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EVALUATION OF THE PROBABILITY CONTENT AS AN INFINITE LINEAR COMBINATION OF WISHART DISTRIBUTIONS

Abstract. The distribution function of the homogeneous generalized quadratic form is represented as an infinite linear combination of the central Wishart distribution functions. The Probability content of the ellipsoid is expressed as an infinite linear combination of the probability contents of spheres, under a central spherical multivariate normal distributions with unit variance, covariance matrix.

1. INTRODUCTION

Let $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ be p-dimensional random vectors. Then a generalized homogeneous quadratic form is defined as

$$\sum_{i=1}^{n} a_i \mathbf{x}_i \mathbf{x}_i' = \mathbf{X}'\mathbf{A}\mathbf{X}, \tag{1}$$

where $\mathbf{X}'$ is $p \times n$ random matrix whose columns are $\mathbf{x}_1, ..., \mathbf{x}_n$ and A is a real diagonal $n \times n$ matrix of constants whose diagonal elements are denoted by $a_1, ..., a_n$. We shall assume for convenience, without loss of generality, that $0 < a_1 \leqslant a_2 \leqslant ... \leqslant a_n$. When $p = 1$, $\mathbf{X}'\mathbf{A}\mathbf{X}$ reduces to a single homogeneous quadratic form and it is equal to $\sum_{i=1}^{n} a_i x_i^2$.

For such single homogeneous and non-homogeneous quadratic functions of normal variables R u b e n (1962) expressed the distribution function as an infinite linear combination of chi-square distribution functions with arbitrary scale parameter. This result was used by R a j a g o p a l a n and

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Broemeling (1983) to find as approximations to the posterior distributions of variances components in univariate mixed linear model. To solve the above problem in multivariate case (see Jelenkowska, Press, 1995) we need the results of this paper.

In this paper we shall find the distribution function of generalized homogeneous non-negative quadratic form of a finite number of correlated normal random variables. This distribution will be expressed as an infinite linear combination of Wishart distribution functions with arbitrary scale parameter.

2. DEFINITIONS AND PRELIMINARY LEMMAS

Let $\underset{n \times p}{\mathbf{M}}$ be a matrix whose rows are $\mu_1', ..., \mu_n'$ where $\underset{p \times 1}{\mu_i} = \mathrm{E}(\mathbf{x}_i)$, $i = 1, ..., n$ and $\mathrm{cov}(\mathbf{x}_i, \mathbf{x}_j) = \Sigma$. Assume

$$\underset{p \times 1}{\mathbf{x}_i} \sim N(\mu_i, \Sigma), \; i = 1, ..., n, \; \Sigma > 0 \text{ and symmetric} \tag{2}$$

Then

$$\underset{n \times p}{\mathbf{X}} \sim N(\mathbf{M}, \mathbf{I}_n \otimes \Sigma).$$

The distribution (2) may be standardized by the transformation

$$\mathbf{u}_i = \Sigma^{-\frac{1}{2}}(\mathbf{x}_i - \mu_i).$$

That is

$$\mathbf{u}_i \sim N(\mathbf{0}, \mathbf{I}_p)$$

and

$$\mathbf{U} \sim N(\mathbf{0}, \mathbf{I}_n \otimes \mathbf{I}_p).$$

Let

$$R = \{\mathbf{U}: \mathbf{U}'\mathbf{A}\mathbf{U} \leqslant \mathbf{T}\},$$

$\mathbf{T} > \mathbf{0}$, $p \times p$, symmetric matrix and

$$H_{n;A}(\mathbf{T}) = (2\pi)^{-\frac{n}{2}} \int_R \exp\left\{-\frac{1}{2}\mathrm{tr}\mathbf{U}'\mathbf{U}\right\} d\mathbf{U} = P\left[\sum_{i=1}^{n} a_i \mathbf{u}_i \mathbf{u}_i' \leqslant \mathbf{T}\right] \tag{3}$$

If U and T are symmetric matrices, $\mathbf{T} \geqslant \mathbf{U}$ means that $\mathbf{T}-\mathbf{U}$ is nonnegative definite. On replacing $\mathbf{U}=\mathbf{A}^{-\frac{1}{2}}\mathbf{Z}$ in (3), we obtain

$$H_{n;A}(\mathbf{T})=(2\pi)^{-\frac{n}{2}}|\mathbf{A}|^{-\frac{1}{2}}\int_{R^*}\exp\left\{-\frac{1}{2}\mathrm{tr}\mathbf{Z}'\mathbf{A}^{-1}\mathbf{Z}\right\}d\mathbf{Z} \tag{4}$$

where

$$R^*=\{\mathbf{Z}\colon\ \mathbf{Z}'\mathbf{Z}\leqslant\mathbf{T}\}.$$

Let $F_n(\cdot)$ denote the nonsingular p-dimensional Wishart distribution with scale matrix $\mathbf{I}_p$ and n degrees of freedom, $p\leqslant n$, i.e.

$$F_n(\mathbf{T})=P\left[\sum_{i=1}^{n}\mathbf{u}_i\mathbf{u}_i'\leqslant\mathbf{T}\right],$$

so that

$$F_n(\mathbf{T})=\gamma^{-1}\int_{T}|\mathbf{T}|^{\frac{1}{2}(n-p-1)}\exp\left(-\frac{1}{2}\mathrm{tr}\mathbf{T}\right)d\mathbf{T},\quad \mathbf{T}>0 \tag{5}$$

and $F_n(\mathbf{T})=0$ otherwise, where γ is a numerical constant defined as

$$\gamma=2^{\frac{np}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{r=1}^{p}\Gamma\left(\frac{n+1-r}{2}\right).$$

Then

$$F_n(\mathrm{T}=H_{n;I}(\mathbf{T}) \tag{6}$$

where $\mathbf{I}$ is the unit $n\times n$ dimensional matrix.

We shall show, that the distribution function of $\sum_{i=1}^{n}a_i\mathbf{u}_i\mathbf{u}_i'$ may be expressed as a linear combination of infinitely many Wishart distribution functions, i.e.

$$H_{n;A}(\mathrm{T}=\sum_{j=0}^{\infty}c_jF_{n+2j}\left(\frac{1}{\omega}\mathbf{T}\right) \tag{7}$$

where $c_j=c_{j;n;A}(\omega)$, and ω is an arbitrary positive constant. Generating functions for the coefficients c_j will be derived.

Now we shall define the norm of matrix **U** as

$$\| \mathbf{U} \| = \left| \sum_{i=1}^{n} \mathbf{u}_i \mathbf{u}_i' \right|^{\frac{1}{2}} \tag{8}$$

Let $\underset{p \times n}{\mathbf{L}'} = (\underset{p \times 1}{\mathbf{l}_1}, \ldots, \underset{p \times 1}{\mathbf{l}_n})$ be a matrix for that $\mathbf{L}'\mathbf{L} = \mathbf{I}$ (orthogonal matrix). If **L** is a uniformly distributed on Ω, $E\Phi(\mathbf{L})$ will be written as M – operator $M\Phi(\mathbf{L})$. If $\sum_{j=0}^{\infty} \Phi_j(\mathbf{L})$ converges uniformly on Ω, we note that

$$M \sum_{j=0}^{\infty} \Phi_j(\mathbf{L}) = \sum_{j=0}^{\infty} M\Phi_j(\mathbf{L}) \tag{9}$$

Next we adopt the convention that $M\Phi(\mathbf{L})$ will be written as $M\Phi$ and $E\Phi(U)$ as $E\Phi$. Thus the argument of Φ will be **L** if the expectation operator is M and **U** if the expectation operator is E. We shall also define that **L** is induced by **U** if $\mathbf{L} = \mathbf{U}\|\mathbf{U}\|^{-1}$ for $\mathbf{U} \neq 0$, we say that **U** has centered spherical distribution if the distribution of **PU** is the same as that of **U** for every orthogonal matrix **P**. Immediately from the above definition it follows that: If **U** has a centered spherical distribution and **L** is induced by **U**, then **L** and $\|\mathbf{U}\|$ are independent. **L** is uniformly distributed on Ω.

Lemma 1. If **U** has a centered spherical distribution, **L** is induced by **U**, $\Phi(\mathbf{U})$ is a generalized homogeneous quadratic form of degree k and $E\|\Phi\mathbf{U}\| < \infty$, then

$$E\Phi(\mathbf{U}) = E\|\mathbf{U}\|^k M\Phi(\mathbf{L}) \tag{10}$$

Proof. Using property of homogeneous function and independence of **L** and $\|\mathbf{U}\|$ we have

$$E\Phi(\mathbf{U}) = E[\Phi(\|\mathbf{U}\|\mathbf{L})] = E[\|\mathbf{U}\|^k \Phi(\mathbf{L})] = E[\|\mathbf{U}\|^k] E\Phi(\mathbf{L}).$$

$E\Phi(\mathbf{L})$ can be replaced by $M\Phi(\mathbf{L})$ since **L** is uniformly distributed on Ω. It yields the result.

Lemma 2. If the matrix U: $n \times p$ has a multivariate standardized spherical normal distribution then

$$M\Phi = \frac{\prod_{r=1}^{p}\Gamma\left(\frac{n+1-r}{2}\right)E\Phi}{2^{\frac{kp}{2}}\prod_{r=1}^{p}\Gamma\left(\frac{n+k+1-r}{2}\right)} \tag{11}$$

Proof. The density function of **U** is $p(\mathbf{U}) = (2\pi)^{-\frac{np}{2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\mathbf{U}'\mathbf{U}\right\}$.
Let

$$\mathbf{V} = \sum_{i=1}^{n}\mathbf{u}_i\mathbf{u}_i'$$

Then

$$E\|\mathbf{U}\|^k = E|\mathbf{V}|^{\frac{k}{2}} = c\int_{V>0}|\mathbf{V}|^{\frac{1}{2}(k+n-p-1)}\exp\left\{-\frac{1}{2}\mathrm{tr}\mathbf{V}\right\}d\mathbf{V},$$

where

$$c^{-1} = 2^{\frac{np}{2}}\pi^{\frac{p(p-1)}{4}}\prod_{r=1}^{p}\Gamma\left(\frac{n+1-r}{2}\right).$$

Since

$$\int_{V>0}|\mathbf{V}|^{\frac{1}{2}(n+k-p-1)}\exp\left\{-\frac{1}{2}\mathrm{tr}\mathbf{V}d\mathbf{V} = 2^{\frac{(n+k)p}{2}}\pi^{\frac{p(p-1)}{\omega}}\prod_{r=1}^{p}\Gamma\left(\frac{n+k+1-r}{2}\right)\right\},$$

we obtain

$$E\|\mathbf{U}\|^k = \frac{2^{\frac{kp}{2}}\prod_{r=1}^{p}\Gamma\left(\frac{n+k+1-r}{2}\right)}{\prod_{r=1}^{p}\Gamma\left(\frac{n+1-r}{2}\right)}.$$

Using (10) in (11) the lemma is proved.

3. MAIN RESULTS

In this section we prove the following fundamental theorem:

Theorem 1. The distribution function of the homogeneous generalized quadratic form (3) is represented as an infinite linear combination of central Wishart distribution functions, i.e.

$$H_{n;A}(\mathbf{T}) = \sum_{j=0}^{\infty} c_j F_{n+2j}\left(\frac{1}{\omega} T\right) \tag{12}$$

where ω is an arbitrary positive constant,

$$c_j = \frac{\omega^{\frac{n}{2}+j} |\mathbf{A}|^{-\frac{1}{2}} E[-(\mathrm{tr}\mathbf{Q})^j]}{(2j)! 2^{n(p-\frac{1}{2})+1} \pi^{\frac{p(p-1)}{4}} \prod_{t=1}^{p} \Gamma\left(\frac{n+1-r}{2}\right)} \tag{13}$$

and

$$Q = Q(\mathbf{U}) = \sum_{i=1}^{n} \left(\frac{1}{a_i} - \frac{1}{\omega}\right) \mathbf{u}_i \mathbf{u}_i'.$$

Proof. Let

$$\mathbf{Z} = \mathbf{A}^{\frac{1}{2}} \mathbf{X} \quad \text{and} \quad \mathbf{Z} = \|\mathbf{Z}\| \mathbf{L}.$$

Then

$$H_{n;A}(\mathbf{T}) = (2\pi)^{-\frac{n}{2}} |\mathbf{A}|^{-\frac{1}{2}} \int_{T>0} \int_{L>0} \exp\left\{-\frac{1}{2}\mathrm{tr}(\mathbf{L}'\mathbf{A}^{-1}\mathbf{L}) \|\mathbf{Z}\|^2\right\} \|\mathbf{Z}\|^{n-1} d\mathbf{Z} d\mathbf{L} =$$

$$= \left[2^{\frac{np}{2}} \pi^{\frac{p(p-1)}{4}} \Gamma\left(\frac{n}{2}\right) \prod_{r=1}^{p} \Gamma\left(\frac{n+1-r}{2}\right)\right]^{-1} |\mathbf{A}|^{-\frac{1}{2}} \int_{T>0} M\left[\exp -\frac{1}{2}\left\{\mathrm{tr}\mathbf{L}'\left(\mathbf{A}^{-1} - \frac{1}{\omega}\mathbf{I}_n\right)\mathbf{L}\right\} \|\mathbf{Z}\|^2\right] \cdot$$

$$\cdot \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\frac{1}{\omega}\mathbf{Z}'\mathbf{Z}\right)\right\} \|\mathbf{Z}\|^{n-1} d\mathbf{Z} \tag{14}$$

Denoting by

$$Q = Q(\mathrm{L}) = \mathrm{L}'\left(\mathbf{A}^{-1} - \frac{1}{\omega}\mathbf{I}\right)\mathrm{L} \tag{15}$$

we can expand $\exp\left\{-\frac{1}{2}\mathrm{tr}Q\|\mathbf{Z}\|^2\right\}$ as a power series in $\|\mathbf{Z}\|$, i.e.

$$\exp\left\{-\frac{1}{2}\mathrm{tr}Q\|\mathbf{Z}\|^2\right\} = \sum_{m=0}^{\infty}(-1)^m(-\mathrm{tr}Q)^{\frac{m}{2}}\|\mathbf{Z}^m/m! \tag{16}$$

Using (9) we can write

$$M\left[\exp\left(-\frac{1}{2}\mathrm{tr}Q\right)\|\mathbf{Z}\|^2\right] = \sum_{m=0}^{\infty}(-1)^m M[-(\mathrm{tr}Q)^{\frac{m}{2}}]\|\mathbf{Z}\|^m/m!$$

By symmetry for odd m

$$M\left[-(\mathrm{tr}Q)^{\frac{m}{2}}\right] = 0 \quad m = 1, 3, \ldots$$

and (16) reduces to

$$M\left[\exp\left(-\frac{1}{2}\mathrm{tr}Q\right)\|\mathbf{Z}\|^2\right] = \sum_{j=0}^{\infty} M[-(\mathrm{tr}Q)^j]\|\mathbf{Z}\|^{2j}/(2j)! \tag{17}$$

Next, using Lemma 2 we obtain

$$M\left[\exp\left(-\frac{1}{2}\mathrm{tr}Q\right)\|\mathbf{Z}\|^2\right] = \sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}\right)}{2^{jp}\Gamma\left(\frac{n}{2}+j\right)}\left\{\frac{E[+(\mathrm{tr}Q)^j]}{(2j)!}\right\}\|\mathbf{Z}\|^{2j} \tag{18}$$

Putting (18) in (14)

$$H_{n;A}(T) = \left[2^{\frac{np}{2}}\pi^{\frac{p(p-1)}{4}}\Gamma\left(\frac{n}{2}\right)\prod_{r=1}^{p}\Gamma\left(\frac{n+1-r}{2}\right)\right]^{-1}|A|^{-\frac{1}{2}}\int_{T>0}\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}\right)}{2^j\Gamma\left(\frac{n}{2}+j\right)}\cdot$$

$$\cdot\left\{\frac{E[-(\mathrm{tr}Q)^j]}{(2j)!}\right\}\exp\left\{-\frac{1}{2}\mathrm{tr}\frac{1}{\omega}\mathbf{Z}'\mathbf{Z}\right\}\|\mathbf{Z}\|^{n+2_j-1}d\mathbf{Z} \tag{19}$$

The result follows by noting that

$$\int_{T>0} \|\mathbf{Z}\|^{n+2_j-1} \exp\left\{-\frac{1}{2}\operatorname{tr}\frac{1}{\omega}\mathbf{Z}'\mathbf{Z}\right\} d\mathbf{Z} = \frac{1}{2}\omega^{\frac{n}{2}+j} 2^{\frac{n}{2}+j} \cdot \Gamma\left(\frac{n}{2}+j\right) F_{n+2j}\left(\frac{1}{\omega}\mathbf{T}\right).$$

Theorem 1 represents the distribution function of the homogeneous generalized quadratic form in terms of central Wishart distribution functions.

The series in (12) converges uniformly on every finite matrix space of $\mathbf{T}$ for each $\overline{\omega} > 0$. In Addition to the formula (13) for c_j of Theorem 1 explicit formulae is expressed as the expectation of a certain homogeneous function of degree 2_j in independent standardized normal variables $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

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