ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 216, 2008

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MULTIVALUED COHERENT RISK MEASURES

ABSTRACT. The concept of coherent risk measures with its axiomatic characterization was discussed in a finite probability spaces. The aim of this paper is to apply a multivalued random variable as a multivalued risk measures, for description the risk of portfolio. This is a study related to aggregation problem. We study to alternative methods of aggregation: coherent aggregation of random portfolios and coherent aggregation of risk.

Key words: Coherent risk measures, risk aggregation.

I. INTRODUCTION

Artzner et al introduced the concept of coherent risk measures together with its axiomatic characterization. In this paper, the risky portfolio under consideration is a given real-valued random variable. A risk measure ρ is then defined as a map from L^∞ into R satisfying some coherency axioms, so that for any $X \in L^\infty$, $\rho(X + \rho(X)) = 0$, i.e., the deterministic amount $\rho(X)$ cancels the risk of X.

We focus on the more realistic situation where the risky portfolio is an R^d-valued random variable. We assume that a partial ordering $\succ$ on R^d is given. The specification of $\succ$ accounts for some frictions on the financial market such as transaction costs, liquidity problems, irreversible transfers, etc. We will notice an extension of the axiomatic characterization to multi-dimensional framework. Given an integer $n \le d$, we define (d, n)-coherent risk measure (consistent with $\succ$) as a multivalued map R from L_d^∞ into R^n satisfying some convenient axioms. When $n = d = 1$, we recover the results of Delbaen [3] by setting $R = [\rho, \infty)$.

Throughout this paper, we shall denote by x_i the i-th component of an element x of a finite dimensional vector space. We shall denote by $\mathbf{1}^i$ the i-th canonical basis vector defined by $\mathbf{1}^i_j = 1$ if $i = j$, zero otherwise, and we set $\mathbf{1} := \sum_i \mathbf{1}^i$ the vector with unit components. The latter notation should not be confused with

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the indicator function $\mathbf{1}^A$ of a set A. The closure, the interior, and the relative interior of a set will be denoted respectively by cl[·], int[·], and ri[·].

Given a subset $A \subset \mathrm{R}^d$, we shall denote by $L^p{}_d(A)$ the collection of A-valued random variables with finite L^p-norm. We shall use the simplified notation $L^p{}_d := L^p{}_d(\mathrm{R}^d)$, $L^p(A) := L^p{}_1(A)$, and $L^p := L^p(\mathrm{R})$. As usual, L^0 and L^∞ stand respectively for the set of all measurable functions, and all essentially bounded functions.

II. MULTIVALUED COHERENT RISK MEASURES ON $L^\infty{}_D$

Let (Ω, F, P) be a probability space. In this paper we study the financial risk induced by a random portfolio from the point of view of the regulator/supervisor. In mathematical words, a (random) portfolio is a vector-valued random variable X on the probability space (Ω, F, P). We shall restrict our attention to portfolios in $L^\infty{}_d$, the space of all equivalence classes of (essentially) bounded R^d-valued random variables. We intend to extend the notion of coherent risk measure to the multidimensional case. Real-valued coherent measures of risk have been introduced by ADEH [2]; see also Delbaen [3] for the general probability space setting.

Portfolios in $L^\infty{}_d$ are (partially) ordered according to the following rule. Let K be a closed convex cone of R^d such that

$$\mathrm{R}^d{}_+ \subset K \text{ and } K \neq \mathrm{R}^d . \qquad (2.1)$$

The closed convex cone K induces the partial ordering $\succ$ on R^d by $x \succ 0$ iff $x \in K$. We extend naturally the partial ordering $\succ$ to $L^\infty{}_d$ by:

$$X \succ 0 \text{ iff } X \in K \; P\text{ - a.s.}$$

With this definition the condition $\mathrm{R}^d{}_+ \subset K$ means that any portfolio x with nonnegative entries is non-negative in the sense of the partial ordering $\succ$. We assume further that K satisfies the *substitutability* condition:

$$\text{for all } i = n + 1, \ldots , d : -\mathbf{1}^i + \alpha\mathbf{1}^1 \text{ and } \mathbf{1}^i - \beta\mathbf{1}^1 \in K \text{ for some } \alpha, \beta > 0 . \qquad (2.2)$$

Condition (2.2) means that any position on each entry $i > n$ can be compensated by some position on the first entry. More precisely, it states that the unitary prices of the assets $i > n$ in terms of the assets $j \leq n$ must be bounded. In the case $n = d$, condition (2.2) is empty.

(d, n)-Coherent risk measures

We extend the notion of coherent risk measure introduced previously in ADEH to allow for random portfolios valued in R^d. Each component of this portfolio corresponds to a specific security market. The motivation is that investors are in general not able to aggregate their portfolio because of liquidity problems and/or transaction costs between the different security markets.

– In order for a random portfolio X to be acceptable in terms of "risk", the regulator/ supervisor recommends that some deterministic portfolio x^0 be added to the position. We then say that x^0 *cancels* the risk induced by X if the aggregate portfolio $X + x^0$ is acceptable by the regulator/supervisor in the sense of the risk measure. The risk measure of the portfolio X will consist of the collection of such deterministic portfolios x^0.

– The integer d, representing the dimensionality of the portfolio $X(\omega)$, is typically large since the firm has positions on many different securities markets. Although regulator/supervisor can possibly recommend any deterministic portfolio $x^0 \in R^d$ which cancels the risk of X, it is natural to restrict x^0 to have a small number $n \leq d$ of non-zero entries. This reduction can be obtained by means of some aggregation procedure either of the initial random portfolio X or of the deterministic portfolio x^0.

– For instance, when an amount of cash in Dollars is recommended to be added to the position, we have $n = 1$. When the regulator/supervisor recommends to add two different amounts of cash in Dollars and in Euros, we are in the situation $n = 2$.

– By possibly rearranging the components of x^0, we shall consider that its last d-n components are zero, for some integer $n \leq d$. This suggests the following (which will be used throughout the paper):

$$\text{for all } x \in R^n,\ x^0 := (x, 0) \in R^d .$$

In conclusion, the notion of (d, n)-risk measure should be defined as a L^∞_d (the set of bounded random portfolios) into the subsets of R^n. We following definition which will be shown to be a convenient extension our context.

Definition 2.1 *A (d, n)-coherent risk measure is a multivalued map R : $L^\infty_d \rightarrow R^n$ satisfying the following axioms :*

A0 – *For all $X \in L^\infty_d$, $R(X)$ is closed, and $0 \in R(0) \neq R^n$;*

A1 – *For all $X \in L^\infty_d$: $X \succ 0$ P-a.s. $\Rightarrow R(0) \subset R(X)$;*

A2 – *For all $X, Y \in L^\infty_d$, $R(X) + R(Y) \subset R(X + Y)$;*

A3 – *For all $t > 0$ and $X \in L^\infty_d$, $R(tX) = t\, R(X)$;*

A4 – *For all $x \in R^n$ and $X \in L^\infty_d$, $R(X + x^0) = \{-x\} + R(X)$.*

Remark 2.2 Let us specialize the discussion to the one-dimensional setting $d = n = 1$. Starting from a multivalued mapping $R: L^{\infty}_{1} \to \mathrm{R}$ satisfying A0, we define $\rho(x) := \min R(x) > -\infty$

Assume that $R(X)$ coincides with $[\rho(X), +\infty)$ (A2 and A3 will guarantee that $R(X)$ is comprehensive which ensures that in the one dimensional case R is of the above form, see Property 3.1 below). Then, it is easily checked that R satisfies A1-A2-A3-A4 if and only if ρ is a coherent risk measure in the sense of ADEH [2] and Delbaen [3].

Before going any further, we briefly comment Axioms A0 through A4 introduced in the previous definition.

– The first requirement in A0 is natural, and only needed for technical reasons. Then, A0 says that 0 is a deterministic portfolio, which allows to cancel the risk of the null portfolio. The condition $R(0) \neq \mathrm{R}^n$ is assumed to avoid the trivial case $R(X) = \mathrm{R}^n$ for all $X \in L^{\infty}_{d}$.

– A1 says that any deterministic portfolio in $R(0)$ allows to cancel the risk of a portfolio X, whenever $X \succ 0$.

– A2 is the usual reduction property by risk aggregation: let x (resp. y) be a deterministic portfolio in R^n which cancels the risk of X (resp. Y). Then $x + y$ cancels the risk of the aggregate risk $X + Y$.

– A3 is the usual positive homogeneity property of the risk measure.

– A4 is the analogue of the translation invariance axiom introduced in ADEH.

(d, n)-acceptance sets

An alternative way of defining risk measures is provided by the notion of acceptance set, i.e., the set of random portfolios $X \in L^{\infty}_{d}$ which are viewed as free from risk by the supervisor/regulator.

Definition 2.2 *A (d, n)-acceptance set is a closed convex cone A of* L^{∞}_{d}, *containing* $L^{\infty}_{d}(K)$, *and such that* $\mathrm{R}^n \times \{0\}^{d-n} \not\subset A$.

Remark 2.3 This definition is motivated by the following observation. Let R be a (d, n)-coherent risk measure. Then $A := \{X \in L^{\infty}_{d} : R(0) \subset R(X)\}$ is a (d, n)-acceptance set in the sense of the above definition. This claim is a direct consequence of the properties stated in the subsequent section.

We now show that the notion of acceptance sets is directly connected to coherent risk measures.

Theorem 2.1 [6] *Let A be a subset of L^{∞}_{d}, and define the multivalued map R_A: $L^{\infty}_{d} \to \mathrm{R}^n$ by*

$$R_A(X) := \{x \in \mathrm{R}^n : X + x^0 \in A\}.$$

Then, A is a (d, n)-acceptance set if and only if R_A is a (d, n)-coherent risk measure.

Example 2.1: Multivalued WCE_α

In ADEH, the authors propose the use of the *worst conditional expectation* measure of risk defined by:

$$\text{For } X \in L^{\infty}\text{: } WCE_\alpha := \inf_{B \in F^\alpha} E[\,X|B\,],$$

where $F^\alpha := \{B \in F : P[B] > \alpha\}$,
and the *level* α is a given parameter in (0, 1). The corresponding acceptance set is given by:

$$A_{WCE_\alpha} := \{\, X \in L^{\infty} : \{\,X|B\,\}\, for\ all\ B \in F^\alpha \,\}.$$

The functional WCE_α is a coherent risk measure, in the sense of ADEH, which appears naturally as a good alternative for the (non-coherent) *Value-at-Risk* measure.

We now provide an extension of this coherent risk measure to our multidimensional framework. Let J be a closed convex cone of R^d such that:

$$K \subset J \text{ and } J \neq \mathrm{R}^d$$

and define the subset of L^{∞}_{d} :

$$A^J_\alpha = \{\, X \in L^{\infty}_d : E[\,X|B\,] \in J\ P-a.s.\ for\ all\ B \in F^\alpha \,\}.$$

Observe that A^J_α coincides with $A_{WCE\alpha}$ when $d = 1$. Clearly A^J_α is a closed convex cone of L^{∞}_{d} containing $L^{\infty}_{d}(K)$. Also, for all positive integer $n \leq d$, A^J_α does not contain the deterministic set $\mathrm{R}^n \times \{0\}^{d-n}$. Hence A^J_α is a (d, n)-acceptance set, and the multivalued map:

$$WCE^J_\alpha(X) := R_{A^J_\alpha}(X) = \{\, x \in R^n : X + x^0 \in A^J_\alpha \,\}$$

defines a (d, n)-coherent risk measure. This is a natural extension of the worst conditional expectation risk measure to the multi-dimensional framework.

Notice that the risk measure WCE_a is shown to coincide with the *Tail VaR* in the one-dimensional case, under suitable conditions, and is therefore as easy to compute in practice as the VaR measure. We leave for future research the possible extensions of these results to our multi-dimensional framework.

III. PROPERTIES OF COHERENT RISK MEASURES

We now derive some properties of (d, n)-coherent risk measures as defined in Definition 2.1.

Property 3.1 *$R(X)$ is a closed convex subset of* R^n, *$R(0)$ is a closed convex cone of* R^n, *and*

$$R(X) = R(X) + R(0) \textit{ for all } X \in L^{\infty}{}_d .$$

The next result requires the following additional notations:

$$K_n := \{x \in \mathrm{R}^n : x^0 \in K\} \text{ and } \mathbf{R}_0 := R(0) \cap -R(0).$$

Observe that $\mathbf{R}_0$ is a vector space.

Property 3.2 (Consistency with $\succ$) $K_n \subset R(0)$ *and :*

$$\mathrm{int}(-K_n) \cap R(0) = (-K_n \setminus \mathbf{R}_0) \cap R(0) = \varnothing .$$

Property 3.3 (Monotonicity)

(i) *Let X, $Y \in L^{\infty}{}_d$ be such that $X \succ Y$. Then, $R(Y) \subset R(X)$.*

(ii) *Let $X \in L^{\infty}{}_d$ be such that $a^0 \succ X \succ b^0$ for some a, $b \in \mathrm{R}^n$. Then:*

$$\{-b\} + R(0) \subset R(X) \subset \{-a\} + R(0) .$$

(iii) *For all $X \in L^{\infty}{}_d$, we have* $\{\|\pi(X)_{\infty} 1\|\} + R(0) \subset R(X)$.

Property 3.4 (Self-consistency) *For all $X \in L^{\infty}{}_d$,*

$$R(X) = \{x \in \mathrm{R}^n : 0 \subset R(X + x^0)\} = \{x \in \mathrm{R}^n : R(0) \subset R(X + x^0)\} .$$

The final property of this section states the continuity of the multivalued map R.

We recall that

– a multivalued map F from a metric vector space U into a metric vector space V is said to be continuous if it is both lower semi-continuous and upper semi-continuous,

– F is lower semi-continuous at some $u \in U$ if for all $v \in F(u)$ and for any sequence
$(u^n)_n \subset \mathrm{dom}(F)$ converging to u, there is a sequence $v^n \in F(u^n)$ such that $v^n \to v$,

– F is upper semi-continuous at some $u \in U$ if for all $\varepsilon > 0$, there exists a constant $\eta > 0$ such that $F(u + \eta B_U) \subset F(u)+\varepsilon B_V$; B_U and B_V are the unit balls of U and V.

Property 3.5 (Continuity)

(i) *For all* $X, Y \in L^{\infty}_{d}$,

$$R(Y) + \{\|\pi(Y - X)_{\infty}1\|\} \subset R(X) \subset R(Y) - \{\|\pi(Y - X)_{\infty}1\|\}.$$

(ii) *The multivalued map* R *is continuous on* L^{∞}_{d}.

IV. COHERENT AGGREGATION OF RANDOM PORTFOLIOS

Definition 4.1 *Let* R *be a* (n, n)*-coherent risk measure. A function* $f : \mathrm{R}^d \to \mathrm{R}^n$ *is an* R*-coherent portfolio aggregator if*

PA1 $f(K) \subset R(0)$;

PA2 *For all* $x, y \in \mathrm{R}^d$: $f(x + y) - f(x) - f(y) \in R(0)$;

PA3 *For all* $x \in \mathrm{R}^d$ *and* $t > 0$: $f(tx) - tf(x) \in \mathbf{R}_0$;

PA4 *For all* $x \in \mathrm{R}^n$ *and* $y \in \mathrm{R}^d$: $f(x^0 + y) - f(y) - x \in \mathbf{R}_0$.

We shall discuss some examples of R-coherent portfolio aggregators at the end of this section.

Our next result requires introducing a **stronger version of A1:**

A1$_s$ For all $X \in L^{\infty}_{d}$, we have: $X \in R(0)$ P-a. s. $\Rightarrow$. $0 \in R(X)$.

Theorem 4.1 [6] *Let* R *be a* (n, n)*-coherent risk measure, and let* f *be a mapping from* R^d *into* R^n.

(i) *Suppose that the multivalued map* $R \circ f : L^{\infty}_{d} \to \mathrm{R}^n$ *is a* (d, n)*-coherent risk measure. Then* f *is an* R*-coherent portfolio aggregator.*

(ii) *Conversely, assume that* A1*s holds, and let f be an R-coherent portfolio aggregator. Then the multivalued map* $R \circ f: L^{\infty}_{d} \to \mathrm{R}^n$ *is a (d, n)-coherent risk measure.*

V. COHERENT AGGREGATION OF RISK

Definition 5.1 *Let R be a (d, d)-coherent risk measure. A function g:* $\mathrm{R}^d \to \mathrm{R}^n$ *is an R-coherent risk aggregator if:*

RA1 $g(R(0)) \neq \mathrm{R}^n$ *and* $0 \in g(R(0))$*;*

RA2 *For all* $x, y \in \mathrm{R}^d$: $g(x) + g(y) \in \mathrm{cl}[g(R(-x - y))]$*;*

RA3 *For all* $x \in \mathrm{R}^d$ *and* $t > 0$*:*

$$g(tx) \in \mathrm{cl}[tg(R(-x))] \text{ and } tg(x) \in \mathrm{cl}[g(tR(-x))] \; ;$$

RA4 *For all* $x \in \mathrm{R}^n$ *and* $y \in \mathrm{R}^d$:

$$g(x^0 + y) \in x + \mathrm{cl}[g(R(-y))] \text{ and } x + g(y) \in \mathrm{cl}[g(R(-x^0 - y))] \, .$$

Some examples of coherent risk aggregators will be discussed at the end of this section.

Theorem 5.1 [6] *Let R be a (d, d)-coherent risk measure, and let g be a mapping from* R^d *into* R^n. *Define the multivalued map*

$$\mathrm{cl}[g \circ R]: L^{\infty}_{d} \to \mathrm{R}^n$$
$$X \to \mathrm{cl}[g(R(X))].$$

Then, $\mathrm{cl}[g \circ R]$ *is a (d, n)-coherent risk measure if and only if g is an R-coherent risk aggregator.*

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WIELOWARTOŚCIOWE KOHERENTNE MIARY RYZYKA

Koncepcja koherentnych miar ryzyka wraz z układem aksjomatów jest dyskutowana w skończonej przestrzeni probabilistycznej. Celem artykułu jest wykorzystanie wielowartościowych zmiennych losowych jako wielowartościowych miar ryzyka do opisu ryzyka portfela aktywów finansowych. Jest to problem agregacji informacji. Rozważamy dwa podejścia: koherentna agregacja losowych stop zwrotu z portfeli oraz koherentna agregacja ryzyka.