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PROPERTIES OF THE COX CONSISTENCY TEST IN THE CASE OF INCOME DISTRIBUTION ANALYSIS

Abstract

Testing the consistency of theoretical income distributions with the empirical ones is a very important problem in income distribution analysis. Most of well known goodness-of-fit tests cannot be used to solve this problem because the parameters of the population are usually not known and the samples are very large.

In the paper we present the main properties of the Cox statistic which is based on likelihood ratio. The presented results were obtained by means of the Monte Carlo experiment. The theoretical distributions most often used in income distribution analysis as the gamma, lognormal, Dagum and Singh-Maddala were taken into consideration.

Key words: statistical inference, income distribution, consistency test.

I. INTRODUCTION

Testing the consistency of empirical distributions with the theoretical ones is a very important problem in wage and income distribution analysis. A lot of consistency tests have been proposed in the literature. Taking into consideration the construction of a test statistic, they can be divided into the following groups:

- tests based on the comparison of density functions,
- tests based on the comparison of cumulative distribution functions,
- tests constructed on the basis of positional statistics,
- tests based on the moments of probability distributions.

In spite of a great variety of consistency tests, one can be faced with many problems trying to apply them to the analysis of income distributions.

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Firstly, the last two groups comprise only the normality tests, so they can hardly ever be used in income distribution analysis. Secondly, the population parameters are usually not known, what limits the possibilities of the application of Kołmogorov–Smirnow statistic based on the comparisons of cumulative distribution functions. Moreover, it has been proved that for very large samples the well known Pearson χ^2 statistic rejects the null hypothesis even when the discrepancies between the distributions are negligible. The samples coming from the Household Budgets or Labour Force Surveys, being the main source of information on personal income, are usually very large.

II. THE COX STATISTICS

Cox (1961) developed a general large-sample test procedure to verify the composite hypothesis about the consistency of a distribution with a theoretical one. This procedure was based on a modification of the Neyman-Pearson maximum likelihood ratio. It was proposed to obtain high power for a composite alternative hypothesis that a distribution is different from the one indicated in the null hypothesis.

Suppose that the observed value of a random vector $\mathbf{Y} = (Y_1, ..., Y_n)$ is to be used to test the null hypothesis, H_f, that the probability density function is $f(y, \theta)$, where $\boldsymbol{\alpha}$ is an unknown vector of parameters. Let it be required to obtain high power for the alternative hypothesis H_g, that the probability density function is $g(y, \eta)$, where $\boldsymbol{\eta}$ is unknown vector of parameters. It is worth mentioning, that the hypothesis H_g serves only to indicate the type of alternative for which high power is required. What is important, $f(y, \theta)$ and $g(y, \eta)$ are separate families of distributions. That means that for an arbitrary parameter θ_0, the density function $f(y, \theta)$ cannot be approximated by $g(y, \eta)$ arbitrarily closely. When the families of considered probability density functions are not separate, it is advisable to use the likelihood ratio test.

The test statistic proposed by Cox is the following:

$$T_f = L_f(\hat{\theta}) - L_g(\hat{\eta}) - E_{\theta}[L_f(\hat{\theta}) - L_g(\hat{\eta})] \tag{1}$$

where: $L_f(\hat{\theta})$, $L_g(\hat{\eta})$ denotes maximum log likelihoods under H_f or H_g respectively.

When H_f is true, T_f statistic is asymptotically normally distributed with expected value equal to zero:

$$T_f \sim as\ N(0, D(T_f))$$

If the roles of H_f and H_g as null and alternative hypothesis are interchanged we obtain a test statistic given in the form:

$$T_g = L_g(\hat{\eta}) - L_f(\hat{\theta}) - E_{\hat{\eta}}[L_g(\hat{\eta}) - L_f(\hat{\theta})] \tag{2}$$

where: $T_g \sim as\ N(0, D(T_g))$.

The statistics T_f and T_g are, in general, different functions of observations. Under H_f, T_g should be approximately zero, whereas under H_g, T_f should be negative. Hence, using T_f statistic three decisions are possible:

- rejection of H_f in direction of H_g, when T_f is significantly different from zero and negative,
- rejection of H_f away from H_g, when T_f is significantly different from zero and positive,
- no reasons for rejection of H_f when T_f is near zero.

III. DISTRIBUTION AND PROPERTIES OF THE COX STATISTIC FOR SELECTED PAIRS OF DISTRIBUTIONS

The aim of our research was to investigate the properties and the distribution of the Cox statistic. We took into consideration the probability density functions most often used in the analysis of income distributions. The problems we were particularily interested in were the following:

- researching the asymptotic distribution of the test statistic for selected pairs of theoretical distributions,
- assessing the influence of sample size on a test decision,
- evaluating the test power for selected alternative distributions.

In all the experiments the H_f hypothesis stated that the distribution is of the Dagum type. The cumulative distribution function of the Dagum distributions can be written in the form (Dagum, 1977):

$$F(y) = (1 + \lambda y^{-\delta})^{-\beta} \tag{3}$$

where: λ, β, δ – distribution parameters.
As alternative distributions gamma, lognormal and Burr type XII distribution were used. The gamma density function is the following:

$$f(y) = \frac{\lambda^{\alpha}}{\Gamma\alpha} y^{\alpha-1} \cdot e^{-\lambda y}, \quad y > 0, \tag{4}$$

where: λ – distribution parameters, while the lognormal density curve takes the form:

$$f(y) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{ -\frac{1}{2\sigma^2} (\ln y - \mu)^2 \right\}, \quad y > 0 \tag{5}$$

where: μ, σ – distribution parameters.

The Burr type XII distribution function, introduced to the income distribution analysis by Singh and Maddala (1976), can be written as follows:

$$F(y) = 1 - \frac{1}{(1 + a_1 y^{a_2})^{a_3}}, \quad y \geqslant 0 \tag{6}$$

where: a_1, a_2, a_3 – distribution parameters.

In case of the theoretical distributions mentioned above, the computation of the test statistic and its standard error is not a trivial matter. Hence, we had to carry out the Monte Carlo experiments. The mean and the standard deviation of T_f statistic were obtained by means of an experiment on the assumption that H_f is true and the empirical maximum likelihood estimate is the true parameter.

The stages of the Monte Carlo experiment. The first stage consisted of generating a random sample under the hypothesis that H_f is true ($f(y)$ is the density function of the Dagum distribution). The generating was made by means of the inversion of the Dagum cumulative distribution function. We dealt with a random sample where individual observations were grouped into intervals. The sample sizes were the following: 200, 500, 1000, 2000, 5000.

The second stage was to calculate the maximum likelihood estimators for the Dagum and alternative distributions (gamma, lognormal, Singh-Maddala). The density of an alternative distribution is denoted by $g(y)$.

The third step was evaluation, on the basis of N repetitions of the first two steps, the expected value and the variance of the following statistic:

$$T_f^* = L_{f(\hat{\theta})} - L_{g(\hat{\eta})} \tag{7}$$

On the basis of the experiment we obtained the parameters and the histograms of the Cox statistic distribution for different sample sizes. They are presented in Tables 1–3 and on figures.

Table 1. Expected values and standard deviations of T_f^* statistic ($N = 500$)

Sample size n	Alternative distribution					
	lognormal		gamma		Burr type XII	
	$E(T_f^*)$	$D(T_f^*)$	$E(T_f^*)$	$D(T_f^*)$	$E(T_f^*)$	$D(T_f^*)$
200	0.0102	0.0132	0.0384	0.0249	0.0054	0.0078
500	0.0085	0.0070	0.0361	0.0139	0.0034	0.0031
1000	0.0081	0.0050	0.0359	0.0099	0.0030	0.0019
2000	0.0078	0.0036	0.0355	0.0073	0.0028	0.0014
5000	0.0076	0.0023	0.0352	0.0046	0.0027	0.0008

Source: Author's calculations.

Table 2. Expected values and standard deviations of T_f^* *statistic* ($N = 1000$)

Sample size n	Alternative distribution					
	lognormal		gamma		Burr type XII	
	$E(T_f^*)$	$D(T_f^*)$	$E(T_f^*)$	$D(T_f^*)$	$E(T_f^*)$	$D(T_f^*)$
200	0.0099	0.0125	0.0377	0.0239	0.0053	0.0075
500	0.0085	0.0071	0.0364	0.0140	0.0035	0.0030
1000	0.0080	0.0057	0.0358	0.0103	0.0030	0.0019
2000	0.0077	0.0042	0.0352	0.0071	0.0028	0.0013
5000	0.0076	0.0022	0.0353	0.0046	0.0027	0.0008

Source: Author's calculations.

Table 3. Standardized values of Cox statistic T_f

Sample size n	Alternative distribution					
	lognormal		gamma		Burr type XII	
	$N = 500$	$N = 1000$	$N = 500$	$N = 1000$	$N = 500$	$N = 1000$
200	0.80	0.64	–	0.61	–	–0.14
500	1.21	1.39	–	2.73	–	0.24
1000	1.62	1.82	–	1.60	–	0.64
2000	2.17	2.54	–	2.38	–	1.06
5000	3.29*	4.88*	–	3.70*	–	1.80

Source: Author's calculations.

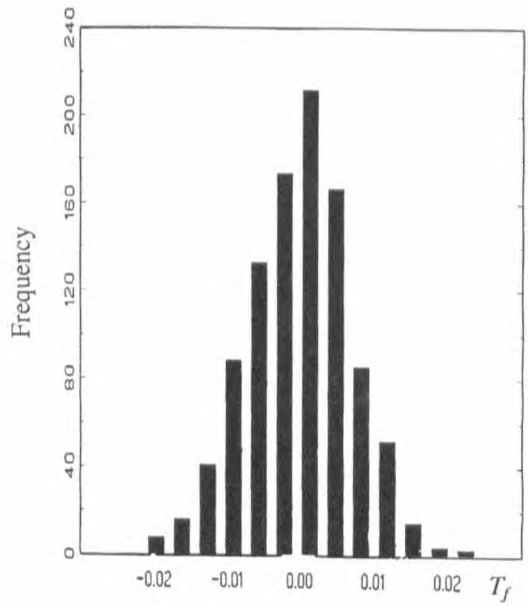

Figure 1. Empirical distribution of T_f statistic ($n = 500$)

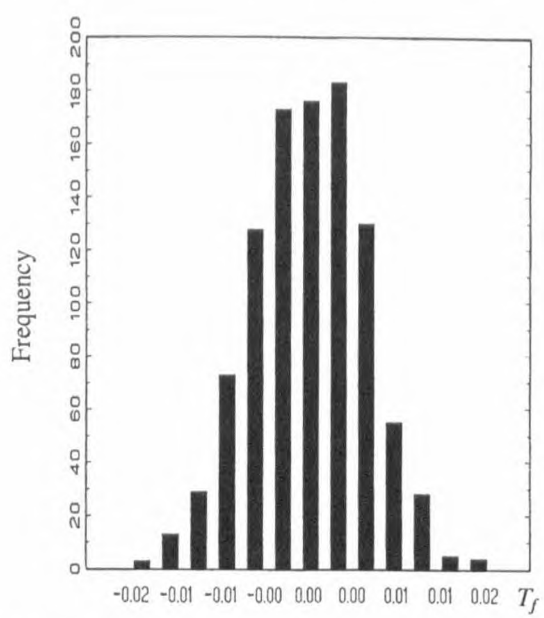

Figure 2. Empirical distribution of T_f statistic ($n = 1000$)

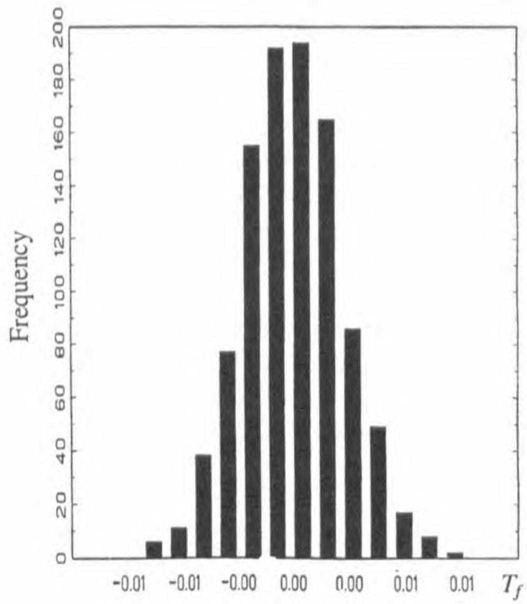

Figure 3. Empirical distribution of T_f statistic ($n = 2000$)

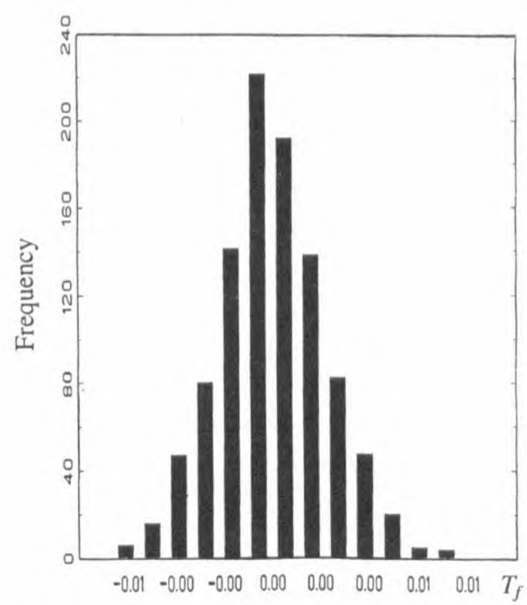

Figure 4. Empirical distribution of T_f statistic ($n = 5000$)

The second Monte Carlo procedure was focused on the evaluation of the Cox test power for selected alternative distributions. First, we had to generate a random sample assuming that the distribution is: a) lognormal, b) gamma, c) Burr type XII.

Then the value of T_f statistic was calculated from this sample, using the value $E(T_f^*)$ estimated in the previous experiment (Tables 1 or 2). For the calculation of the standardized value of T_f statistic, standard deviations calculated in the first experiment were necessary. When $|T_{f\,STAND}| > T_\alpha$ we rejected the null hypothesis that the general population distribution is of the Dagum form.

After N repetitions of the experiment we got the empirical test power. It was calculated as the numbers of good decisions (rejections of H_0) divided by the total number of random samples generated from the appropriate alternative distribution. The results are presented in Tables 4–5.

Table 4. Cox test power for selected alternative distributions (N = 500)

n \ α	0.1	0.05	0.025	0.1	0.005
Lognormal distribution					
200	0.350	0.188	0.106	0.032	0.006
500	0.758	0.616	0.504	0.336	0.246
1 000	0.964	0.910	0.836	0.378	0.660
2 000	1.000	1.000	1.000	1.000	1.000
5 000	1.000	1.000	1.000	1.000	1.000
Gamma distribution					
200	0.978	0.948	0.908	0.830	0.754
500	1.000	1.000	1.000	0.998	0.988
1 000	1.000	1.000	1.000	1.000	1.000
2 000	1.000	1.000	1.000	1.000	1.000
5 000	1.000	1.000	1.000	1.000	1.000
Burr type XII distribution					
200	0.104	0.066	0.050	0.032	0.024
500	0.182	0.142	0.086	0.050	0.038
1 000	0.408	0.254	0.158	0.114	0.086
2 000	0.718	0.524	0.336	0.200	0.150
5 000	0.998	0.986	0.964	0.918	0.846

Source: Author's calculations.

Table 5. Cox test power for selected alternative distributions ($N = 1000$)

n \ α	0.1	0.05	0.025	0.1	0.005
Lognormal distribution					
200	0.362	0.218	0.119	0.047	0.020
500	0.780	0.643	0.516	0.357	0.252
1 000	0.935	0.846	0.747	0.594	0.483
2 000	0.997	0.991	0.976	0.921	0.876
5 000	1.000	1.000	1.000	1.000	1.000
Gamma distribution					
200	0.976	0.951	0.917	0.848	0.784
500	1.000	1.000	1.000	0.999	0.998
1 000	1.000	1.000	1.000	1.000	1.000
2 000	1.000	1.000	1.000	1.000	1.000
5 000	1.000	1.000	1.000	1.000	1.000
Burr type XII distribution					
200	0.096	0.072	0.048	0.030	0.021
500	0.200	0.139	0.092	0.057	0.45
1 000	0.427	0.286	0.187	0.123	0.090
2 000	0.792	0.624	0.454	0.289	0.204
5 000	0.995	0.985	0.959	0.901	0.826

Source: Author's calculations.

IV. CONCLUSIONS

Figures 1–4 present empirical frequencies of T_f statistic for different sample sizes. They were calculated on the basis of the Monte Carlo experiment conducted for the Dagum ($f(y)$) and lognormal ($g(y)$) distributions. It can be easily noticed that with increasing sample size the shapes of the presented empirical distributions become similar to the normal distribution. Simultaneously, the standard deviations of the considered test statistic are very small, tending to zero as the sample size increase (Tables 1–2) and as discrepancies between the compared distributions become relatively small (the case of Dagum and Burr type XII distribution).

The results of the second experiment are presented in Tables 4–5. They show empirical test power calculated for the selected pairs of distributions. The best results were obtained when the alternative distribution was the gamma curve – empirical test power are near one for all sample sizes. Very high test power for relatively small samples was obtained also for the lognormal distribution. For the Burr type XII distribution empirical test power tends to one for sample size 5000 and more. It can be explained by high similarity at the compared distributions.

Summing up, one can say, that the Cox statistic can be useful in the analysis of income distributions in Poland. The test power for the alternatives most often used is very high for relatively small samples.

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WŁASNOŚCI TESTÓW ZGODNOŚCI COXA W PRZYPADKU BADANIA ZGODNOŚCI ROZKŁADÓW DOCHODÓW

Streszczenie

W analizie rozkładów płac i dochodów istotnym problemem jest badanie zgodności rozkładów empirycznych z teoretycznymi. Większość znanych testów zgodności nie może być stosowana do badania tego zagadnienia ze względu na fakt, że parametry zbiorowości generalnej nie są na ogół znane, a rozważane próby są często bardzo liczne.

W artykule przedstawione zostały podstawowe własności testu zgodności Coxa opartego na ilorazie wiarygodności. Prezentowane wyniki otrzymano metodą Monte Carlo. Rozważane były rozkłady teoretyczne najczęściej wykorzystywane w analizie płac i dochodów: gamma, logarytmiczno-normalny, Daguma i Singha–Maddali.