Analytic and Algebraic Geometry
Łódź University Press 2013, 33 – 40
http://dx.doi.org/10.18778/7969-017-6.03

NECESSARY CONDITIONS FOR IRREDUCIBILITY OF ALGEBROID PLANE CURVES

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let $f \in \mathbb{K}[[X]][Y]$ be monic. Using the properties of approximate roots given in [J. Algebra 343 (2011), pp. 143–159] we propose some necessary conditions for irreducibility of f in $\mathbb{K}[[X]][Y]$. The result is expressed only in terms of intersection multiplicities of f with its approximate roots.

1. Introduction

We recall that for a monic polynomial $f \in R[Y]$ of degree k, where R is a commutative ring with unity, and for a positive integer $l|k$ satisfying $\gcd(l, \operatorname{char} R) = 1$, there exists a unique monic polynomial $g \in R[Y]$ with the property

$$\deg_Y(f - g^l) < k - \frac{k}{l}.$$

The polynomial g is called an *approximate l-th root of f* and is denoted by $\sqrt[l]{f}$ (cf. [Abh77, Definition (4.3)]).

Now, let $\mathbb{K}$ be an algebraically closed field of *characteristic* 0, $\mathbb{K}[[X]]$ – the ring of power series in one variable X with coefficients in $\mathbb{K}$ and $\mathbb{K}((X))$ – its field of fractions. Let $f \in \mathbb{K}((X))[Y]$ be a *monic* and *irreducible* polynomial. In [Brz11] we proved an extension of the results of Abhyankar and Moh concerning approximate roots of f (see [AM73]) to the case of so-called 'non-characteristic' approximate roots of f. The necessary excerpt from [Brz11, Theorem 5] is given in Theorem 1. In the present work, we use this theorem and the properties of characteristic sequences to give some necessary conditions for the irreducibility of $f \in \mathbb{K}[[X]][Y]$ when $\operatorname{char} \mathbb{K} = 0$ (one can think $\mathbb{K} = \mathbb{C}$). These conditions are effective in the case

2010 *Mathematics Subject Classification.* Primary 12E05, Secondary 12E10, 14C17.

Key words and phrases. Approximate root, irreducibility condition, characteristic sequence, intersection multiplicity.

of $f \in \mathbb{K}[X, Y]$. Namely, Theorem 4 below can be easily turned into a test algorithm for reducibility, main point of which is the process of division with remainder (it serves to compute the intersection multiplicity (cf. [GP13]) and approximate roots (cf. [Brz11, Remark 1])).

Let us remark that the problem of testing irreducibility has been fully solved by Abhyankar in [Abh89], but his criterion is more technical than our numeric conditions as it involves analyzing the form of G-adic expansions of polynomials. From this criterion one can easily deduce necessary conditions for irreducibility ([Abh90, p. 183], presented in Theorem 2 below) similar in nature to ours (Theorem 4). We show by example (Example 2) that in general our necessary conditions are stronger than those in Theorem 2.

For an interesting combinatorial criterion of irreducibility see the recent work [GG10].

2. Characteristic Sequences (cf. [Abh77, § 6])

Let $\mathbb{K}$ be an algebraically closed field (for simplicity – of characteristic 0) and let $f \in \mathbb{K}((X))[Y]$ be a monic and irreducible polynomial. By Newton Theorem ([Abh77, Theorem (5.19)]), f can be written in the form

$$f(t^k, Y) = \prod_{\varepsilon \in U_k(\mathbb{K})} (Y - y(\varepsilon t)), \tag{2.1}$$

where $U_k(\mathbb{K}) := \{\varepsilon \in \mathbb{K} : \varepsilon^k = 1\}$ and $y(t) = \sum_{j \in \mathbb{Z}} y_j t^j \in \mathbb{K}((t))$. We recall that the *support* $\operatorname{Supp}_t y(t)$ of $y(t)$ is the set of those exponents of the powers of t that occur with a non-zero coefficient in the Laurent expansion of $y(t)$. Note also that from the irreducibility of f it follows that $\gcd(\{k\} \cup \operatorname{Supp}_t y(t)) = 1$.

The basic characteristic sequences of f. To begin with, we put $m_0 := k$, $d_1 := k$ and $m_1 := \operatorname{ord}_t y(t)$. If, now, $y(t) = 0$ then putting $h := 0$ we end the construction. In the opposite case, let $d_2 := \gcd(m_0, m_1)$. Inductively, if $m_0, \dots, m_i$ and $d_1, \dots, d_{i+1}$ are already defined for some $i \geqslant 1$, put

$$m_{i+1} := \inf\{j \in \operatorname{Supp}_t y(t) : j \not\equiv 0 (\operatorname{mod} d_{i+1})\}.$$

If, now, $m_{i+1} < +\infty$, we also define

$$d_{i+2} := \gcd(m_0, \dots, m_{i+1}),$$

whereas in the case $m_{i+1} = +\infty$ we put $h := i$ and finish the inductive definition.

Since in the above construction there is always $0 < d_{j+1} < d_j$ for $j \geqslant 2$, the process ends after finitely many steps. Thus we end up with two sequences:

$$m := (m_0, m_1, \dots, m_{h+1})$$

and

$$d := (d_1, \dots, d_{h+1}).$$

We call them, respectively: *the characteristic of* f and *the sequence of characteristic divisors of* f.

Using the sequences m and d we also define the following **derived characteristic sequence of** f:

$$r = (r_0, \ldots, r_{h+1}),$$

where $r_0 := m_0$, $r_i := \frac{1}{d_i}(m_1 d_1 + \sum_{2 \leqslant j \leqslant i} (m_j - m_{j-1}) d_j)$ for $1 \leqslant i \leqslant h$, and $r_{h+1} := +\infty$.

Note that the characteristic sequences defined above do not depend on the choice of a particular $y(t)$ satisfying (2.1).

Immediately from the definitions we get:

Property 1. *The sequences* m, d, r *are integer-valued (or* $+\infty$*). What is more,*

1. $h \geqslant 1$ *unless* $f = Y$,
2. $m_1 < m_2 < \ldots < m_{h+1} = +\infty$,
3. $d_{i+1} = \gcd(m_0, \ldots, m_i) = \gcd(d_i, m_i) = \gcd(d_i, r_i) = \gcd(r_0, \ldots, r_i)$ *for* $1 \leqslant i \leqslant h$,
4. $1 = d_{h+1} | d_h | \ldots | d_1 = k$ *and* $d_{h+1} < d_h < \ldots < d_2 \leqslant d_1$,
5. *if* $M \in \mathbb{Z} \cup \{+\infty\}$ *and* $m_{i-1} < M \leqslant m_i$ *for some* $i \in \{2, \ldots, h+1\}$ *(or only* $M \leqslant m_i$ *if* $i = 1$*), then*
$$\gcd(\{k\} \cup (\operatorname{Supp}_t y(t) \cap (-\infty, M))) = \gcd(m_0, \ldots, m_{i-1}) = d_i,$$
6. $r_i d_i = r_{i-1} d_{i-1} + (m_i - m_{i-1}) d_i$ *for* $2 \leqslant i \leqslant h$,
7. $r_1 d_1 < r_2 d_2 < \ldots < r_{h+1} d_{h+1} = +\infty$.

3. The Preliminary Result

We start with the following (here m, d, r are the characteristic sequences of f with $h + 1$ equal to the length of the divisor sequence d).

Theorem 1. *Let* $\mathbb{K}$ *be an algebraically closed field,* $\operatorname{char} \mathbb{K} = 0$, *let* $f \in \mathbb{K}((X))[Y]$ *be of the form (2.1) and let* l *be a positive divisor of* k. *Define* $i := \max\{1 \leqslant j \leqslant h+1 : l | d_j\}$. *Then*

$$\operatorname{ord}_t(\sqrt[l]{f}(t^k, y(t))) = r_i \frac{d_i}{l}. \tag{3.1}$$

Proof. The case $l \neq d_i$ is the non-characteristic case stated in [Brz11, Theorem 5, item 5]; if $l = d_i$ and $l \neq k$ then $2 \leqslant i$, and this is the characteristic case proved in [Abh77, Theorem (8.2)].

It remains to prove the case of $l = k$. Now, if $k = 1$ then $\sqrt[l]{f} = f$, $i = h + 1$ and $r_{h+1} = \infty$, so (3.1) is valid by the very definitions (cf. Section 2). Hence, in the following we may assume that $k \geqslant 2$. Property 1 implies that in this case

$$i \in \{1, 2\} \text{ and } d_1, \ldots, d_i = k; \tag{3.2}$$

also $h \geqslant i$ since $k \geqslant 2$. Let $f(t^k, Y) = Y^k + v(t^k)Y^{k-1} + \dots$. From Viète's formulas it follows that

$$v(t^k) = -\sum_{\varepsilon \in U_k(\mathbb{K})} y(\varepsilon t) = -\sum_{\varepsilon \in U_k(\mathbb{K})} \Big(\sum_{j<m_i} (y_j \varepsilon^j t^j) + y_{m_i}\varepsilon^{m_i} t^{m_i}\Big) + \text{ terms of order} > m_i.$$

By the definitions of i and the characteristic sequences of f, we have $d_{i+1} = \gcd(d_i, m_i) < d_i = k$ and also $\gcd(\{k\} \cup (\operatorname{Supp}_t y(t) \cap (-\infty, m_i))) = d_i = k$ (by Property 1). Consequently, for a k-th primitive root of unity $\varepsilon_0 \in U_k(\mathbb{K})$,

$$\begin{cases} \varepsilon_0^j = 1, & \text{if } j < m_i \\ \varepsilon_0^j \neq 1, & \text{if } j = m_i \end{cases}$$

and so

$$\sum_{\varepsilon \in U_k(\mathbb{K})} \varepsilon^j = \begin{cases} k, & \text{if } j < m_i \\ 0, & \text{if } j = m_i \end{cases} .$$

It follows that

$$v(t^k) = -k \cdot \sum_{j<m_i} y_j\, t^j + \text{ terms of order} > m_i.$$

Now, by the definition of an approximate root, one sees easily that $\sqrt[l]{f} = \sqrt[k]{f} = Y + \frac{v(t)}{k}$. Thus we have

$$\sqrt[l]{f}(t^k, y(t)) = y(t) + \frac{v(t^k)}{k} = y_{m_i}\, t^{m_i} + \text{ terms of order} > m_i,$$

and since $y_{m_i} \neq 0$,

$$\operatorname{ord}_t(\sqrt[l]{f}(t^k, y(t))) = m_i.$$

It remains to see that (according to (3.2))

$$m_i = \left\{ \begin{array}{ll} r_1, & \text{if } i = 1 \\ \frac{m_1 d_1 + (m_2 - m_1) d_2}{d_2} = r_2, & \text{if } i = 2 \end{array} \right\} = r_i \frac{d_i}{l}.$$

□

4. Necessary Conditions for Irreducibility

Throughout this section $\mathbb{K}$ denotes an algebraically closed field of characteristic 0.

Notation 1. *For monic polynomials $f, g \in \mathbb{K}[[X]][Y]$ we write $\mathcal{I}(f, g)$ to denote the* intersection multiplicity *of f and g at $0 = (0, 0)$, which is, by definition, equal to the dimension of the $\mathbb{K}$-vector space $\mathbb{K}[[X, Y]]/(f, g)$ (see e.g.* [Pło13, Section 3]*).*

We recall that a monic $f \in \mathbb{K}[[X]][Y]$ with $f(0) = 0$ is called *Y-distinguished* if $f = Y^k + a_1(X)\, Y^{k-1} + \ldots + a_k(X)$ and $a_1(0) = \ldots = a_k(0) = 0$.

The simplest test for reducibility is the following well-known

Property 2. *If a monic $f \in \mathbb{K}[[X]][Y]$, $f(0) = 0$, is not distinguished, then f is reducible in $\mathbb{K}[[X]][Y]$.*

Proof. This can be deduced from Hensel's Lemma. An alternative proof is the following. Suppose that f is irreducible. It is clear that f is also irreducible in $\mathbb{K}((X))[Y]$. By Newton Theorem we can assume that f is of the form (2.1). Since $f \in \mathbb{K}[[X]][Y]$ we have $y(t) \in \mathbb{K}[[t]]$ and since: $f(0) = 0$, $f(t^k, 0) = \pm \prod_{\varepsilon \in U_k(\mathbb{K})} y(\varepsilon t)$ — we have $y(0) = 0$. This means that f is distinguished. □

The above property implies that the only interesting case to deal with is that of a distinguished polynomial. Hence in the following we will consider only such polynomials. The starting point for our further considerations is:

Theorem 2 (**Abhyankar's Necessary Conditions for Irreducibility** [Abh90, p. 183])**.** *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree $k \geqslant 2$. Put $r'_0 := d'_1 := k$, $r'_1 := \mathcal{I}(f, Y)$, $d'_2 := \gcd(d'_1, r'_1)$ and then $r'_e := \mathcal{I}(f, \sqrt[d_e]{f})$, $d'_{e+1} := \gcd(d'_e, r'_e)$, for $e = 2, \ldots, h' + 1$, where the number $h' \geqslant 1$ is defined in such a way that $d'_{h'} > d'_{h'+1} = d'_{h'+2}$ and where (by convention) every integer divides ∞. If either*

(A1) $\quad d'_{h'+1} \neq 1$

or

(A2) $\quad$ *the sequence $(r'_1\, d'_1, \ldots, r'_{h'+1}\, d'_{h'+1})$ is not strictly increasing,*

then the polynomial f is reducible in $\mathbb{K}[[X]][Y]$.

Proof. If f is irreducible, one can use the Abhyankar-Moh result on characteristic approximate roots (cf. Theorem 1) and Property 1 item 3 to see that in such a case none of the above conditions hold. Indeed, it is enough to note that the sequences $(d'_1, \ldots, d'_{h'+1})$, $(r'_0, \ldots, r'_{h'+1})$ are in fact the characteristic sequences d, r (respectively) defined in section 2. □

Theorem 1 of section 3 can be restated as follows.

Theorem 3. *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree k. Let $(l_1, \ldots, l_a)$ be the strictly decreasing sequence of all the positive divisors of the number k. Define $\Delta := \{\delta_j : j = 0, \ldots, a\}$ where*

$$\delta_j := \mathcal{I}(f, \sqrt[l_j]{f}) \cdot l_j \ (j = 1, \ldots, a)$$

and

$$\delta_0 := \mathcal{I}(f, Y) \cdot k.$$

If f is irreducible in $\mathbb{K}[[X]][Y]$ and (m, d, r) denote the characteristic sequences of f with $h + 1$ equal to the length of the divisor sequence d, then

$$\Delta = \{r_e \cdot d_e : e = 1, \ldots, h + 1\}.$$

Proof. By the same argument as in the proof of Property 2, we can assume that f is of the form (2.1), where $y(t) \in \mathbb{K}[[t]]$ and $y(0) = 0$. Hence $(t^k, y(t))$ is a normalization of the algebroid curve $f = 0$. By the well-known property of intersection multiplicity, for any $g \in \mathbb{K}[[X]][Y]$ we have (cf. [Cam80, Chapter 2.3] or [Pło13])

$$\mathcal{I}(f,g) = \operatorname{ord}_t g(t^k, y(t)).$$

Thus, $\delta_0 = \mathcal{I}(f,Y)\cdot k = \operatorname{ord}_t y(t)\cdot k = m_1 k = r_1 d_1$. Moreover, from Theorem 1 and the definition of the derived sequence r it follows that

$$\delta_j = \mathcal{I}(f, \sqrt[l_j]{f})\cdot l_j = \operatorname{ord}_t \sqrt[l_j]{f}(t^k, y(t))\cdot l_j \in \{r_e\cdot d_e : e = 1,\dots,h+1\},$$

for $j = 1,\dots,a$. In particular, if $l_j = d_e < k$ we have $\delta_j = r_e d_e$, for $e = 2,\dots,h+1$; if $l_j = d_2 = k$, we still have $\delta_j = r_2 d_2$. Consequently, $\Delta = \{r_e\cdot d_e : e = 1,\dots,h+1\}$. □

Now we can strengthen Abhyankar's criterion.

Theorem 4. *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree $k \geqslant 2$. Define the sequences d', r' as in Theorem 2 and the set Δ as in Theorem 3. If any of the conditions $(A1)$, $(A2)$,*

$(B1)$ $\quad \Delta \neq \{r'_e\, d'_e : 1 \leqslant e \leqslant h'+1\}$

or

$(B2)$ *there exists $j \in \{1,\dots,a\}$ such that for $i := \max\{1 \leqslant e \leqslant h'+1 : l_j | d'_e\}$ it is*

$$\delta_j \neq r'_i\, d'_i$$

holds, then f is reducible in $\mathbb{K}[[X]][Y]$.

Proof. As in the proof of Theorem 2, if f is irreducible then the sequences $(d'_1,\dots, d'_{h'+1})$, $(r'_0,\dots,r'_{h'+1})$ are in fact the characteristic sequences d and r of f. Hence the condition $(B1)$ is fulfilled by Theorem 3. As for condition $(B2)$, putting $i(l_j) := \max\{1 \leqslant e \leqslant h'+1 : l_j | d'_e\}$ for $j = 1,\dots,a$, thanks to Theorem 1 we get

$$\delta_j = \mathcal{I}(f, \sqrt[l_j]{f})\cdot l_j = r'_{i(l_j)}\frac{d'_{i(l_j)}}{l_j}\cdot l_j = r'_{i(l_j)} d'_{i(l_j)}, \text{ for } j = 1,\dots,a.$$

This finishes the proof. □

We illustrate Theorem 4 with some examples.

Example 1. *Take Kuo's example considered in* [Abh89]*:*

$$f := (Y^2 - X^3)^2 - X^7.$$

We easily compute $\sqrt[4]{f} = Y$, $\sqrt[2]{f} = (Y^2 - X^3)$ and, naturally, $\sqrt[1]{f} = f$. Hence $(r'_1 d'_1,\dots,r'_{h'+1} d'_{h'+1}) = (6\cdot 4, 14\cdot 2)$. By the condition $(A1)$ of Theorem 2 we deduce that f is reducible. *Now we change f a little:*

$$f := (Y^2 - X^3)^2 - 4X^5Y - X^7.$$

The approximate roots are as before but now $(r'_1 d'_1,\dots,r'_{h'+1} d'_{h'+1}) = (6\cdot 4, 13\cdot 2, \infty\cdot 1)$. Moreover, $(\delta_j)_{j=0}^3 = (24, 24, 26, \infty)$. This easily implies that none of the *conditions* $(A1)$–$(B2)$ *of Theorem 4 is fulfilled and we may suspect (which is indeed the case) that* f *is* irreducible.

The next example shows that the conditions $(B1)$–$(B2)$ of Theorem 4 are sometimes stronger than Abhyankar's conditions $(A1)$–$(A2)$.

Example 2. *Consider* $f \in \mathbb{C}[[X]][Y]$ *of the form*

$$\begin{aligned} f :=& (Y^2 - X)^6 - 2\,X^3Y(Y^2 - X)^3 - 24\,X^4Y(Y^2 - X)^2 \\ &+ (-32\,X^5Y + X^6)(Y^2 - X) + 64\,X^8Y. \end{aligned}$$

One easily checks that

$$\begin{aligned} \sqrt[2]{f} &= (Y^2 - X)^3 - X^3Y \\ \sqrt[3]{f} &= (Y^2 - X)^2 \\ \sqrt[4]{f} &= Y^3 - \frac{3}{2}XY \\ \sqrt[6]{f} &= Y^2 - X \\ \sqrt[12]{f} &= Y \end{aligned}$$

and then $\delta_0 = \delta_1 = \mathcal{I}(f, Y) \cdot 12 = 6 \cdot 12 = 72$, $\delta_2 = \mathcal{I}(f, \sqrt[6]{f}) \cdot 6 = 17 \cdot 6 = 102$, $\delta_3 = \mathcal{I}(f, \sqrt[4]{f}) \cdot 4 = 18 \cdot 4 = 72$, $\delta_4 = \mathcal{I}(f, \sqrt[3]{f}) \cdot 3 = 34 \cdot 3 = 102$, $\delta_5 = \mathcal{I}(f, \sqrt[2]{f}) \cdot 2 = 40 \cdot 2 = 80$, $\delta_6 = \mathcal{I}(f, \sqrt[1]{f}) = \infty$. *Hence* $\Delta = \{72, 80, 102, \infty\}$.

On the other hand, performing the test of Theorem 2, we have $(r'_e\, d'_e)_{e=1,\dots,h'+1} = (6 \cdot 12, 17 \cdot 6, \infty \cdot 1) = (72, 102, \infty)$ *which easily shows that the conditions* $(A1)$–$(A2)$ *are not fulfilled. Hence in this case one cannot decide reducibility of* f *using the criterion of Theorem 2. But since* $\Delta \supsetneqq \{r'_e\, d'_e : e = 1, \dots, h' + 1\}$, *the condition* $(B1)$ *of Theorem 4 is fulfilled and we may conclude that* f *is reducible.*

Remark. *Abhyankar's criterion (Theorem 2) is valid over any algebraically closed field* $\mathbb{K}$ *of characteristic* $\operatorname{char}\mathbb{K} =: p$ *as long as* $k \not\equiv 0 \pmod{p}$. *Theorem 4, however, requires even more assumptions in such generality. Namely, in the notations of Theorem 4, for every positive divisor* l *of the number* k *one has to assume that* $\binom{\frac{d'_{i+1}}{l} - 1}{u} \cdot \mathbf{1} \neq 0$ *in* $\mathbb{K}$, *where* $i := \max\{1 \leqslant e \leqslant h' + 1 : l | d'_e\}$ *and* $u := \max\{0 \leqslant e \leqslant \frac{d'_{i+1}}{l} : \binom{\frac{d'_{i+1}}{l}}{e} \cdot \mathbf{1} \neq 0$ *in* $\mathbb{K}\}$. *This follows from* Theorem 11 *in* [Brz08] *which generalizes* Theorem 5 *of* [Brz11], *the main ingredient for the results of the present paper.*

Acknowledgement. *We wish to thank the anonymous referee whose remarks have led to a significant simplification of the exposition.*

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