get that the exponent $\mathcal{L}_0^{\mathbb{C}}(\nabla g)$ is attained on the set*

$$S = \{z \in \mathbb{C}^2 : \frac{\partial g}{\partial z_1}(z) \frac{\partial g}{\partial z_2}(z) = 0\}.$$

It is easy to check that $S = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = \mathbb{C} \times \{0\}, \qquad S_2 = \{0\} \times \mathbb{C},$$
$$S_3 = \{(-i\sqrt{5}t^2, t) \in \mathbb{C}^2 : t \in \mathbb{C}\}, \quad S_4 = \{(i\sqrt{5}t^2, t) \in \mathbb{C}^2 : t \in \mathbb{C}\}.$$

Then

$$\nabla g|_{S_1}(t, 0) = (0, t^2), \qquad \nabla g|_{S_2}(0, t) = (0, 5t^4),$$
$$\nabla g|_{S_3}(-i\sqrt{5}t^2, t) = (-2i\sqrt{5}t^3), \quad \nabla g|_{S_1}(i\sqrt{5}t^2, t) = (-2i\sqrt{5}t^3).$$

Hence, we get $\mathcal{L}_0^{\mathbb{C}}(\nabla g) = 4$. *In particular* $\mathcal{L}_0(\nabla g) \leqslant 4$. *Since* $\nabla g(0,t) = (0, 5t^4)$ *for* $t \in \mathbb{R}$, *we deduce that* $\mathcal{L}_0(\nabla g) = 4$.

The polynomial $f = x_1^2 x_2 + a x_2^5$, $a \in \mathbb{C}$, *is a* $\mathcal{C}^4$*-realization of* 4*-jet* v *of polynomial* g. *Since* $\mathcal{L}_0(\nabla g) = 4 = 5-1$, *the Łojasiewicz inequality* (Ł) *and Theorem 2 implies that the* 4*-jet* v *is not* $\mathcal{C}^0$*-sufficient. It agrees with the statement in Example 2, that for* $a \leqslant 0$ *the functions* f *and* g *are not equivalent at zero. By Theorem 2,* 5*-jet of function* g *is* $\mathcal{C}^0$*-sufficient in* $\mathcal{C}^5$ *class. This means that the addition to* g *any terms of degree at least* 6, *leads to an equivalent at zero function* g.

4. Proof of Theorem 2

Implication (c)⇒(a). Let us begin with a simple lemma.

Lemma 1. *Let* $G \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set and* $W : G \to \mathbb{R}^n$ *be a continuous mapping. If a system*

$$\frac{dy}{dt} = W(t,y) \tag{6}$$

has a global uniqueness of solutions property in $G \setminus (\mathbb{R} \times \{0\})$ *and if*

$$|W(t,x)| \leqslant C|x| \quad \textit{for} \quad (t,x) \in U, \tag{7}$$

for some constant $C > 0$ *and some neighbourhood* $U \subset G$ *of* $(\mathbb{R} \times \{0\}) \cap G$, *then* (6) *has a global uniqueness of solutions property in* G.

Proof. By the uniqueness of solutions of (6) in $G \setminus (\mathbb{R} \times \{0\})$, it suffices to prove that there exists a locally unique solution of a system (6) that passes through the point 0. Assume that $(t_0, 0) \in G$. Condition (7) implies that the mapping $y_0(t) = 0$, defined in some neighbourhood of t_0, is a solution of (6). Suppose that there exists another solution $y_1 : (a,b) \to \mathbb{R}^n$ of (6) such that $y_1(t_0) = 0$. Then y_0 and y_1 fulfill the following system of integral equations

$$y(t) = \int_{t_0}^{t} W(\xi, y(\xi)) d\xi. \tag{8}$$

Let $0 < \varepsilon < \frac{1}{C}$ be small enough to guarantee that graphs of $y_0, y_1 : I \to \mathbb{R}^n$, where $I = [t_0 - \varepsilon, t_0 + \varepsilon] \subset (a,b)$ lie in U. Then there exists $\eta \in I$ such that

$$\varrho := \sup_{t \in I} |y_0(t) - y_1(t)| = |y_0(\eta) - y_1(\eta)|.$$

In view of the assumption we get that $\varrho > 0$. Therefore (8) and assumption (7) give

$$\varrho = \left| \int_{t_0}^{\eta} [W(\xi, y_0(\xi)) - W(\xi, y_1(\xi))] d\xi \right| \leqslant \left| \int_{t_0}^{\eta} C |y_1(\xi))| d\xi \right| \leqslant C \varrho \varepsilon < \varrho,$$

which is impossible. □

Proof of implication (c)⇒(a). In the case $k = 1$ this is a consequence of the inverse function theorem. Let us assume that $k > 1$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the k-th

Taylor polynomial of a k-jet v and let g be a $\mathcal{C}^k$-realization of jet v. It suffices to show that mappings f and g are $\mathcal{C}^0$-equivalent. From the choice of g we have

$$\lim_{x\to 0}\frac{g(x)-f(x)}{|x|^k}=0,$$

which implies that for every $\varepsilon_0>0$ there exists $\delta_0>0$ such that

$$|g(x)-f(x)|\leqslant\varepsilon_0|x|^k\qquad\text{for}\quad |x|<\delta_0. \tag{9}$$

We may assume that g is defined in $\mathbb{R}^n$. Therefore we have a well-defined mapping $F:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, where

$$F(\xi,x)=f(x)+\xi(g(x)-f(x)),\qquad \xi\in\mathbb{R},\quad x\in\mathbb{R}^n.$$

We note that (cf. Kuo [12], Lemma 1, p. 168) there exist ε and $\delta>0$ such that

$$|\nabla F(\xi,x)|\geqslant\varepsilon|x|^{k-1}\quad\text{for}\quad |x|<\delta\quad\text{and}\quad -2<\xi<2. \tag{10}$$

Indeed, since f and g are $\mathcal{C}^k$ functions, $\nabla(g-f)$ is a $\mathcal{C}^{k-1}$ mapping. The choice of g shows that the $(k-1)$-th Taylor polynomial centered at zero of mapping $\nabla(g-f)$ vanishes identically. Hence

$$\lim_{x\to 0}\frac{|\nabla(g-f)(x)|}{|x|^{k-1}}=0.$$

Therefore there exists $\delta>0$ such that

$$|\nabla(g-f)(x)|\leqslant\frac{C}{4}|x|^{k-1}\quad\text{for}\quad |x|<\delta,$$

where C comes from the condition (c) of Theorem 2. Since

$$\nabla F(\xi,x)=[(g-f)(x),\nabla f(x)+\xi\nabla(g-f)(x)], \tag{11}$$

then by taking $\varepsilon=\frac{C}{2}$, we have from assumption (c)

$$|\nabla F(\xi,x)|\geqslant|\nabla f(x)+\xi\nabla(g-f)(x)|\geqslant|\nabla f(x)|-2|\nabla(g-f)(x)|\geqslant\varepsilon|x|^{k-1}$$

provides $|x|<\delta$ and $-2<\xi<2$. This gives (10). One can of course assume that $\varepsilon=\varepsilon_0$ and $\delta=\delta_0<\frac{1}{2}$.

Define $G=\{(\xi,x)\in\mathbb{R}\times\mathbb{R}^n : |x|<\delta,\ -2<\xi<2\}$, where ε and δ are as above. Let $X:G\to\mathbb{R}^n\times\mathbb{R}$ be a mapping of the form

$$X(\xi,x)=(X_1,\dots,X_{n+1})=\frac{(g(x)-f(x))}{|\nabla F(\xi,x)|^2}\nabla F(\xi,x),\qquad\text{provided}\quad x\neq 0$$

and $X(\xi,0)=0$. By (9) and (10), we have

$$|X(\xi,x)|\leqslant\frac{\varepsilon|x|^k}{|\nabla F(\xi,x)|}\leqslant\frac{\varepsilon|x|^k}{\varepsilon|x|^{k-1}}=|x|\quad\text{for}\quad(\xi,x)\in G,x\neq 0. \tag{12}$$

It is easy to see that the above inequality holds also for $x=0$, so X is continuous.

Let us define a vector field $W:G\to\mathbb{R}^n$ by

$$W(\xi,x)=\frac{1}{X_1(\xi,x)-1}[X_2(\xi,x),\dots,X_{n+1}(\xi,x)].$$

Inequality (12) implies that

$$|X_1(\xi,x)-1| \geqslant 1-|X(\xi,x)| \geqslant 1-|x| > 1-\delta > \frac{1}{2} \quad \text{for} \quad (\xi,x)\in G,$$

whence W is well-defined. Moreover it is continuous and

$$|W(\xi,x)| \leqslant 2|x| \qquad \text{for} \quad (\xi,x)\in G. \tag{13}$$

Consider now a system of differential equations

$$\frac{dy}{dt} = W(t,y). \tag{14}$$

Since $k>1$, then W is at least of class $\mathcal{C}^1$ on $G\setminus(\mathbb{R}\times\{0\})$, so it is a lipschitzian vector field. As a consequence, the above system has a uniqueness of solutions property in $G\setminus(\mathbb{R}\times\{0\})$. Hence, inequality (13) and Lemma 1 implies the global uniqueness of solutions of the system (14) throughout G. Since $y_0(t)=0$, $t\in(-2,2)$ is one of the solutions of (14), then the above implies the existence of a neighbourhood $U\subset\mathbb{R}^n$ of 0 such that every integral solution y_x of (14) with $y_x(0)=x$, where $x\in U$, is defined at least in $[0,1]$.

Now, let us define a mapping $\varphi : U\to\mathbb{R}^n$ by the formula

$$\varphi(x) = y_x(1),$$

where y_x stands for an integral solution of (14) with $y_x(0)=x$. This mapping is continuous and bijective. It gives a homeomorphism of some neighbourhoods of the origin. Indeed, considering solution $\overline{y}_x : [0,1]\to\mathbb{R}^n$ of (14) with $\overline{y}_x(1)=x$, where x is from some neighbourhood of the origin, we get that $\varphi(\overline{y}_x(0))=x$. Similar reasoning shows that the mapping $x\mapsto\overline{y}_x(0)$ is continuous in the neighbourhood of the origin. Consequently $\varphi : (\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ maps homeomorphically a neighbourhood of the origin onto a neighbourhood of the origin.

Finally, note that for every $x\in U$,

$$F(t,y_x(t)) = \text{const.} \quad \text{in} \quad [0,1]. \tag{15}$$

Indeed, from definition of W we derive the formula

$$[1,W(\xi,x)] = \frac{1}{X_1(\xi,x)-1}(X(\xi,x)-e_1) \qquad \text{for} \quad (\xi,x)\in G,$$

where $e_1=[1,0,\dots,0]\in\mathbb{R}^{n+1}$ and $[1,W] : G\to\mathbb{R}\times\mathbb{R}^n$. Thus, if we denote by $\langle a,b\rangle$ the scalar product of two vectors a, b, then according to (11) for $t\in[0,1]$, we have

$$\begin{aligned}\frac{dF(t,y_x(t))}{dt} &= \langle(\nabla F)(t,y_x(t)),[1,W(t,y_x(t))]\rangle\\ &=\frac{1}{X_1(t,y_x(t))-1}\Big(\langle(\nabla F)(t,y_x(t)),X(t,y_x(t))\rangle-\frac{\partial F}{\partial\xi}(t,y_x(t))\Big)\\ &=\frac{1}{X_1(t,y_x(t))-1}\big(g(y_x(t))-f(y_x(t))-g(y_x(t))+f(y_x(t))\big)=0.\end{aligned}$$

This gives (15). Finally, (15) yields

$$f(x) = F(0, x) = F(0, y_x(0)) = F(1, y_x(1)) = F(1, \varphi(x)) = g(\varphi(x))$$

for $x \in U$. This ends the proof of the implication (c)⇒(a) in Theorem 2. □

Proof of Corollary 1. Let $k = \deg f$. It suffices to prove the corollary assumming that

$$f(x) = (\alpha_1 x_1 + \alpha_2 x_2)h(x) \quad \text{i} \quad g(x) = (\beta_2 x_1 + \beta_2 x_2)h(x),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $h \in \mathbb{R}[x_1, x_2]$ is a form of degree $k - 1$. Moreover, it can be assumed that f and g differ only by a constant factor and that the region $\{(x_1, x_2) \in \mathbb{R}^n : \alpha_1 x_1 + \alpha_1 x_2 > 0,\ \beta_1 x_1 + \beta_2 x_2 > 0\}$ is disjoint from $h^{-1}(0)$. Then there is an interval (a, b) containing the interval $[0, 1]$ such that for every $\xi \in (a, b)$ a linear mapping

$$L_\xi(x) = (\alpha_1 x_1 + \beta_1 x_2) + (1 - \xi)[(\alpha_2 - \alpha_1)x_1 + (\beta_2 - \beta_1)x_2]$$

does not divide h. Let $F(\xi, x) = f(x) + \xi(g(x) - f(x))$. Then $F(\xi, x) = L_\xi(x)h(x)$, so for every $\xi \in (a, b)$, function F does not have multiple factors. Therefore after eventually diminishing the interval (a, b) such that still $[0, 1] \subset (a, b)$, and using the curve selection lemma, we easily show that F satisfies (10) for $\xi \in (a, b)$. Since $g - f$ is a form of degree k, it satisfies (9) for some $\varepsilon_0 > 0$. Repeating now the rest of the proof of the implication (c)⇒(a) in Theorem 2, we get the assertion. □

Implication (b)⇒(c). In developing this proof we used the original Bochnak and Łojasiewicz proof [1]. Assuming that the implication fails, the proof consists in the construction of an appropriate $\mathcal{C}^k$-realization of jet, whose set of zeros is not a topological manifold in any neighbourhood of the point 0. In fact there is the following

Lemma 2. *Let v be a k-jet and let f be its $\mathcal{C}^k$-realization. If v is V-sufficient in $\mathcal{C}^k$, then there is a neighbourhood $U \subset \mathbb{R}^n$ of 0 such that $f^{-1}(0) \cap (U \setminus \{0\})$ is a $(n-1)$-dimensional topological manifold or an empty set.*

Proof. Let g be a k-th Taylor polynomial of jet v. Then

$$h = g + x_1^{k+1} + \cdots + x_n^{k+1}$$

is a $\mathcal{C}^k$-realization of jet v. Moreover ∇h has no zeros at infinity (even over $\mathbb{C}$), so its set of zeros is finite. Therefore the assertion follows from the implicit function theorem and from the definition of V-sufficiency. □

A key point in the proof of considered implication is Proposition 2 given below. In the proof of mentioned proposition we will use the following Morse lemma, which follows from the previously proven implication (c)⇒(a) in Theorem 2 (cf. [18] Lemma 2.2).

Corollary 3. (Morse lemma). *Let f be a function of class $\mathcal{C}^2$ in a neighbourhood of $a \in \mathbb{R}^n$, $n > 1$, such that*

$$f(a) = 0, \qquad \nabla f(a) = 0 \qquad \text{and} \qquad \det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right] \neq 0. \tag{16}$$

Then there is a homeomorphism $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$ and there is an integer $0 \leqslant l \leqslant n$ such that

$$f \circ \varphi(x) = \sum_{i=1}^{l}(x_i - a_i)^2 - \sum_{i=l+1}^{n}(x_i - a_i)^2 \quad \textit{in a neighbourhood of } a.$$

Proof. It suffices to consider the case $a = 0$. Then, from (16), 2-nd Taylor polynomial of function f is a quadratic form: $h(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(0) x^i x^j$. It can be assumed, from the assumption (16), by the appropriate selection of linear coordinate system, that

$$h(x) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{n} x_i^2 \quad \text{for some} \quad l \in \mathbb{Z}, \quad 0 \leqslant l \leqslant n.$$

We can directly verify that $|\nabla h(x)| = 2|x|^{2-1}$ for $x \in \mathbb{R}^n$. Hence and from the implication (c)⇒(a) in Theorem 2, 2-jet of function h is $\mathcal{C}^0$-sufficient in $\mathcal{C}^2$. Since f is $\mathcal{C}^2$-realization of this jet, there is a homeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ \varphi = h$ in a neighbourhood of 0. □

In the proof of Proposition 2 we will also need two known topological facts. Let's start with the definition.

The set $S^l = \{(x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} : x_1^2 + \cdots + x_{l+1}^2 = 1\}$ as well as any set homeomorphic to S^l will be called a *sphere* of dimension l.

Let A be a topological manifold and S — a sphere in A. The mappings $\varphi, \psi : S \to A$ will be called *homotopic in A*, if there is a continuous mapping $H : S \times [0, 1] \to A$ such that

$$H(x, 0) = \varphi(x) \qquad \text{and} \qquad H(x, 1) = \psi(x) \qquad \text{for} \qquad x \in S.$$

The mapping H will be called *a homotopy of φ and ψ in A*.

We will say that a sphere S is *contractible in A*, if there is a point $a \in A$ such that the mapping $\varphi : S \ni x \mapsto x \in A$ is homotopic in A to a constant map $\psi : S \ni x \mapsto a$. The homotopy of mappings φ and ψ will be called *a null-homotopy in A*.

Lemma 3. *Let A be a topological manifold of dimension k and $a \in A$. If $1 \leqslant l \leqslant k - 2$, then there exists a neighbourhood $U \subset A$ of a such that every l-dimensional sphere $S \subset U \setminus \{a\}$ is contractible in $U \setminus \{a\}$.*

Proof. We may assume, by choosing a neighbourhood $U \subset A$ of a homeomorphic with $\mathbb{R}^k$, that $U = \mathbb{R}^k$ and $a = 0$. Let $S \subset \mathbb{R}^k \setminus \{0\}$ be an arbitrary l-dimensional sphere and $\varphi : S^l \to S$ be a homeomorphism. Approximating φ

by a polynomial mapping $\psi : S^l \to \mathbb{R}^k \setminus \{0\}$, we may assume that φ and ψ are homotopic in $\mathbb{R}^k \setminus \{0\}$. It is easy to find a line $E \subset \mathbb{R}^k \setminus \psi(S^l)$ such that $0 \in E$. The mappings ψ and $a+\psi$ are homotopic in $\mathbb{R}^k \setminus \{0\}$ for every $a \in E$. Moreover there is $a \in E$ such that 0 is not in the convex hull of $(a+\psi(S^l))$. Therefore $a+\psi$ is contractible in $\mathbb{R}^k \setminus \{0\}$. □

Lemma 4. *The sphere* $S = \{(x_1, \dots, x_l) \in \mathbb{R}^l : x_1^2 + \dots + x_l^2 = r^2\}$, *where* $r > 0$ *is not contractible in* $\mathbb{R}^l \setminus \{0\}$.

Proof. Assume to the contrary that there is a null-homotopy $H : S \times [0,1] \to \mathbb{R}^l \setminus \{0\}$. It can be assumed that $r = 1$ and that $H(S \times [0,1]) \subset S$. Therefore a mapping h defined by $h(x) = H(\frac{x}{|x|}, 1-|x|)$ for $0 < |x| \leqslant 1$ and $h(0) = H(y,1)$, where $y \in S$, is a continuous mapping of a ball $D = \{x \in \mathbb{R}^l : |x| \leqslant r\}$ onto a sphere S, whereas $h(x) = x$ for $x \in S$. Thus S is a deformation retract of ball D, which is impossible. □

Proposition 2. *Let* $n > 1$ *and* $f : (\mathbb{R}^n, a) \to (\mathbb{R}, 0)$ *be a function of class* $\mathcal{C}^2$ *fulfilling the assumptions* (16) *of Morse lemma. Then* $f^{-1}(0)$ *is not a topological manifold of dimension* $n-1$ *in any neighbourhood of point* a.

Proof. In view of Corollary 3 (Morse lemma), it suffices to reduce our considerations to the case $a = 0$,

$$f(x) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{n} x_i^2$$

and $f^{-1}(0) \neq \{0\}$. Then $1 \leqslant l < n$. It can be assumed, of course, that $l \leqslant \frac{n}{2}$.

The theorem is clearly true for $l = 1$, since then a set $f^{-1}(0) \setminus \{0\}$ has at least four topological components in every neighbourhood of the origin for $n = 2$, and at least two such components for $n > 2$. It can therefore be assumed that $n > 2$ and $l > 1$ and then

$$1 \leqslant l - 1 \leqslant (n-1) - 2. \tag{17}$$

Assume now that for some neighbourhood $\Omega \subset \mathbb{R}^n$ of the point $0 \in \mathbb{R}^n$,

$A = f^{-1}(0) \cap \Omega$ is a topological manifold of dimension $n-1$.

Therefore (17) and Lemma 3 implies that there is a neighbourhood $U \subset A$ of the origin such that every $(l-1)$-dimensional sphere $S \subset U \setminus \{0\}$ is contractible in $U \setminus \{0\}$. However, by taking a $(l-1)$-dimensional sphere

$$S = \{(x_1, \dots, x_l) \in \mathbb{R}^l : x_1^2 + \dots + x_l^2 = r^2\}$$

for sufficiently small $r > 0$ and a point $\overset{o}{x} = (\overset{o}{x}_{l+1}, \dots, \overset{o}{x}_n) \in \mathbb{R}^{n-l}$ such that $\overset{o}{x}{}^2_{l+1} + \dots + \overset{o}{x}{}^2_n = r^2$, we see that $S \times \{\overset{o}{x}\} \subset U \setminus \{0\}$. The sphere $S \times \{\overset{o}{x}\}$ is contractible in $U \setminus \{0\}$ by the assumption. Let $H = (h_1, \dots, h_n) : S \times \{\overset{o}{x}\} \times [0,1] \to U \setminus \{0\}$ be a null-homotopy of $S \times \{\overset{o}{x}\}$ in $U \setminus \{0\}$. Then

$$h_1^2 + \dots + h_l^2 = h_{l+1}^2 + \dots + h_n^2 \quad \text{in} \quad S \times \{\overset{o}{x}\} \times [0,1].$$

Hence $h_1^2 + \cdots + h_l^2$ does not vanish anywhere in $S \times \{\overset{o}{x}\} \times [0,1]$, so $(h_1, \ldots, h_l)$ is a null-homotopy of S in $\mathbb{R}^l \setminus \{0\}$. This contradicts the assertion of Lemma 4. $\square$

Remark 2. *The assumption* $\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right] \neq 0$ *in Corollary 2 may not be omitted, because a polynomial* $f(x_1, x_2) = x_1^3 - x_2^3$ *does not satisfy this assumption for* $a = 0$ *and* $f^{-1}(0) = \{(t,t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ *is a topological manifold of dimension* 1.

In the proof of the considered implication the well known Bochnak and Łojasiewicz inequality [1] play the dominant role.

Lemma 5. (Bochnak-Łojasiewicz inequality) *Let* $0 < \theta < 1$. *If the function* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is analytic, then*

$$|x||\nabla f(x)| \geqslant \theta |f(x)| \qquad \text{in some neighbourhood of } 0.$$

Proof of implication (b)⇒(c). The assumption (b) implies that k-th Taylor polynomial h of function f is nonzero. Otherwise the functions $f_1(x) = 0$, $f_2(x) = x_1^{k+1}$ would be the $\mathcal{C}^k$-realizations of a k-jet which is V-sufficient in the class $\mathcal{C}^k$, which is impossible. Hence, in case $n = 1$, $\mathcal{L}_0(\nabla f) = \operatorname{ord}_0 f' \leqslant k - 1$. This gives (c) in this case. Assume therefore that $n > 1$.

In the case $k = 1$ from (b) it follows $\nabla f(0) \neq 0$. In fact, otherwise for the two $\mathcal{C}^1$ realizations $f_1(x) = x_1^2$ and $f_2(x) = x_1 x_2$ of the 1-jet v the sets $f_1^{-1}(0)$ and $f_2^{-1}(0)$ would be homeomorphic, in some neighbourhoods of zero, which is impossible. The condition $\nabla f(0) \neq 0$ obviously implies (c). Therefore we may assume that $k > 1$.

Since

$$\lim_{x \to 0} \frac{\nabla f(x) - \nabla h(x)}{|x|^{k-1}} = 0,$$

$\mathcal{L}_0(\nabla f) \leqslant k - 1$ if and only if $\mathcal{L}_0(\nabla h) \leqslant k - 1$. Hence, it is sufficient to verify the implication for $f = h$.

Assume to the contrary that (c) is not satisfied. Then, for a sequence $(a_\nu) \subset \mathbb{R}^n \setminus \{0\}$ such that $a_\nu \to 0$ as $\nu \to \infty$, we have

$$\frac{|\nabla f(a_\nu)|}{|a_\nu|^{k-1}} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{18}$$

Therefore, the Bochnak-Łojasiewicz inequality (Lemma 5) gives

$$\frac{|f(a_\nu)|}{|a_\nu|^k} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{19}$$

Taking a subsequnce of (a_ν), we may suppose that $|a_{\nu+1}| \leqslant \frac{1}{2}|a_\nu|$ for $\nu \in \mathbb{N}$. Then

$$B_\nu = \{x \in \mathbb{R}^n : |x - a_\nu| \leqslant \frac{1}{4}|a_\nu|\}, \quad \nu \in \mathbb{N}, \quad \text{is a family of disjoint closed balls.}$$

Let us take an arbitrary sequence $(\lambda_\nu) \subset \mathbb{R}$ such that

$$\frac{\lambda_\nu}{|a_\nu|^{k-2}} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{20}$$

Since $k > 1$, we may assume that

$$\lambda_\nu \quad \text{is not an eigenvalue of the matrix} \quad \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a_\nu)\right]. \tag{21}$$

Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ such that $\alpha(x) = 0$ for $|x| \geqslant \frac{1}{4}$ and $\alpha(x) = 1$ in some neighbourhood of 0. Consider a mapping $F : \mathbb{R}^n \to \mathbb{R}$ defined by the formulas

$$F(x) = \alpha\left(\frac{x - a_\nu}{|a_\nu|}\right)\left(f(a_\nu) + d_{a_\nu} f(x - a_\nu) + \frac{1}{2}\lambda_\nu |x - a_\nu|^2\right) \quad \text{for} \quad x \in B_\nu$$

and $F(x) = 0$ for $x \notin \bigcup_{\nu=1}^\infty B_\nu$. Then F is of class $\mathcal{C}^k$ (even of class $\mathcal{C}^\infty$) and $F(0) = 0$. Moreover $f(a_\nu) = F(a_\nu)$ and $\nabla f(a_\nu) = \nabla F(a_\nu)$, so

$$(f - F)(a_\nu) = 0 \quad \text{i} \quad \nabla(f - F)(a_\nu) = 0 \qquad \text{for} \quad \nu \in \mathbb{N}. \tag{22}$$

Let $M > 0$ be a constant such that $|\alpha(x)| \leqslant M$ for $x \in \mathbb{R}^n$. Then for $x \in B_\nu$,

$$\begin{aligned}\frac{|F(x)|}{|x|^k} &\leqslant M \frac{|f(a_\nu) + d_{a_\nu} f(x - a_\nu) + \frac{1}{2}\lambda_\nu |x - a_\nu|^2|}{|x|^k} \\ &\leqslant 2^k M \frac{|f(a_\nu)| + |\nabla f(a_\nu)||a_\nu| + \frac{1}{2}|\lambda_\nu||a_\nu|^2}{|a_\nu|^k}.\end{aligned}$$

Hence, (18), (19) and (20) implies

$$\frac{|F(x)|}{|x|^k} \to 0 \qquad \text{as} \qquad x \to 0.$$

In consequence, $f - F$ is a $\mathcal{C}^k$-realization of k-jet v. In view of (22) and the assumption (b), Lemma 2 implies that $(f - F)^{-1}(0)$ is a $(n-1)$-dimensional topological manifold in every sufficiently small neighbourhood of the point $0 \in \mathbb{R}^n$. On the other hand, (21) gives

$$\det\left[\frac{\partial^2 (f - F)}{\partial x_i \partial x_j}(a_\nu)\right] \neq 0 \qquad \text{for} \quad \nu \in \mathbb{N}.$$

This with (22) and Proposition 2 implies that $(f - F)^{-1}(0)$ is not a topological manifold of dimension $n - 1$ in any neighbourhood of a_ν. In particular it is not a topological manifold in any neighbourhood of 0 (because $a_\nu \to 0$). This contradiction yields the truth of the considered implication. □

5. Equivalence of mappings at infinity

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $f : \mathbb{K}^n \to \mathbb{K}$. By the *Łojasiewicz exponent at infinity* of gradient ∇f, denoted by $\mathcal{L}_\infty(\nabla f)$, we mean the supremum of exponents $\nu \in \mathbb{R}$ in the following *Łojasiewicz inequality*:

$$|\nabla f(x)| \geq C|x|^\nu \text{ as } |x| > R \text{ for some constants } C > 0 \text{ and } R > 0.$$

It is known that for a polynomial function f we have $\mathcal{L}_\infty(\nabla f) \in \mathbb{Q} \cup \{-\infty\}$ and $\mathcal{L}_\infty(\nabla f) > -\infty$ if and only if the set $(\nabla f)^{-1}(0)$ is finite.

Similar considerations (as in the above sections of this paper) may be carried out for functions in neighbourhoods of infinity. In the case of polynomials in two complex variables P. Cassou-Noguès and H. H. Vui [2, Theorem 5] proved that:

Let $f \in \mathbb{C}[z_1, z_2]$, $\mathcal{L}_\infty(\nabla f) \geq 0$ and $k \in \mathbb{Z}$, $k \geq 1$. The following conditions are equivalent:
(i) $\mathcal{L}_\infty(\nabla f) \geq k-1$,
(ii) there exists $\varepsilon > 0$, such that for every polynomial $P \in \mathbb{C}[z_1, z_2]$ of degree $\deg P \leq k$, whose modules of coefficients of monomials of degree k are less or equal ε, the links at infinity of almost all fibers $f^{-1}(\lambda)$ and $(f+P)^{-1}(\lambda)$, $\lambda \in \mathbb{C}$ are isotopic.

Recall that by link at infinity of the fiber $P^{-1}(\lambda)$ of a polynomial $P : \mathbb{C}^2 \to \mathbb{C}$ we mean the set $P^{-1}(\lambda) \cap \{(x,y) \in \mathbb{C}^2 : \ |x|^2 + |y|^2 = r^2\}$ for sufficiently large r.

The above result of P. Cassou-Noguès and H. H. Vui was generalized by G. Skalski [27, Theorems 3, 7]:

Let $f \in \mathbb{K}[x_1, \ldots, x_n]$, let $k \in \mathbb{Z}$, $k \geq 0$, and let $\mathcal{L}_\infty(\nabla f) \geqslant k-1$. Then there exists $\varepsilon > 0$, such that for each polynomial $P \in \mathbb{K}[x_1, \ldots, x_n]$ of degree $\deg P \leq k$, whose modules of coefficients of monomials of degree k does not exceed ε, polynomials f and $f+P$ are analytically equivalent at infinity.

We say that functions $f, g : \mathbb{K}^n \to \mathbb{K}$ are *analytically equivalent at infinity* when there exists an analytic diffeomorphism φ of neighbourhoods of infinity, such that $|\varphi(x)| \to \infty$ if and only if $|x| \to \infty$ and there exists an analytic diffeomorphism $\psi : \mathbb{K} \to \mathbb{K}$, such that

$$f \circ \varphi = \psi \circ g \quad \text{in a neighbourhood of infinity.}$$

The inverse to the Skalski theorem is false (see [27, Remark 2]) .

The method of proof of this theorem is slightly similar to the proof of Theorem 2 in this article. It consists in integrating the appropriate vector field

$$W(\xi, x) = \frac{1}{X_1(\xi, x) - 1}[X_2(\xi, x), \ldots, X_{n+1}(\xi, x)],$$

where

$$X(\xi, x) = (X_1, \ldots, X_{n+1}) = \frac{P(x)}{|\nabla F(\xi, x)|^2} \nabla F(\xi, x)$$

and $F(\xi, x) = f(x) + \xi P(x)$ with $\overline{\nabla F(\xi, x)}$ instead of $\nabla F(\xi, x)$ in the complex case.

The method of integration of the field was used also in the result by Rodak and Spodzieja [26, Theorem 1]:

Let $f\colon \mathbb{K}^n \to \mathbb{K}^m$, where $m \leq n$, be a C^2 mapping (holomorphic if $\mathbb{K} = \mathbb{C}$). Assume that there exist $k \in \mathbb{R}$ and positive constants C, R such that

$$\nu(df(x)) \geq C|x|^{k-1}, \quad |x| \geq R. \tag{23}$$

Then there exists $\varepsilon > 0$ such that for any $P \in \mathcal{P}_{k,\varepsilon}$ the mappings f and $f+P$ are isotopic at infinity, where the symbol $\mathcal{P}_{k,\varepsilon}$ (for $k \in \mathbb{R}$, $\varepsilon > 0$) denotes all C^2 mappings $P\colon \mathbb{K}^n \to \mathbb{K}^m$, for which there exists $R > 0$ such that

$$|P(x)| \leq \varepsilon |x|^k \text{ and } |dP(x)| \leq \varepsilon |x|^{k-1} \text{ for any } |x| \geq R, \tag{24}$$

where $dP(x)$ is the diferential of P at $x \in \mathbb{K}^n$. The symbol ν stands for

$$\nu(A) = \inf\{\|A^*\varphi\| : \varphi \in Y', \|\varphi\| = 1\},$$

where A^* is the adjoint operator in the space of linear continuous mappings from Y' to X' and X', Y' are the dual spaces of Banach spaces X an Y respectively.

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