A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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SOME ASPECTS OF ESTIMATION AND PREDICTION EFFICIENCY OF SELECTED NONLINEAR TRENDS

1. INTRODUCTORY

Despite of development of more and more complex methods of prediction there is a big demand for the simpliest of them - methods based on extrapolation of trend. Among nonlinear trend models two trends dominate:
- the exponential trend, which describes "expanding development";
- the logistic trend with a horizontal asymptote, which can be interpreted as the saturation level of the examined process.

It is rather easy to compute the parameters of logistic curve when we have 3 or more points laying exactly on this curve. The situation is more complicated when the observed values of variable contain an error.

There are a lot of techniques of calculating the parameters of logistic trend. They can be compared from the point of view of
- numerical effectiveness i.e. if (how often) the method leads to resonable results;
- efficiency in the statistical sense - then we need the statistical model of the process.

In this paper, we analyse the efficiency of extrapolation of the logistic trend, assuming that the errors in data are random

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"white noise". We compare the logistic curve with two its "competitors".

2. THE PROBLEM

Let us consider three following trend models

$$y_t = f_h(t) + \varepsilon_t \tag{1}$$

where ε_t is the "white noise" (h = 0, 1, 2) and

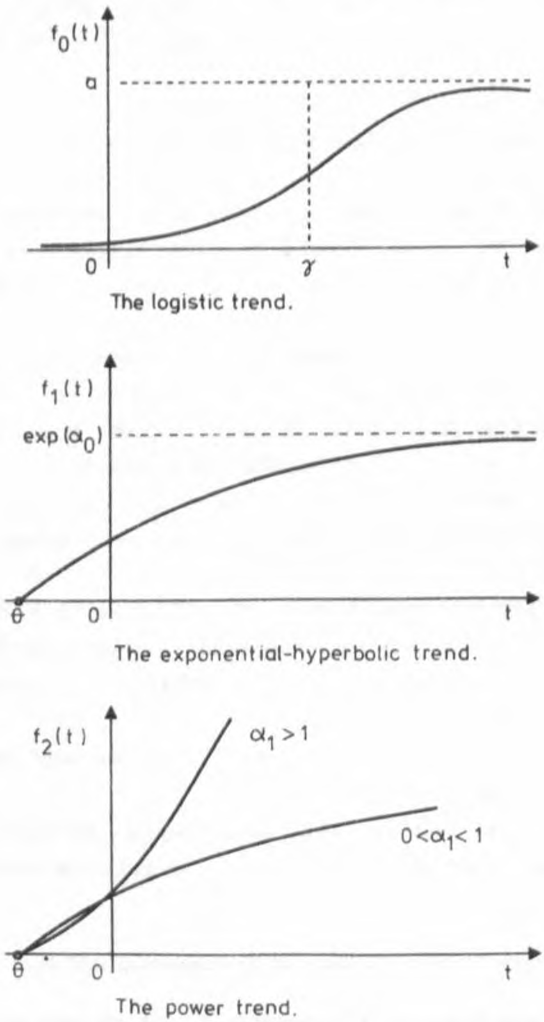

Fig. 1. The shapes of three nonlinear trends

$$f_0(t) = \frac{a}{1 + b\exp(-ct)} \tag{2}$$

(logistic trend),

$$f_1(t) = \exp(\alpha_0 + \frac{\alpha_1}{t + \theta}) \tag{3}$$

(exponential-hyperbolic trend),

$$f_2(t) = \exp(\alpha_0 + \alpha_1 \ln(t + \theta)) \tag{4}$$

(power trend).

We limit our study to increasing trends i.e. when

$$a > 0,\ b > 0,\ c > 0 \quad \text{for logistic curve}$$

$$\alpha_1 < 0,\ \ \theta > -1 \quad \text{for } f_1,\ \ \alpha_1 > 0,\ \theta > -1 \text{ for } f_2. \tag{5}$$

The shapes of f_0, f_1, f_2 functions can be seen on the Figure 1.

3. ESTIMATION METHODS

Let us assume that we have a sample consisting of n observations $y_1, y_2, \ldots, y_n$ generated by the proccess $\{\varepsilon_t\}$.

We use least-squares criterion of estimation of the logistic function parameters

$$\min \sum_{t=1}^{n} (y_t - f_0(t))^2. \tag{6}$$

From numerical techniques of minimization we have chosen the Gauss-Newton method.

According to the suggestions, of the paper by G o r y l and W a l k o s z (1985), we have used three sums methods as a main technique of calculating the starting point. When it failes, we can use following methods:

- the Hotelling method,
- the Stanisz method,
- three means method,
- three medians method,
- three harmonic means method.

We use two methods of estimating f_1 and f_2 functions. The first one is based on another criterion

$$\min \sum_{t=1}^{n} (\ln y_t - \ln f_h(t))^2. \tag{7}$$

To calculate the minimum with respect to α_0 and α_1 for fixed θ, we apply Ordinary Least Square Method for logarithms y_t (in this case we have linear model). This method is optimal for the model with a multiplicative lognormal random term

$$y_t = f_h(t)\eta_t,$$

but it is not efficient for the trend (1) with the additive random term. We have chosen this method because of some traditions and because it is easier to apply then the Gauss-Newton method.

The second method used for estimation of f_1 and f_2 functions is the Gauss-Newton method, with the starting point calculated as above.

The first method is described in the paper by C z y ż e w s k i and T o m a s z e w i c z (1977), and in monography by T o m a s z e w i c z (1975) p. 344-354 too. The analysis was limited there to integer values of θ and $\theta \leq 512$ only. It was caused by the fact, that the sum of squares was decreasing together with the growth of θ, and large changes of θ hardly influenced the values of function (7).

Both methods are used in two variants:

$$\theta \leq 2n, \tag{8}$$

$$\theta \leq 500. \tag{9}$$

4. THE PREDICTION EFFICIENCY CRITERION

Let s be a natural number, s > n. As the basic criterion of prediction we assumed the mean square error, which (with the fixed number of observations n and model (1)) is as follows

$$d^2_{hjs} = E(\hat{y}_{hjs} - y_{hs})^2 \tag{10}$$

where

$$\hat{y}_{hjs} = \hat{f}_{hj}(s)$$

is a forecast for s period calculated for h model using j method ($\hat{f}_{hj}$ denotes the estimate of h function by means of j method), and

$$y_{hs} = f_h(s)$$

denotes expected value of variable y forecasted in s period, when the h model is true.

5. THE RANGE OF RESEARCH

To study the efficiency of prediction on the basis of functions f_0, f_1, and f_2 we have used the Monte-Carlo method.

To take a large number of parameters into account we should consider 5-6 values of each of four parameters: n, c, R^2, γ. It would lead us to about one thousand variants, so in order to save computer time the experiment was planned as follows.

The parameter a is equal to 1 in all variants.

The values of parameter b depends on segment of the logistic curve, from which the observations came. We fixed a point in the sample (represented by parameter γ), where the logistic curve had the inflexion point. The logistic curve has the inflexion point for $t = \gamma$, where

$$\gamma = \frac{\ln(b)}{c},$$

hence, we can calculate parameter b for fixed γ:

$$b = \exp(\gamma c).$$

The variance of random term

$$D^2\varepsilon_t = \sigma^2$$

in all variants depended on fixed value of the determination coefficient R^2:

$$R^2 = 1 - \frac{\sigma^2}{\sigma_y^2},$$

where

$$\sigma_y^2 = \frac{1}{n}\sum_{t=1}^{n}(y_t - \bar{y})^2$$

and

$$\bar{y} = \frac{1}{n}\sum_{t=1}^{n} y_t,$$

$$y_t = \frac{a}{1 + b\exp(-ct)}.$$

R^2 denotes fixed determination coefficient (multiple correlation coefficient in the population), while we denoted its estimate in the sample by $\hat{R}^2$.

So, the variants of experiment differed from each other in values of four parameters:

n - the sample size;

c - the parameter of the logistic curve;

R^2 - fixed value of determination coefficient;

γ - the number defining, which part of the logistic curve is included in the sample (the position of the inflexion point).

We separated 6 variants (we shall call them basic variants) denoted by a1, a2, a3, a4, a5, a6. The values of control parameters for these variants are contained in Table 1.0.

Table 1.0

Data in the experiment - basic variants

c	Gamma	R^2	n	Variant
0.10	0.00	0.95	20	a1
0.10	0.50	0.95	20	a2
0.10	1.00	0.95	20	a3
0.20	0.00	0.95	20	a4
0.20	0.50	0.95	20	a5
0.20	1.00	0.95	20	a6

Source: The author´s calculations.

Next we made each of the four controlled parameters "variable", keeping three remaining parameters fixed as in the basic variants. Using this procedure we have got additionally

- 30 variants b1-b30 (variable n),
- 30 variants c1-c30 (variable R^2),
- 18 variants d1-d18 (variable c),
- 12 variants e1-e12 (variable γ).

Thus the total number of planned variants was 96.

Parameters assumed in all variants of the experiment are contained in Tables 1.1-1.4. For each variant we calculated

1^0 Values of the logistic curve

$$y_t^{(0)} = f_0(t) = \frac{a}{1 + b\exp(-ct)}.$$

Table 1.1

Data in the experiment - variants with variable n

c	Gamma	R^2	n = 10	n = 15	n = 20	n = 30	n = 40	n = 50
0.10	0.00	0.95	b1	b2	a1	b3	b4	b5
0.10	0.50	0.95	b6	b7	a4	b8	b9	b10
0.10	1.00	0.95	b11	b12	a2	b13	b14	b15
0.20	0.00	0.95	b16	b17	a5	b18	b19	b20
0.20	0.50	0.95	b21	b22	a3	b23	b2	b25
0.20	1.00	0.95	b26	b27	a6	b28	b29	b30

Source: As Table 1.0.

Table 1.2

Data in the experiment - variants with variable R^2

c	Gamma	R^2						
		n	0.80	0.90	0.95	0.98	0.99	0.995
0.10	0.00	20	c1	c2	a1	c3	c4	c5
0.10	0.50	20	c6	c7	a4	c8	c9	c10
0.10	1.00	20	c11	c12	a2	c13	c14	c15
0.20	0.00	20	c16	c17	a5	c18	c19	c20
0.20	0.50	20	c21	c22	a3	c23	c2	c25
0.20	1.00	20	c26	c27	a6	c28	c29	c30

Source: As Table 1.0.

Table 1.3

Data in the experiment - variants with variable c

Gamma	R^2	n	c							
			0.01	0.05	0.10	0.15	0.20	0.25	0.30	0.50
0.00	0.95	20	d1	d2	a1	d3	a4	d4	d5	d6
0.50	0.95	20	d7	d8	a2	d9	a5	d10	d11	d12
1.00	0.95	20	d13	d14	a3	d15	a6	d16	d17	d18

Source: As Table 1.0.

2^0 Values

$$y_t^{(1)} = f_1(t) = \exp(\alpha_0 + \frac{\alpha_1}{t + \theta})$$

$$y_t^{(2)} = f_2(t) = \exp(\alpha_0 + \alpha_1 \ln(t + \theta)).$$

Table 1.4

Data in the experiment - variants with variable gamma

c	R^2	n	Gamma = -0.50	-0.25	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.10	0.95	20	e1	e3	a1	e5	a2	e7	a3	e9	e11
0.20	0.95	20	e2	e4	a4	e6	a5	e8	a6	e10	e12

Source: As Table 1.0.

We chose the parameters α_0, α_1, Θ in order to minimize

$$\Sigma \ (y_t^{(h)} - y_t^{(0)})^2$$

(h = 1, 2).

3^0 We generated q = 100 samples

$$y_{qt}^{(h)} = y_t^{(h)} + \varepsilon_t^{(h)}$$

for every model (h = 0, 1, 2), where $\varepsilon_t^{(h)}$ is a random value drawn from the normal distribution

$N(0, \sigma^2)$.

4^0 In each sample for every sequence $y_t^{(h)}$, h = 0, 1, 2 we calculated estimates of the parameters of

- the logistic function f_0 (method 0);
- the f_1 function
  = using linear transformation with the condition $\Theta \le 2n$ (method 1),
  = using linear transformation with the condition $\Theta \le 500$ (method 2),
  = using the Gauss-Newton method (without linear approximation) with the condition $\Theta \le 2n$ (method 3),
  = using the Gauss-Newton method (without linear approximation) with the condition $\Theta \le 500$ (method 4),
- f_2 function, using the methods analogous to f_1 function (methods 5-8);

and estimate of the determination coefficient R^2. In result, for every sample we got 27 function estimates, on the basis of which we calculated forecasts of the variable y.

We have defined 9 methods of trend model estimation. Formally each of them is the method of model (1) estimation for suitable

h. When we do not know the shape of f_h function, but we forecast, using one of three: f_0, f_1, f_2, the choice of the method is relevant to the choice of the model. From this point of view, we can treat each method as the method of extrapolation y_t sequence (the method of prediction), no matter which model we actually deal with. (Of course, properties of each method depend on the model).

In the further part of the text, we denote described methods by m0, m1, ..., m8. For the models (2)-(4) we use symbols f_0, f_1, f_2.

We must pay attention to the fact, that the Gauss-Newton method (like most procedures of nonlinear function estimation) in some cases may appear ineffective (divergent or convergent to the estimates) not complying with conditions (2). When such a situation took place during the choice of sequences $y_t^{(1)}$ or $y_t^{(2)}$ (alternative to the logistic trend - see 2^0), then, the calculations for this variant could not be done - those variants were eliminated from the experiment. The list of them is as follows

b18, b19, b20, b28, b29, b30,
c26,
d8, d10, d11, d12, d16, d17, d18,
e10, e11, e12.

As a result, there were 79 variants left in the experiment.

6. RESULTS AND CONCLUSIONS

The experiment was quite wide. Thus, we cannot present results concerning all variants. In this paper we shall show only some of them.

If the Gauss-Newton procedure did not give a result for a certain sample, then

- for functions 1 and 2 (the methods 1-8), we took the "substitute" estimates of the parameters;
- for the logistic function (the method 0) this fact was noted and we did not use any substitute procedure.

The number of samples, for which the Gauss-Newton method was inefficient, is displayed in Table 2 (in the table we omitted those variants for which the Gauss-Newton method gave result for all samples).

Table 2

Number of samples, for which the procedure of the logistic curve estimation (method 0) was inefficient

Var.	c	Gamma	R^2	n	f_0	f_1	f_2
a2	0.10	0.50	0.95	20	1	1	1
a3	0.10	1.00	0.95	20	17	18	18
a5	0.20	0.50	0.95	20	0	1	0
a6	0.20	1.00	0.95	20	17	15	14
b1	0.10	0.00	0.95	10	18	19	20
b2	0.10	0.00	0.95	15	3	3	3
b6	0.20	0.00	0.95	10	2	2	3
b11	0.10	0.50	0.95	10	26	13	26
b12	0.10	0.50	0.95	15	6	6	6
b16	0.20	0.50	0.95	10	9	10	10
b21	0.10	1.00	0.95	10	32	32	33
b22	0.10	1.00	0.95	15	16	16	17
b23	0.10	1.00	0.95	30	4	7	7
b24	0.10	1.00	0.95	40	3	5	4
b25	0.10	1.00	0.95	50	9	22	25
b26	0.20	1.00	0.95	10	17	18	19
b27	0.20	1.00	0.95	15	8	9	10
c1	0.10	0.00	0.80	20	12	14	16
c2	0.10	0.00	0.90	20	0	1	1
c11	0.10	0.50	0.80	20	23	26	26
c12	0.10	0.50	0.90	20	10	12	13
c16	0.20	0.50	0.80	20	7	9	12
c17	0.20	0.50	0.90	20	0	1	4
c21	0.10	1.00	0.80	20	35	36	36
c22	0.10	1.00	0.90	20	24	25	25
c23	0.10	1.00	0.98	20	1	1	1
c27	0.20	1.00	0.90	20	25	25	28
c28	0.20	1.00	0.98	20	5	4	3
d1	0.01	0.00	0.95	20	47	47	47
d2	0.05	0.00	0.95	20	15	15	15
d7	0.01	0.50	0.95	20	48	48	47
d13	0.01	1.00	0.95	20	45	44	43
d14	0.05	1.00	0.95	20	29	28	29
d15	0.15	1.00	0.95	20	13	15	16
e7	0.10	0.75	0.95	20	5	5	6
e8	0.20	0.75	0.95	20	2	2	3
e9	0.10	1.25	0.95	20	22	22	21

Source: As Table 1.0.

Results in Table 2 suggest that the estimation of the logistic trend is the most difficult when

- we observe a segment, in which it is approximately linear;
- we observe its first segment;
- the sample is small;
- the determination coefficient is relatively small.

The difference between the numbers of samples, in which the Gauss-Newton method did not give a result for the models f_0, f_1, f_2 is generally small with a some superiority of the model f_0 (this superiority proved to be significant only in variant b25).

In practice, we generally do not know which model of f_0, f_1, f_2 is the best for the process which is being examined. It is worth having discrimination procedure, which allows to determine correct model. The discrimination can be based (in practice of the econometric research) on the value of the determination coefficient $\hat{R}^2$. In our analisis, we used this procedure in the following way.

In each sample from among these three methods 0, 4, 8, we choose this, for which the value of determination coefficient $\hat{R}^2$ is the largest. In this way the new method of trend estimation and prediction was determined, and we denote it by 10.

Analogously, we define the method 11 as the chosen one from the methods 0, 2, 6 (with the linear transformation), based on the value $\hat{R}^2$.

To estimate accurateness of the procedures of discrimination being used, we counted the samples, in which, for a given model, each of the mentioned methods was chosen. Results concerning a2 and some other variants are shown in Table 3. For instance, in the third row of Table 3, columns 6-8 contain numbers 59, 7, 34. This means that in variant a2 from three methods m0, m4, m8 following methods were chosen by means of the discrimination procedure based on $\hat{R}^2$:

m0 in 59 samples,
m4 in 7 samples,
m8 in 34 samples.

Here we estimate the discrimination of the methods 0, 4, 8 only. As to methods 0, 3, 7 general conclusions are the same.

In most samples the procedure of discrimination based on $\hat{R}^2$ results in choosing of the suitable model (i.e. for f_0 the lar-

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Number of samples when R^2 appeared maximum for method m0, m4, m8 respectively

Var.	c	Gamma	R^2	n	f_0			f_1			f_2			f_0			f_1			f_2		
					m0	m4	m8	m0	m4	m8	m0	m4	m8	m0	m3	m7	m0	m3	m7	m0	m3	m7
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
b11	0.10	0.50	0.95	10	51	2	47	51	3	46	49	2	49	52	1	47	55	2	43	50	1	49
b12	0.10	0.50	0.95	15	58	7	35	50	6	44	50	5	45	60	6	34	54	4	42	54	4	42
a2	0.10	0.50	0.95	20	59	7	34	46	13	41	47	12	41	60	6	34	48	11	41	50	10	40
b13	0.10	0.50	0.95	30	62	22	16	30	28	42	26	19	55	62	22	16	30	28	42	26	19	55
b14	0.10	0.50	0.95	40	77	17	6	35	34	31	20	30	50	77	17	6	35	34	31	20	30	50
b15	0.10	0.50	0.95	50	84	15	1	25	52	23	9	35	56	84	15	1	25	52	23	9	35	56
c11	0.10	0.50	0.80	20	45	9	46	39	11	50	39	11	50	47	6	47	40	7	53	39	7	54
c12	0.10	0.50	0.90	20	54	7	39	46	13	41	43	15	42	55	5	40	47	9	44	44	11	45
a2	0.10	0.50	0.95	20	59	7	34	46	13	41	47	12	41	60	6	34	48	11	41	50	10	40
c13	0.10	0.50	0.98	20	60	9	31	43	14	43	39	13	49	62	6	32	43	13	44	39	10	51
c14	0.10	0.50	0.99	20	64	9	27	39	17	44	34	17	49	64	8	28	40	14	46	34	15	51
c15	0.10	0.50	0.99	20	65	17	18	37	20	43	24	23	53	66	16	18	37	18	45	25	22	53
d7	0.01	0.50	0.95	20	30	0	70	30	0	70	31	0	69	37	0	63	37	0	63	38	0	62
a2	0.10	0.50	0.95	20	59	7	34	46	13	41	47	12	41	60	6	34	48	11	41	50	10	40
d9	0.15	0.50	0.95	20	63	10	27	42	16	42	35	19	46	63	10	27	42	16	42	35	18	47
a5	0.20	0.50	0.95	20	67	15	18	33	28	39	26	24	50	67	15	18	33	28	39	26	24	50
e1	0.10	-0.50	0.95	20	59	11	30	47	12	41	37	9	54	59	11	30	47	12	41	37	9	54
e3	0.10	-0.25	0.95	20	59	10	31	47	12	41	39	10	51	59	10	31	47	12	41	39	9	52
a1	0.10	0.00	0.95	20	57	11	32	46	12	42	40	11	49	58	9	33	47	11	42	40	11	49
e5	0.10	0.25	0.95	20	57	10	33	49	8	43	40	14	46	57	8	35	50	7	43	40	13	47
a2	0.10	0.50	0.95	20	59	7	34	46	13	41	47	12	41	60	6	34	48	11	41	50	10	40
e7	0.10	0.75	0.95	20	59	10	31	51	10	39	49	10	41	60	7	33	53	6	41	51	7	42
a3	0.10	1.00	0.95	20	52	9	39	44	10	46	43	10	47	52	7	41	45	7	48	44	7	49
e9	0.10	1.25	0.95	20	47	8	45	45	7	48	43	9	48	48	4	48	46	6	48	45	7	48

S o u r c e: As Table 1.0.

gest $\hat{R}^2$ is observed most often for m0, for f_1 - for m4, for f_2 - for m8). The only exception is m4, which is dominated by m0 in a lot of cases.

If the sample is small, m4 is distinguished the most rarely for all models f_0, f_1, f_2 (unless R^2 is large - variants c8-c10, or the parameter c is large - variants d4-d6). Moreover, m0 was choosen more often in the models f_0 and f_1, and m8 for f_2.

As the estimates of the bias of sample determination coefficient, there were taken differences

$$\frac{1}{q} \Sigma \hat{R}^2_{hj} - R^2_h$$

(h = 0, 1, 2 - model number, j = 0, ..., 8 - method number). These estimates for the methods m0, m4, m8, m10, for all models and some variants are given in Table 4 (the bias estimates were multiplied

Table 4

Estimates of bias of R^2 (per mille)

Var.	c	Gamma	R^2	n	f_0				f_1				f_2			
					m0	m4	m8	m10	m0	m4	m8	m10	m0	m4	m8	m10
a1	0.10	0.00	0.95	20	6	6	5	7	5	6	6	7	5	6	6	6
a2	0.10	0.50	0.95	20	6	5	5	6	6	6	6	6	5	5	6	6
a3	0.10	1.00	0.95	20	7	5	5	6	7	5	5	6	7	5	5	6
a4	0.20	0.00	0.95	20	6	2	-4	7	2	6	4	8	-3	3	5	6
a5	0.20	0.50	0.95	20	6	3	1	7	4	6	6	8	2	5	6	7
a6	0.20	1.00	0.95	20	9	-1	4	7	-25	-1	4	8	-87	-1	5	7
b11	0.10	0.50	0.95	10	20	17	17	17	19	17	17	18	20	17	17	17
b12	0.10	0.50	0.95	15	9	9	9	9	9	9	9	10	9	9	9	10
a2	0.10	0.50	0.95	20	6	5	5	6	6	6	6	6	5	5	6	6
b13	0.10	0.50	0.95	30	5	4	3	5	4	5	4	6	3	4	4	5
b14	0.10	0.50	0.95	40	2	-1	-3	3	0	2	1	3	-1	2	2	3
b15	0.10	0.50	0.95	50	2	-3	-10	2	-0	3	-1	3	-3	2	1	2

Source: As Table 1.0.

by 1000). Generally, the results are in accordance with expectations. The greater value of $\hat{R}^2$ is observed when the method suits the model (m0 for f_0, m4 for f_1, m8 for f_2) and (but a little smaller) for m10 although in these cases, particulary for small

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Estimates of mean square errors of prediction

Var.	c	Gamma	R^2	n	s	f_0				f_1				f_2			
						m0	m4	m8	m10	m0	m4	m8	m10	m0	m4	m8	m10
a1	0.10	0.00	0.95	20	1	0.019	0.019	0.019	0.018	0.020	0.018	0.017	0.018	0.021	0.018	0.017	0.018
					10	0.051	0.067	0.079	0.058	0.058	0.048	0.049	0.051	0.069	0.049	0.043	0.054
a2	0.10	0.50	0.95	20	1	0.026	0.028	0.028	0.025	0.027	0.026	0.025	0.025	0.027	0.026	0.025	0.025
					10	0.103	0.145	0.153	0.111	0.109	0.104	0.105	0.095	0.119	0.102	0.096	0.098
a3	0.10	1.00	0.95	20	1	0.024	0.023	0.024	0.023	0.025	0.023	0.023	0.023	0.025	0.023	0.023	0.023
					10	0.120	0.162	0.173	0.143	0.141	0.125	0.127	0.126	0.148	0.123	0.123	0.127
a4	0.20	0.00	0.95	20	1	0.019	0.027	0.040	0.021	0.027	0.019	0.023	0.023	0.033	0.024	0.019	0.021
					10	0.027	0.075	0.126	0.047	0.066	0.032	0.061	0.054	0.102	0.056	0.033	0.048
a5	0.20	0.50	0.95	20	1	0.044	0.066	0.070	0.048	0.055	0.042	0.042	0.043	0.059	0.044	0.040	0.044
					10	0.106	0.363	0.403	0.207	0.227	0.138	0.176	0.174	0.302	0.157	0.135	0.189
a6	0.20	1.00	0.95	20	1	0.038	0.053	0.047	0.038	0.046	0.053	0.043	0.037	0.055	0.050	0.037	0.037
					10	0.324	0.900	0.941	0.535	0.523	0.656	0.603	0.447	0.587	0.609	0.517	0.467
b11	0.10	0.50	0.95	10	1	0.018	0.017	0.017	0.018	0.017	0.017	0.017	0.018	0.019	0.017	0.016	0.018
					10	0.102	0.125	0.127	0.126	0.095	0.110	0.110	0.111	0.121	0.108	0.104	0.117
b12	0.10	0.50	0.95	15	1	0.022	0.022	0.021	0.022	0.023	0.021	0.021	0.022	0.023	0.021	0.021	0.022
					10	0.108	0.130	0.134	0.119	0.117	0.106	0.105	0.111	0.123	0.105	0.100	0.113
a2	0.10	0.50	0.95	20	1	0.026	0.028	0.028	0.025	0.027	0.026	0.025	0.025	0.027	0.026	0.025	0.025
					10	0.103	0.145	0.153	0.111	0.109	0.104	0.105	0.095	0.119	0.102	0.096	0.098
b13	0.10	0.50	0.95	30	1	0.028	0.032	0.036	0.029	0.034	0.026	0.025	0.028	0.037	0.027	0.024	0.029
					10	0.060	0.117	0.149	0.087	0.114	0.058	0.063	0.081	0.138	0.068	0.054	0.092
b14	0.10	0.50	0.95	40	1	0.030	0.044	0.056	0.034	0.045	0.029	0.030	0.038	0.049	0.033	0.028	0.037
					10	0.051	0.141	0.189	0.082	0.126	0.056	0.070	0.097	0.156	0.073	0.054	0.095
b15	0.10	0.50	0.95	50	1	0.027	0.059	0.097	0.033	0.050	0.027	0.048	0.036	0.056	0.032	0.029	0.033
					10	0.041	0.158	0.247	0.069	0.129	0.047	0.101	0.083	0.163	0.069	0.048	0.077

S o u r c e: As Table 1.0.

samples, the bias is generally the greatest. The bias of $\hat{R}^2$ decreases together with the growth of $\hat{R}^2$ and with the growth of n. If the method does not suit the model, we can generally observe the decrease of bias together with the growth of its absolute value when the size of sample n, parameters c, γ and r^2 grow.

As the measure of efficiency, we took the mean square error. Its estimate is

$$\hat{d}^2_{hjs} = \frac{1}{q} \Sigma (\hat{y}_{hjs} - y_{hs})^2.$$

The values

$$\hat{d}_{hjs} = \sqrt{\hat{d}^2_{hjs}}$$

which are the estimates of prediction errors are shown in Table 5 for some variants.

As far as the choice of the correct prediction method is concerned, values of relative measure of efficiency in relation to m0 are much more interesting. Quotients

$$e_{hjs} = d^2_{hjs}/d^2_{h0s} \tag{11}$$

for s = 1, 2, 3, 5, 10, 20 for basic variants are shown in Table 6 ($e_{hjs} > 1$, when the extrapolation of logistic function m_0 proved to be better). For the f_0 model, in accordance with expectations, the m_0 method turned out to be the most efficient. Only variants c1 and d1, d7, d13 with small n and c are exceptions (particulary for large γ - when the sample includes the first segment of logistic curve: variants a2, a3). The decline of efficiency is not large (exception: very small c: d1, d7, d13) and the longer period s of the forecast, the smaller the decline is. One should notice that the m10 method based on discrimination procedure is as good as m0 or is worse a little, if only s is not too large ($s \leq 5$). The m4 method is significantly superior of m8 one, since the latter is based on the model without asymptote.

For the f_1 and f_2 models the m0 method of extrapolation of logistic curve proves to be much worse. The only exceptions here are long-term forecasts when the model is chosen correctly.

In our opinion, it is worth comparing the efficiency of estimation methods based on criterion (7) - LSM after linearization of the model, i.e. m1, m2, m5, m6 with respective methods based

Table 6

Estimates of efficiency of methods m4, m8, m10 in relation to method 0

Var.	c	Gamma	R^2	n	s	f_0			f_1			f_2		
						m4/m0	m8/m0	m10/0	m4/m0	m8/m0	m10/m0	m4/m0	m8/m0	m10/m0
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a1	0.10	0.00	0.95	20	1	1.0169	1.0850	0.9237	0.8223	0.7640	0.8464	0.7739	0.6834	0.7970
					2	1.0743	1.2024	0.9434	0.7837	0.7392	0.8202	0.7168	0.6167	0.7531
					3	1.1426	1.3346	0.9729	0.7534	0.7080	0.8020	0.6699	0.5642	0.7190
					5	1.2985	1.6229	1.0495	0.7112	0.6870	0.7817	0.5996	0.4894	0.6714
					10	1.7460	2.4176	1.2881	0.6672	0.7056	0.7771	0.5019	0.3939	0.6128
					20	2.7995	4.3133	1.8727	0.6871	0.8670	0.8339	0.4316	0.3363	0.5783
a2	0.10	0.50	0.95	20	1	1.1939	1.1517	0.9383	0.9487	0.8842	0.8457	0.9170	0.8212	0.8386
					2	1.2685	1.2549	0.9492	0.9174	0.8602	0.8126	0.8672	0.7689	0.7985
					3	1.3489	1.3659	0.9682	0.8952	0.8469	0.7894	0.8279	0.7298	0.7682
					5	1.5215	1.5993	1.0194	0.8734	0.8436	0.7625	0.7750	0.6807	0.7279
					10	2.0101	2.2317	1.1791	0.9056	0.9216	0.7526	0.7330	0.6519	0.6850
					20	3.2929	3.8266	1.5653	1.2035	1.3297	0.8378	0.8316	0.7803	0.6868
a3	0.10	1.00	0.95	20	1	0.9408	0.9768	0.9364	0.8210	0.8253	0.8620	0.8126	0.8100	0.8523
					2	0.9917	1.0438	0.9604	0.7898	0.7903	0.8344	0.7728	0.7632	0.8179
					3	1.0601	1.1303	0.9963	0.7682	0.7671	0.8148	0.7428	0.7288	0.7920
					5	1.2335	1.3451	1.0909	0.7465	0.7470	0.7929	0.7046	0.6876	0.7584
					10	1.8257	2.0699	1.4154	0.7763	0.7996	0.7990	0.6911	0.6855	0.7364
					20	3.5102	4.1723	2.3230	1.1077	1.2534	0.9927	0.8761	0.9495	0.8436

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
a4	0.20	0.00	0.95	20	1	2.1361	4.6824	1.2531	0.4912	0.6960	0.7087	0.5094	0.3154	0.4031
					2	2.6625	6.1822	1.4069	0.4294	0.6963	0.6880	0.4497	0.2544	0.3500
					3	3.2331	7.8160	1.5780	0.3825	0.7037	0.6748	0.4072	0.2123	0.3133
					5	4.4732	11.442	1.9602	0.3182	0.7315	0.6617	0.3529	0.1597	0.2677
					10	7.9713	22.500	3.0893	0.2401	0.8401	0.6650	0.2966	0.1035	0.2200
					20	15.883	52.911	5.8713	0.1915	1.1384	0.7252	0.2791	0.0720	0.1972
a5	0.20	0.50	0.95	20	1	2.3207	2.5615	1.2278	0.5904	0.5924	0.6124	0.5520	0.4584	0.5514
					2	3.0686	3.5095	1.4181	0.5198	0.5511	0.5762	0.4661	0.3725	0.4865
					3	3.8729	4.5316	1.6327	0.4715	0.5304	0.5584	0.4079	0.3168	0.4478
					5	5.6572	6.8004	2.1244	0.4147	0.5239	0.5500	0.3379	0.2534	0.4098
					10	11.592	14.339	3.7760	0.3715	0.6014	0.5847	0.2702	0.1989	0.3929
					20	34.400	42.774	9.7404	0.4071	0.9490	0.7264	0.2497	0.1976	0.4186
a6	0.20	1.00	0.95	20	1	1.9478	1.5569	0.9982	1.3105	0.8509	0.6483	0.8155	0.4601	0.4531
					2	2.2670	2.0117	1.0969	1.2565	0.8211	0.6366	0.8265	0.4619	0.4713
					3	2.6345	2.5142	1.2258	1.2193	0.8181	0.6387	0.8364	0.4763	0.4972
					5	3.5690	3.6924	1.5424	1.2111	0.8717	0.6549	0.8647	0.5257	0.5443
					10	7.7260	8.4446	2.7322	1.5708	1.3298	0.7308	1.0752	0.7765	0.6334
					20	53.143	57.648	12.677	5.7097	5.7698	1.3054	2.9073	2.5908	0.9629

Source: As Table 1.0.

Table 7

Estimates of efficiency of linearized methods m2, m6, m1, m5

Var.	c	Gamma	R^2	n	s	f_0				f_1				f_2			
						m2/m4	m6/m8	m1/m3	m5/m7	m2/m4	m6/m8	m1/m3	m5/m7	m2/m4	m6/m8	m1/m3	m5/m7
a1	0.10	0.00	0.95	20	1	1.137	1.203	1.026	1.203	1.124	1.167	1.065	1.167	1.158	1.170	1.121	1.170
					10	1.147	1.176	1.027	1.176	1.116	1.175	1.026	1.175	1.149	1.165	1.115	1.165
a2	0.10	0.50	0.95	20	1	1.105	1.194	0.996	1.124	1.174	1.217	1.173	1.179	1.193	1.259	1.252	1.232
					10	1.076	1.112	0.925	1.071	1.126	1.090	1.106	1.059	1.142	1.150	1.237	1.146
a3	0.10	1.00	0.95	20	1	1.234	1.256	2.228	1.314	1.371	1.413	2.251	1.576	1.392	1.444	2.254	1.622
					10	1.046	1.048	1.278	0.920	1.138	1.135	1.848	1.259	1.158	1.166	1.769	1.356
a4	0.20	0.00	0.95	20	1	1.342	1.381	1.342	1.381	1.184	1.306	1.184	1.306	1.408	1.128	1.408	1.128
					10	1.286	1.262	1.286	1.262	1.173	1.274	1.173	1.274	1.343	1.102	1.343	1.102
a5	0.20	0.50	0.95	20	1	2.603	3.130	2.536	3.131	4.974	3.827	4.937	3.827	5.529	3.389	5.515	3.389
					10	1.904	2.388	1.881	2.390	3.761	3.295	3.647	3.295	3.449	3.130	3.411	3.130
a6	0.20	1.00	0.95	20	1	12.03	17.71	19.81	21.24	12.29	23.05	19.04	35.03	13.91	29.47	18.20	34.89
					10	4.845	6.344	4.198	6.020	6.872	12.13	7.885	21.76	7.437	15.31	6.345	23.41
b11	0.10	0.50	0.95	10	1	0.941	0.949	1.145	0.966	0.966	0.976	1.021	0.976	0.951	0.959	1.164	0.994
					10	0.948	0.957	1.084	0.894	0.969	0.979	1.002	0.982	0.947	0.947	1.106	0.958
b12	0.10	0.50	0.95	15	1	1.001	1.033	1.093	0.977	1.047	1.042	1.180	1.011	1.058	1.054	1.206	1.029
					10	1.002	1.036	0.845	0.998	1.020	1.020	1.132	0.981	1.025	1.025	1.163	0.998
a2	0.10	0.50	0.95	20	1	1.105	1.194	0.996	1.124	1.174	1.217	1.173	1.179	1.193	1.259	1.252	1.232
					10	1.076	1.112	0.925	1.071	1.126	1.090	1.106	1.059	1.142	1.150	1.237	1.146
b13	0.10	0.50	0.95	30	1	1.940	2.180	1.876	2.180	1.652	1.857	1.638	1.857	1.766	1.685	1.765	1.685
					10	1.823	1.821	1.766	1.821	1.811	2.061	1.782	2.061	1.716	1.769	1.718	1.769
b14	0.10	0.50	0.95	40	1	3.612	3.731	3.593	3.731	6.008	5.408	5.969	5.408	6.622	4.683	6.604	4.683
					10	2.759	2.645	2.755	2.645	5.195	4.821	5.114	4.821	4.663	4.857	4.637	4.857
b15	0.10	0.50	0.95	50	1	8.749	6.195	8.756	6.195	32.89	26.93	32.89	26.93	26.59	50.54	26.59	50.54
					10	4.677	4.356	4.675	4.356	22.41	20.00	22.41	20.00	13.42	48.64	13.42	48.64

Source: As Table 1.0.

Table 8

Efficiency of methods m3, m7, m1, m5

Var.	c	Gamma	R^2	n	s	f_0				f_1				f_2			
						m3/m4	m7/m8	m1/m2	m5/m6	m3/m4	m7/m8	m1/m2	m5/m6	m3/m4	m7/m8	m1/m2	m5/m6
a1	0.10	0.00	0.95	20	1	0.871	1.000	0.786	1.000	0.869	1.000	0.824	1.000	0.875	1.000	0.847	1.000
					10	0.833	1.000	0.746	1.000	0.742	1.000	0.682	1.000	0.776	1.000	0.753	1.000
a2	0.10	0.50	0.95	20	1	0.524	0.800	0.473	0.754	0.653	0.825	0.652	0.799	0.691	0.835	0.725	0.817
					10	0.310	0.726	0.266	0.699	0.296	0.633	0.291	0.615	0.397	0.619	0.430	0.616
a3	0.10	1.00	0.95	20	1	0.569	0.664	1.027	0.695	0.784	0.744	1.286	0.830	0.822	0.761	1.330	0.855
					10	0.089	0.412	0.109	0.362	0.400	0.362	0.650	0.402	0.526	0.394	0.804	0.458
a4	0.20	0.00	0.95	20	1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
					10	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
a5	0.20	0.50	0.95	20	1	0.907	0.998	0.884	0.998	0.998	1.000	0.991	1.000	0.998	1.000	0.996	1.000
					10	0.852	0.996	0.842	0.997	0.994	1.000	0.964	1.000	0.992	1.000	0.981	1.000
a6	0.20	1.00	0.95	20	1	0.469	0.719	0.772	0.862	0.526	0.588	0.814	0.894	0.634	0.759	0.829	0.899
					10	0.245	0.575	0.212	0.546	0.213	0.315	0.245	0.566	0.313	0.374	0.267	0.572

Source: As Table 1.0.

on criterion (6): m3, m4, m7, m8. Values of measure of the efficiency, defined analogously to (11):

$$e^{(2)}_{hjs} = \hat{d}^2_{hjs}/\hat{d}^2_{h,j+2,s}$$

for $h = 0, 1, 2$, $j = 1, 2, 5, 6$, $s = 1, 10$ for some variants are shown in Table 7. $e^{(2)}_{hjs} < 1$ is seldom observed. Therefore, methods based on criterion (7) are less efficient. Nevertheless, in basic variants and in prevailing majority of other variants (exceptions: large n, large c and large R^2), we observe $e^{(2)}_{hjs} <$ < 1.2, seldom $e^{(2)}_{hjs} < 1.3$. It means, that the decline of efficiency is not generally very large in these cases.

Let us take into consideration the influence of conditions concerning θ parameter on the efficiency of prediction. Let us define, like before,

$$e^{(1)}_{hjs} = \hat{d}^2_{hjs}/\hat{d}^2_{h,j+1,s}$$

(results for $h = 0, 1, 2$, $j = 1, 3, 5, 7$, $s = 1, 10$ are shown in Table 8) as a measure of efficiency in prediction methods with stronger condition concerning θ (5) than it was with condition (6).

In a lot of variants $e^{(1)}_{hjs} = 1$, which means, that a priori condition concerning θ parameter has no influence on the result of estimation and consequently on prediction error. However, values $e^{(1)}_{hjs} < 1$ prevails in Table 8. It means that mean square error of prediction, when conditions are stronger, (methods m1, m3, m5, m7) is less than in the case of weaker conditions, when obviously the determination coefficient $\hat{R}^2$ is greater or equal. This result is rather surprising. It can be explained by the fact that maximization $\hat{R}^2$ at the sacrifice the growth of θ causes better adjustment of parameters to empirical data y_t, what does not mean the less error of prediction. The results shown in Table 8 confirm this: if $e^{(1)}_{hj1} < 1$ that $e^{(1)}_{hj10} < e^{(1)}_{hj1}$ and, therefore the efficiency of prediction with stronger condition is much greater than that for the long-term forecasts.

7. RECAPITULATION

The general recapitulation of the results is difficult because the conclusions drawn from the analysis of particular variants are different. Especially, the superiority of any function as to three functions being taken into consideration cannot be stated without doubts.

Conclusions concerning certain details were presented in the previous point. Here we repeat these, which can be of more general character.

Using logistic function as the only model relevant to the phenomenon and, consequently, using the only prediction method connected with it does not seem to be correct generally. One should take into account the possibility that the estimation method of this function will prove to be inefficient and should keep a substitute procedure in reserve.

The method 10 based on discrimination by means of determination coefficient $\hat{R}^2$ which has been described by us, can be considered as such a procedure. Although, the correct model is chosen more rarely than one could expect, but from the point of view of the prediction efficiency good results can be reached.

The Gauss-Newton method should be used for estimation of f_1 and f_2 competitors of logistic function. The method based on linearization is significantly less efficient. Upper limit of the parameter Θ of these functions generally has a little effect on the efficiency of prediction. One should pay attention that stronger condition on this parameter leads to higher efficiency. In this case the criterion of maximization of $\hat{R}^2$ from the sample is not conformable to minimization of the predictor variance.

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EFEKTYWNOŚĆ ESTYMACJI NIEKTÓRYCH POSTACI NIELINIOWYCH TRENDÓW

Wśród krzywoliniowych modeli tendencji rozwojowej dominuje trend wykładniczy, opisujący "wybuchowy" rozwój, oraz trend logistyczny charakteryzujący się asymptotą poziomą, interpretowaną jako "poziom" nasycenia badanego zjawiska.

Celem tego artykułu jest ocena efektywności ekstrapolacji trendu logistycznego przy założeniu, że błędy w danych mają charakter składnika losowego - białego szumu. Analizę przeprowadzamy w porównaniu z dwiema "konkurencyjnymi" krzywymi.

Zatem przedmiotem naszych rozważań są trzy modele trendu:

$$y_t = f_h(t) + \varepsilon_t \tag{1}$$

gdzie ε_t jest "białym szumem", zaś h = 0, 1, 2, przy czym

$$f_0(t) = \frac{a}{1 + b\exp(-ct)} \tag{2}$$

(trend logistyczny),

$$f_1(t) = \exp(\alpha_0 + \frac{\alpha_1}{T + \Theta}) \tag{3}$$

(trend wykładniczy z odwrotnością),

$$f_2(t) = \exp(\alpha_0 + \alpha_1 \ln(t + \Theta)) \tag{4}$$

(trend potęgowy).

Ograniczamy się tylko do trendów rosnących.