A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S

FOLIA OECONOMICA 68, 1987

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SOME RESULTS CONCERNING EFFICIENCY OF LINEAR TREND ESTIMATION UNDER HETEROSCEDASTICITY

1. Introduction

Consider a single - equation econometric model of the form

$$y = X\alpha + \varepsilon,$$

where X is a fixed $n \times (k + 1)$ matrix of rank $k + 1 < n$, and $E\varepsilon = 0$. One of the alternatives for the classical assumption $D^2\varepsilon = \sigma^2 I$ about spherical random component is the assumption of lack of autocorrelation allowing heteroscedasticity:

(1) $$D^2\varepsilon = \sigma^2\Omega,$$

where Ω is a diagonal matrix with positive diagonal elements.

If Ω is a known matrix one can use the best linear unbiased estimator of the vector α, which is A i t k e n's [1] GLS-estimator

(2) $$a^* = (X^T\Omega^{-1}X)^{-1}X^T\Omega^{-1}y$$

with the variance-covariance matrix

(3) $$D^2a^* = \sigma^2(X^T\Omega^{-1}X)^{-1}.$$

The OLS-estimator

(4) $$\dot{a} = (X^TX)^{-1}X^Ty$$

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is generally inefficient. Moreover the estimator

$$S_e^2 (X^T X)^{-1}$$

of the OLSE variance-covariance matrix

$$D^2 a = \sigma^2 (X^T X)^{-1} X^T \Omega X (X^T X)^{-1} \quad (5)$$

is biased in general.

We can take into account the class of estimators of the form

$$a(W) = (X^T W X)^{-1} X^T W y \quad (6)$$

where W is a symmetrical and positive definite matrix to which the estimators (2) and (4) belong:

$$a(\Omega^{-1}) = a^* \text{ and } a(I) = a. \quad (6a)$$

The estimator (6) will be named as WLSE (weighted LS) with weight matrix W. If the matrix W is non-stochastic (at least not correlated with y) it is easy to derive a formula

$$D^2 a(W) = \sigma^2 (X^T W X)^{-1} X^T W \Omega W X (X^T W X)^{-1}$$

which may be reduced to (3) for $W = \Omega^{-1}$ and to (5) if $W = I$.

For practical applications, generally, so-called two-step estimator

$$a(\hat{\Omega}^{-1}), \quad (7)$$

is suggested, where $\hat{\Omega}$ is the estimate of matrix Ω, expressed as a function of OLS-residuals

$$e = y - Xa$$

(see f.e. Park [13], Glejser [7] and generalization for three-step estimator - Froehlich [5], and Jarmuż [8] p. 47-54 as well). The weight matrix is, in this case, a stochastic matrix and it is difficult to examine estimator's (7) properties for small samples.

We can, however, prove that this estimator is asymptotically equally efficient as a^*. The latest results for models with heteroscedasticity of random component concerning maximum likelihood method optimal method for large samples, can be found for instance in Fuller, Rao [6], Jobson, Fuller [9], Carroll, Ruppert [3].

It seems to us that the most general results referring to OLSE efficiency are Watson inequalities (see Watson [17], Bloomfield and Watson [2], Knott [10], Milo [12]) which define its lower limit in dependence on the matrix eigen values - but independently on matrix X, similarly as Sathe - Vinod [15] inequalities for bias of variance's estimator of OLSE. The inequalities of Watson and Sathe - Vinod, can be generalized to WLSE with fixed (independent on y weight matrix). These results, extremely important from theoretical point of view, don't have great practical meaning, because in a concrete problem the matrix X is known and information about Ω is poor. Therefore, methods allowing to evaluate estimation efficiency when the matrix X is known (or belongs to the defined, relatively narrow class) are more useful for direct applications.

2. Problem formulation

Concepts connected with a typical autocorrelation structure for the linear trend model

$$(8) \qquad y_t = \alpha_0 + \alpha_1 t + \varepsilon_t,$$

that is, for the model (1) in which k = 1 and

$$(9) \qquad X = X_n = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & n \end{bmatrix}$$

were the topic of many works, see i.e. Tomaszewicz [16], Park, Mitchell [14], Chipman [4], Krämer [11]. In this work we undertake the topic of WLSE-efficiency for the model with heteroscedasticity.

As in the case of autocorrelation we consider the models in which "non-sphericity" - in our case heteroscedasticity - is defined by one parameter denoted by β.

When we fix the heteroscedasticity model as

$$(10)\qquad \sigma_t^2 = \sigma_0^2\,\varphi(\beta, t), \qquad t = 1, 2, \ldots, n$$

(that is when the function φ is given), where σ_t^2 are the diagonal elements of Ω, this matrix is also a function of β, so we will use the symbol $\Omega(\beta)$. Let's assume that we use the estimator $a(\Omega(\gamma)^{-1})$ where γ is a fixed number (which should be interpreted as an estimate of β). A measure of WLSE-efficiency can be defined (see Tomaszewicz [16], Chipman [4]) as a ratio of variances of slope α_1 estimators of model (8). This ratio depends on the sample size, so it is a function of three parameters:

$$(11)\qquad \psi(n, \beta, \gamma) = \frac{a_1(\Omega(\beta)^{-1})}{a_1(\Omega(\gamma)^{-1})},$$

where $a_1(\Omega(\gamma)^{-1})$ is the second diagonal element of the matrix $D^2 a(\Omega(\gamma)^{-1})$, so (see (6a))

$$D^2 a_1(\Omega(\gamma)^{-1}) =$$

$$(12)\qquad = c^T(X^T\Omega(\gamma)^{-1}X)^{-1}X^T\Omega(\gamma)^{-1}\Omega(\beta)\Omega(\gamma)^{-1}X(X^T\Omega(\gamma)^{-1}X)^{-1}c'$$

for $\quad c^T = [0 \quad 1]$.

At this stage of work we limited our interest to investigating some heteroscedasticity models including a proposition of heteroscedasticity measure (Section 3) and numerical analysis of efficiency (12) (see: Section 4). Some problems connected with generalization and a series of detailed problems which appeared during the work we hope to solve later.

3. Heteroscedasticity measure

We limit our consideration to the most often, up to now, used heteroscedasticity models (see, for example Jarmuż [8] p. 40, 43). These are the following exemplifications of (10):

- linear heteroscedasticity

$$(13)\qquad \sigma_t^2 = \sigma_o^2\,\varphi_L(\beta, t) = \sigma_o^2\,(1 + \beta\frac{t}{n}),$$

- parabolic heteroscedasticity

$$(14)\qquad \sigma_t^2 = \sigma_o^2\varphi_P(\beta, t) = \sigma_o^2\,(1 + \beta(\frac{t}{n} - 1)^2),$$

- exponential heteroscedasticity

$$(15)\qquad \sigma^2 = \sigma_o^2\,\varphi_E(\beta, t) = \sigma_o^2\,e^{\beta t/n},$$

- group heteroscedasticity

$$(16)\qquad \sigma_t^2 = \sigma_o^2\,\varphi_G(\beta, t) = \begin{cases} 1 & \text{for } t \leq n/2 \\ 1 + \beta & \text{for } t > n/2, \end{cases}$$

- single-observation heteroscedasticity

$$(17)\qquad \sigma_t^2 = \sigma_o^2\,\varphi_S(\beta, t) = \begin{cases} 1 & \text{for } t < n, \\ 1 + \beta & \text{for } t = n. \end{cases}$$

The parameter β must be chosen so that $\sigma_t^2 > 0$ for $t = 1, 2, 3, \ldots, n$. To make possible a comparison of different models we need common heteroscedasticity measure. The most natural measure seems to be the coefficient of variation, which expresses standard deviation of a set $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ in mean value units. This measure is independent of scale coefficient σ_o^2 and is of the following form:

$$(18)\qquad v = \frac{\frac{1}{n}\sum\limits_t (\sigma_t^2 - \bar{\sigma}^2)^2}{\sigma^2}$$

where

$$\bar{\sigma}^2 = \frac{1}{n} \sum_t \sigma_t^2.$$

For heteroscedasticities (13), (14), (16) and (17) we can obtain explicit formulas of (18). These are as follows:

$$(18a) \quad v_L(\beta, n) = \frac{\frac{\beta}{2n}\sqrt{\frac{(n-1)(n+1)}{3}}}{1 + \frac{n+1}{2n}\beta},$$

$$(18b) \quad v_P(\beta, n) = \frac{\frac{\beta}{6n^2}\sqrt{\frac{1}{5}(n-1)(2n-1)(n+1)(8n-11)}}{1 + \frac{(n-1)(2n-1)}{6n^2}\beta},$$

$$(18c) \quad v_G(\beta, n) = \frac{\frac{k\beta}{n}\sqrt{\frac{n}{k} - 1}}{1 + \frac{k\beta}{n}}$$

where $k = \text{integer}\left(\frac{n+1}{2}\right)$ and

$$(18d) \quad v_S(\beta, n) = \frac{\frac{\beta}{n}\sqrt{n-1}}{1 + \frac{\beta}{n}}$$

The range of the functions (18)-(21) and analogous function $v_E(\beta, n)$ for exponential heteroscedasticity (15), were illustrated on the Figures 1-5.

We do not introduce here analysis of these functions. We call only one's attention to upper limits of variations coefficient connected with restrained values of functions (18)-(21) when $\beta \longrightarrow \infty$

$$v_L(\infty, n) = \sqrt{\frac{1}{3}\,\frac{n-1}{n+1}},$$

$$v_P(\infty, n) = \sqrt{\frac{\frac{2}{3}(n+1)(8n+11)}{5(n-1)(2n-1)}},$$

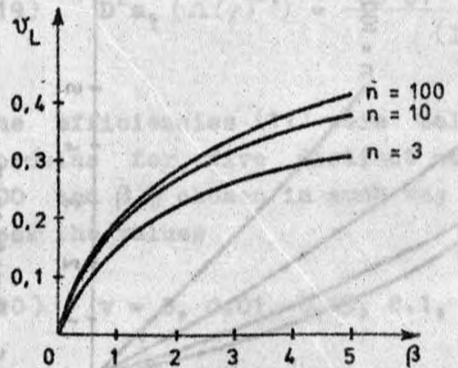

Fig. 1. Function v_L

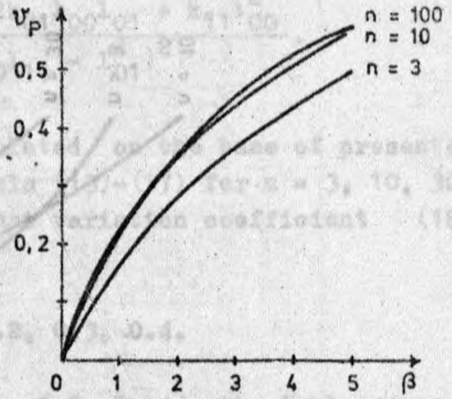

Fig. 2. Function v_P

$$v_G(\infty, n) = \sqrt{\frac{n}{k} - 1} = \begin{cases} 1 & \text{for even } n, \\ \sqrt{\frac{n-1}{n+1}} & \text{for odd } n, \end{cases}$$

$$v_S(\infty, n) = \sqrt{n - 1}.$$

4. Conditional efficiency of the estimator of slope coefficient

Derivation of explicite formulas for the efficiency (11) seems to be difficult in general case. However in quite simple way we can calculate these numerical quantities for chosen values of β and γ for some levels of sample size. Under the notations

$$L = \begin{bmatrix} l_{00} & l_{01} \\ l_{01} & l_{11} \end{bmatrix} = (X^T \Omega(\gamma)^{-1} X)^{-1},$$

$$K = \begin{bmatrix} k_{00} & k_{01} \\ k_{01} & k_{11} \end{bmatrix} = X^T \Omega(\gamma)^{-1} \Omega(\beta) \Omega(\gamma)^{-1} X,$$

and simple transformations we obtain from (12)

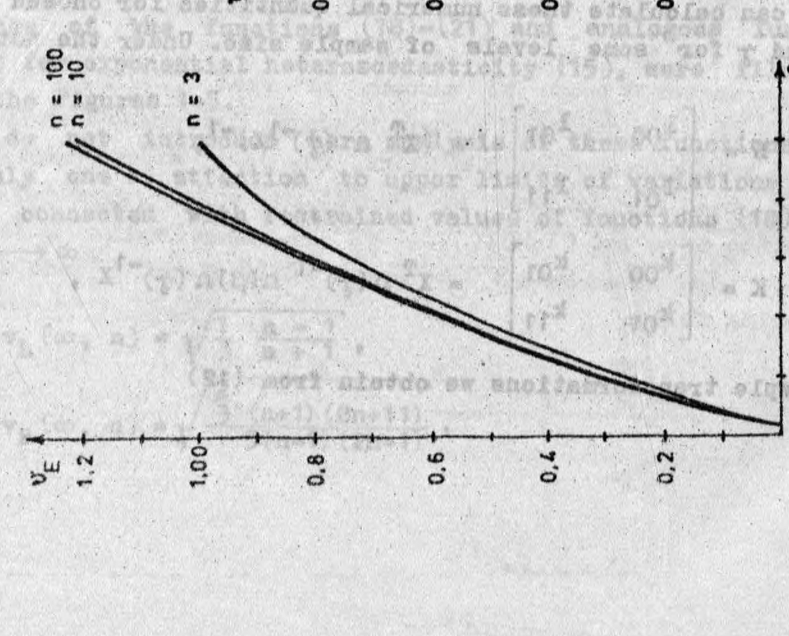

Fig. 3. Function v_E

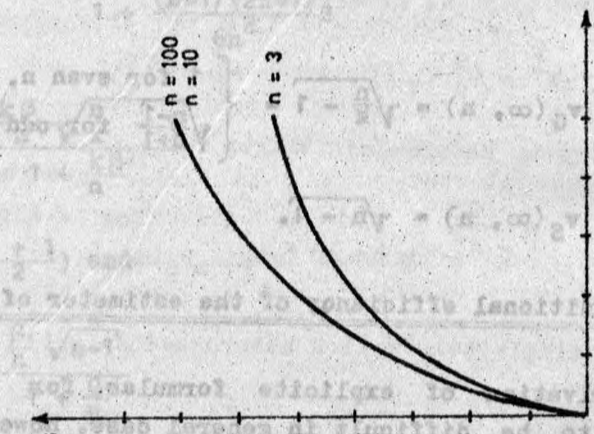

Fig. 4. Function v_G

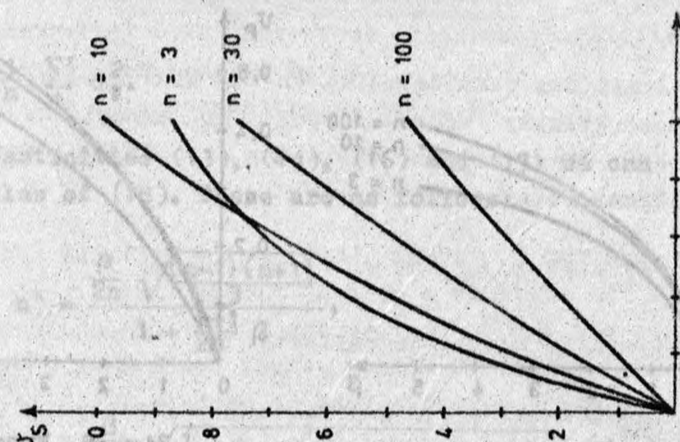

Fig. 5. Function v_S

$$D^2 a_1 (\Omega(\gamma)^{-1}) = \frac{k_{00} l_{01}^2 - 2k_{01} l_{00} l_{01} + k_{11} l_{00}^2}{(l_{00} l_{11} - l_{01}^2)^2}. \tag{19}$$

The efficiencies (11) were calculated on the base of presented formulas for five distinct models (13)-(17) for n = 3, 10, 30, 100 and β, γ chosen in such way that variation coefficient (18) took the values

$$v = 0,\ 0.01,\ 0.05,\ 0.1,\ 0.2,\ 0.3,\ 0.4. \tag{20}$$

The results are collected in Tables 1-5. Inside the tables (row h, column j) are efficiencies (11) calculated for $\beta = g(v_h)$, $\gamma = g(v_j)$, where g denotes an inverse function in v corresponding to one of the functions (18a)-(18d), and $v_h (v_j)$ is respectively h-th (j-th) value from the list (20). On the base of obtained results we can formulate some, as it seems, quite interesting conclusions.

1° The numbers in the tables above the main diagonal are slightly smaller then corresponding ones below this diagonal. It means that overestimation of our heteroscedasticity measure v causes greater efficiency losses than underestimation.

2° Underestimation or overestimation of heteroscedasticity measure causes the greatest losses in the case of smooth linear heteroscedasticity, then exponential and parabolic. Respective losses for group and single-observations heteroscedasticity are much smaller.

3° With the increase of sample size n, the losses in efficiency caused by misestimation of heteroscedasticity measure are generally growing up, excluding the case of single-observation heteroscedasticity, when we observe falling down of efficiency.

Conclusions formulated above, especially third, seems to us to be not trivial. They point out some ways in choosing an estimation method for a given heteroscedasticity model. This problem needs more detailed, further investigations.

Table 1

Efficiency (11) for heteroscedasticity φ_L

		1	2	3	4	5	6	7
n = 5	1	1.00000	0.99993	0.99825	0.99294	0.97104	0.93204	0.87198
	2	0.99993	1.00000	0.99888	0.99427	0.97373	0.93602	0.87707
	3	0.99825	0.99888	1.00000	0.99821	0.98328	0.95005	0.89679
	4	0.99296	0.99429	0.99822	1.00000	0.99235	0.96727	0.91987
	5	0.97140	0.97406	0.98348	0.99243	1.00000	0.99102	0.95977
	6	0.93386	0.93773	0.95223	0.96806	0.99115	1.00000	0.99844
	7	0.87743	0.88231	0.90115	0.92310	0.96106	0.98865	1.00000
n = 10	1	1.00000	0.99992	0.99805	0.99214	0.96718	0.92030	0.84009
	2	0.99992	1.00000	0.99875	0.99361	0.97020	0.92481	0.84594
	3	0.99806	0.99875	1.00000	0.99800	0.98093	0.94184	0.86882
	4	0.99219	0.99365	0.99801	1.00000	0.99121	0.96071	0.89615
	5	0.96804	0.97078	0.98141	0.99139	1.00000	0.98888	0.94536
	6	0.92509	0.92934	0.94532	0.96292	0.98930	1.00000	0.98329
	7	0.85763	0.86292	0.88342	0.90757	0.95066	0.98437	1.00000
n = 20	1	1.00000	0.99992	0.99801	0.99194	0.99622	0.91719	0.83023
	2	0.99992	1.00000	0.99872	0.99345	0.96932	0.92183	0.83628
	3	0.99801	0.99873	1.00000	0.99795	0.98034	0.93940	0.86001
	4	0.99200	0.99350	0.99796	1.00000	0.99092	0.95892	0.88853
	5	0.96722	0.97022	0.98090	0.99113	1.00000	0.99827	0.94051
	6	0.92290	0.92724	0.94357	0.96160	0.98880	1.00000	0.98143
	7	0.85231	0.85769	0.87857	0.90324	0.94757	0.98295	1.00000

		1	2	3	4	5	6	7
n = 50	1	1.00000	0.99992	0.99799	0.99189	0.96595	0.91630	0.82725
	2	0.99992	1.00000	0.99871	0.99341	0.96907	0.92098	0.83335
	3	0.99800	0.99872	1.00000	0.99794	0.98018	0.93870	0.85733
	4	0.99185	0.99346	0.99795	1.00000	0.99084	0.95841	0.88620
	5	0.96699	0.97001	0.98075	0.99106	1.00000	0.98810	0.93900
	6	0.92228	0.92665	0.94309	0.96123	0.98866	1.00000	0.98084
	7	0.85079	0.85619	0.87717	0.90197	0.94665	0.98252	1.00000
n = 100	1	1.00000	0.99992	0.99799	0.99188	0.96591	0.91617	0.82681
	2	0.99992	1.00000	0.99871	0.99340	0.96904	0.92086	0.83292
	3	0.99800	0.99872	1.00000	0.99793	0.98015	0.93860	0.85694
	4	0.99194	0.99345	0.99794	1.00000	0.99083	0.95834	0.88586
	5	0.96696	0.96998	0.98073	0.99105	1.00000	0.98807	0.93878
	6	0.92219	0.92657	0.94301	0.96117	0.98864	1.00000	0.98075
	7	0.85057	0.85598	0.87697	0.90179	0.94652	0.98245	1.00000

Table 2

Efficiency (11) for heteroscedasticity φ_P

		1	2	3	4	5	6	7
n = 5	1	1.00000	0.99993	0.99835	0.99362	0.97594	0.94818	0.91048
	2	0.99993	1.00000	0.99885	0.99486	0.97853	0.95158	0.91474
	3	0.99834	0.99894	1.00000	0.99844	0.98659	0.96400	0.93075
	4	0.99354	0.99479	0.99843	1.00000	0.99409	0.97690	0.94847
	5	0.97538	0.97786	0.98638	0.99403	1.00000	0.99415	0.97634
	6	0.94679	0.95035	0.96329	0.97659	0.99413	1.00000	0.99383
	7	0.90833	0.91281	0.92957	0.94791	0.97630	0.99684	1.00000
n = 10	1	1.00000	0.99992	0.99817	0.99300	0.97398	0.94442	0.90431
	2	0.99992	1.00000	0.99884	0.99436	0.97658	0.94808	0.90886
	3	0.99814	0.99882	1.00000	0.99829	0.98554	0.96142	0.92593
	4	0.99282	0.99423	0.99827	1.00000	0.99936	0.97524	0.94481
	5	0.97291.	0.97569	0.98515	0.99353	1.00000	0.99673	0.97456
	6	0.94182	0.94580	0.96013	0.97469	0.99368	1.00000	0.99333
	7	0.90017	0.90516	0.92366	0.94370	0.97442	0.99634	1.00000
n = 20	1	1.00000	0.99992	0.99812	0.99284	0.97337	0.94303	0.90164
	2	0.99992	1.00000	0.99881	0.99423	0.97603	0.94677	0.90629
	3	0.99810	0.99879	1.00000	0.99825	0.98519	0.96041	0.92374
	4	0.99264	0.99409	0.99823	1.00000	0.99347	0.97456	0.94308
	5	0.97222	0.97507	0.98477	0.99337	1.00000	0.99353	0.97365
	6	0.94028	0.94437	0.95007	0.97399	0.99349	1.00000	0.99305
	7	0.89739	0.90251	0.92147	0.94201	0.97355	0.99307	1.00000

		1	2	3	4	5	6	7
n = 50	1	1.00000	0.99992	0.99811	0.99278	0.97311	0.94236	0.90023
	2	0.99992	1.00000	0.99880	0.99418	0.97579	0.94614	0.90492
	3	0.99808	0.99878	1.00000	0.99824	0.98503	0.95990	0.92255
	4	0.99258	0.99404	0.99822	1.00000	0.99340	0.97421	0.94211
	5	0.97196	0.97484	0.98462	0.99329	1.00000	0.99343	0.97312
	6	0.93965	0.94377	0.95859	0.97366	0.99339	1.00000	0.99289
	7	0.89614	0.90129	0.92041	0.94115	0.97307	0.99292	1.00000
n = 100	1	1.00000	0.99992	0.99811	0.99277	0.97304	0.94217	0.89980
	2	0.99992	1.00000	0.99880	0.99417	0.97573	0.94595	0.90450
	3	0.99808	0.99878	1.00000	0.99823	0.98499	0.95976	0.92217
	4	0.99257	0.99403	0.99821	1.00000	0.99338	0.97410	0.94180
	5	0.97190	0.97478	0.98458	0.99327	1.00000	0.99639	0.97295
	6	0.93949	0.94362	0.95846	0.97357	0.99336	1.00000	0.99284
	7	0.89579	0.90096	0.92011	0.94089	0.97292	0.99287	1.00000

Table 3

Efficiency (11) for heteroscedasticity φ_E

		1	2	3	4	5	6	7
n = 5	1	1.00000	0.99993	0.99825	0.99302	0.97231	0.93855	0.89285
	2	0.99993	1.00000	0.99888	0.99434	0.97494	0.94233	0.89759
	3	0.99825	0.99888	1.00000	0.99824	0.98421	0.95645	0.91573
	4	0.99302	0.99474	0.99824	1.00000	0.99290	0.97159	0.93647
	5	0.97237	0.97499	0.98424	0.99291	1.00000	0.99266	0.97039
	6	0.93882	0.94259	0.95664	0.97171	0.99267	1.00000	0.99228
	7	0.89361	0.89832	0.91633	0.93691	0.97055	0.99230	1.00000
n = 10	1	1.00000	0.99922	0.99806	0.99227	0.96938	0.93224	0.88232
	2	0.99902	1.00000	0.99876	0.99373	0.97229	0.93641	0.88751
	3	0.99806	0.99876	1.00000	0.99805	0.98254	0.95197	0.90743
	4	0.99227	0.99373	0.99806	1.00000	0.99215	0.96868	0.93023
	5	0.96947	0.97237	0.98259	0.99217	1.00000	0.99192	0.96753
	6	0.93267	0.93682	0.95227	0.96886	0.99195	1.00000	0.99156
	7	0.88353	0.88867	0.90839	0.93093	0.96779	0.99160	1.00000
n = 20	1	1.00000	0.99992	0.99802	0.99209	0.96868	0.93073	0.87981
	2	0.99992	1.00000	0.99873	0.99358	0.97165	0.93500	0.88512
	3	0.99802	0.99873	1.00000	0.99801	0.98214	0.95090	0.90546
	4	0.99209	0.99359	0.99801	1.00000	0.99197	0.96799	0.92875
	5	0.96878	0.97174	0.98220	0.99199	1.00000	0.99174	0.96685
	6	0.93120	0.93544	0.95123	0.96818	0.99177	1.00000	0.99139
	7	0.88114	0.88638	0.90651	0.92951	0.96714	0.99144	1.00000

		1	2	3	4	5	6	7
n = 50	1	1.00000	0.99992	0.99800	0.99204	0.96848	0.93032	0.87912
	2	0.99992	1.00000	0.99872	0.99354	0.97147	0.93461	0.88445
	3	0.99800	0.99872	1.00000	0.99800	0.98203	0.95061	0.90491
	4	0.99204	0.99355	0.99800	1.00000	0.99192	0.96779	0.92834
	5	0.96858	0.97157	0.98209	0.99194	1.00000	0.99169	0.96667
	6	0.93080	0.93506	0.95094	0.96800	0.99173	1.00000	0.99135
	7	0.88047	0.88575	0.90599	0.92912	0.96696	0.99139	1.00000
n = 100	1	1.00000	0.99992	0.99800	0.99203	0.96845	0.93026	0.87902
	2	0.99992	1.00000	0.99872	0.99354	0.97145	0.93455	0.88436
	3	0.99800	0.99872	1.00000	0.99800	0.98201	0.95056	0.90484
	4	0.99204	0.99354	0.99800	1.00000	0.99191	0.96777	0.92828
	5	0.96856	0.97154	0.98207	0.99193	1.00000	0.99169	0.96664
	6	0.93074	0.93500	0.95090	0.96797	0.99172	1.00000	0.99134
	7	0.88038	0.88566	0.90591	0.92907	0.96694	0.99139	1.00000

Table 4

Efficiency (11) for heteroscedasticity φ_G

		1	2	3	4	5	6	7
n = 5	1	1.00000	0.99997	0.99929	0.99724	0.98922	0.97531	0.95226
	2	0.99997	1.00000	0.99955	0.99776	0.99025	0.97684	0.95431
	3	0.99927	0.99954	1.00000	0.99931	0.99387	0.98247	0.96212
	4	0.99710	0.99767	0.99929	1.00000	0.99723	0.98848	0.97102
	5	0.98837	0.98954	0.99355	0.99715	1.00000	0.99692	0.98579
	6	0.97338	0.97516	0.98159	0.98817	0.99693	1.00000	0.99598
	7	0.95088	0.95326	0.96210	0.97169	0.98655	0.99618	1.00000
n = 10	1	1.00000	0.99998	0.99939	0.99753	0.98949	0.97386	0.94669
	2	0.99998	1.00000	0.99961	0.99799	0.99047	0.97543	0.94892
	3	0.99939	0.99961	1.00000	0.99937	0.99396	0.98127	0.95750
	4	0.99756	0.99802	0.99938	1.00000	0.99725	0.98759	0.96739
n = 20	1	1.00000	0.99998	0.99938	0.99747	0.98925	0.97330	0.94565
	2	0.99998	1.00000	0.99960	0.99795	0.99026	0.97490	0.94793
	3	0.99938	0.99960	1.00000	0.99936	0.99382	0.98087	0.95667
	4	0.99750	0.99797	0.99903	1.00000	0.99718	0.98733	0.96674
	5	0.98977	0.99073	0.99410	0.99728	1.00000	0.99657	0.98370
	6	0.97605	0.97748	0.98278	0.98848	0.99676	1.00000	0.99544
	7	0.95487	0.95675	0.96387	0.97199	0.98576	0.99578	1.00000

		1	2	3	4	5	6	7
n = 50	1	1.00000	0.99998	0.99937	0.99745	0.98918	0.97315	0.94537
	2	0.99998	1.00000	0.99960	0.99793	0.99020	0.97476	0.94765
	3	0.99937	0.99960	1.00000	0.99936	0.99378	0.98076	0.95644
	4	0.99748	0.99796	0.99936	1.00000	0.99717	0.98726	0.96656
	5	0.98970	0.99067	0.99406	0.99727	1.00000	0.99655	0.98361
	6	0.97590	0.97734	0.98267	0.98841	0.99674	1.00000	0.99541
	7	0.95460	0.95648	0.96365	0.97182	0.98567	0.99575	1.00000
n = 100	1	1.00000	0.99998	0.99937	0.99745	0.98917	0.97313	0.94553
	2	0.99998	1.00000	0.99960	0.99793	0.99019	0.97474	0.94762
	3	0.99937	0.99960	1.00000	0.99935	0.99378	0.98075	0.95641
	4	0.99748	0.99796	0.99936	1.00000	0.99716	0.98725	0.96654
	5	0.98969	0.99066	0.99406	0.99727	1.00000	0.99655	0.98360
	6	0.97588	0.97732	0.98266	0.98840	0.99674	1.00000	0.99541
	7	0.95456	0.95645	0.96362	0.97180	0.98566	0.99575	1.00000

Table 5

Efficiency (11) for heteroscedasticity φ_S

		1	2	3	4	5	6	7
n = 5	1	1.00000	0.99993	0.99827	0.99281	0.96935	0.92772	0.86778
	2	0.99993	1.00000	0.99888	0.99412	0.97203	0.93161	0.87257
	3	0.99826	0.99888	1.00000	0.99811	0.98177	0.94653	0.89144
	4	0.99275	0.99407	0.99810	1.00000	0.99142	0.96347	0.91421
	5	0.96838	0.97115	0.98122	0.99121	1.00000	0.93951	0.95544
	6	0.92208	0.92626	0.94237	0.96078	0.98899	1.00000	0.98677
	7	0.84763	0.85304	0.87457	0.90098	0.94941	0.98562	1.00000
n = 10	1	1.00000	0.99994	0.99844	0.99399	0.97771	0.95680	0.92469
	2	0.99994	1.00000	0.99900	0.99512	0.97980	0.95661	0.92800
	3	0.99838	0.99897	1.00000	0.99847	0.98719	0.96707	0.94064
	4	0.99343	0.99471	0.99840	1.00000	0.99417	0.97833	0.95499
	5	0.97305	0.97576	0.98511	0.99353	1.00000	0.99423	0.97853
	6	0.93753	0.94172	0.95705	0.97289	0.99348	1.00000	0.99417
	7	0.88509	0.89074	0.91208	0.93570	0.97202	0.99525	1.00000
n = 20	1	1.00000	0.99996	0.99896	0.99610	0.98624	0.97261	0.95679
	2	0.99996	1.00000	0.99933	0.99684	0.98755	0.97433	0.95879
	3	0.99889	0.99930	1.00000	0.99902	0.99216	0.98068	0.96634
	4	0.99551	0.99641	0.99894	1.00000	0.99646	0.98740	0.97476
	5	0.98161	0.98358	0.99019	0.99587	1.00000	0.99670	0.98818
	6	0.95717	0.96034	0.97162	0.98270	0.99609	1.00000	0.99684
	7	0.92024	0.92475	0.94128	0.95863	0.98319	0.99618	1.00000

		1	2	3	4	5	6	7
n = 50	1	1.00000	0.99998	0.99951	0.99819	0.99381	0.98796	0.98134
	2	0.99998	1.00000	0.99969	0.99854	0.99441	0.98874	0.98224
	3	0.99947	0.99966	1.00000	0.99955	0.99649	0.99157	0.98559
	4	0.99785	0.99829	0.99950	1.00000	0.99842	0.99454	0.98927
	5	0.99116	0.99215	0.99540	0.99811	1.00000	0.99858	0.99504
	6	0.97921	0.98087	0.98661	0.99204	0.99828	1.00000	0.99868
	7	0.96063	0.96311	0.97193	0.98078	0.99256	0.99837	1.00000
n = 100	1	1.00000	0.99999	0.99974	0.99605	0.99679	0.99380	0.99044
	2	0.99999	1.00000	0.99983	0.99923	0.99710	0.99420	0.99090
	3	0.99972	0.99982	1.00000	0.99976	0.99818	0.99567	0.99264
	4	0.99886	0.99909	0.99974	1.00000	0.99919	0.99720	0.99454
	5	0.99530	0.99583	0.99758	0.99901	1.00000	0.99928	0.99748
	6	0.98990	0.98981	0.99293	0.99583	0.99911	1.00000	0.99933
	7	0.97881	0.98020	0.98508	0.98989	0.99616	0.99917	1.00000

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PEWNE WYNIKI NA TEMAT EFEKTYWNOŚCI ESTYMACJI LINIOWEGO TRENDU W PRZYPADKU HETEROSKEDASTYCZNOŚCI

W pracy rozważa się model trendu postaci:

$$y_t = \alpha_0 + \alpha_1 t + \varepsilon_t \qquad t = 1, \ldots, n$$

przy założeniu heteroskedastyczności rozkładu składników losowych, określonej jako:

$$\sigma_t^2 = \sigma_0^2 \varphi(\beta, t) \qquad t = 1, \ldots, n,$$

gdzie φ oznacza daną funkcję, β - parametr nieznany, σ_t^2 - diagonalne elementy macierzy kowariancji Ω wektora składników losowych $\Omega = \Omega(\beta)$.

Rozważane jest pięć modeli heteroskedastyczności związanych z funkcjami φ.

Jako miarę heteroskedastyczności przyjmuje się współczynnik zmienności zbioru $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Bada się efektywność estymatorów współczynnika kierunkowego α_1. Wykorzystywana miara efektywności będąca stosunkiem wariancji badanych estymatorów jest funkcją trzech parametrów: wielkości próby n, parametru β oraz jego oceny γ. Przedstawiono numeryczne wyniki dla wybranych β, γ oraz kilku poziomów wielkości próby.