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DYNAMIC PROGRAMMING WITH RETURNS IN RANDOM VARIABLES SPACES

Abstract. This paper presents a model of dynamic, discrete decision-making problem (finite number of periods, states and decision variables). Described process has returns in random variables spaces equipped with partial order. The model can be applied for many multi-stage, multi-criteria decision making problems. There are a lot of order relations to compare random variables. Properties of those structures let us apply Bellman's Principle of dynamic programming. The result of using this procedure is obtainment of a whole set of optimal values (in the sense of order relation). For illustration, there is presented a numerical example.

Key words: dynamic programming, partially ordered set, stochastic dominance.

1. INTRODUCTION

The paper presents an optimal model involved into a shape of dynamic programming. This theory was introduced by R. Bellman (a method of solving such tasks is named Bellman's principle). Next T. A. Brown and R. E. Strauch (1965) generalized Bellman's principle to a class of multi-criteria dynamic programming with a lattical order. Then, the use of optimality principle was the interest of L. Mitten (1974) who considered preferences relation, M. I. Henig (1985) – who developed the theory of infinite dynamic process with values of criteria function in a partially ordered set. Others who took interest in the use of multicriterial methods in dynamic programming have been: T. Trzaskalik (1998), D. Li and Y. Y. Haimes (1989).

In the meantime some theories of comparing random variables has developed as well T. Rolski (1976), M. Shaked and J. G. Shanthikumar (1993). It enabled us to use such structures in our dynamic model, which is the essence of the paper. In the example, there is shown

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a combination of fields, namely, dynamic programming and random variables with stochastic dominance.

2. DYNAMIC MODEL

We consider multistage dynamic process with finite number of periods, states and decision variables. To describe the process we will use the following notation:

T – the number of periods,

S_t – the set of all feasible state variables at the beginning of period $t \in \{1, ..., T\}$,

$D_t(s_t)$ – the set of all feasible decision variables for period t and state $s_t \in S_t$, we assume that all these sets are finite,

P – denotes the process, where all sets: T, S_t, $D_t(s_t)$ are identified, on the base of these terms we define:

$r_t = (s_t, s_{t+1})$ – the period realization, $s_t \in S_t$ and $s_{t+1} \in D_t(s_t)$,

R_t – the set of all period realizations in period t,

$(s_t, ..., s_{T+1})$ – the partial realization in period t and $s_w \in S_w$, and $s_{w+1} \in D_w(s_w)$ for $w \in (t, ..., T)$,

$R_t(s_t)$ – the set of all partial realizations which begin at state s_t, $R_t(S_t) = R_t(s_t) : s_t \in S_t$ – the set of all partial realizations which begin at the beginning of period t,

$R = R_1(S_1)$ – the set of all process realizations,

We consider the following structure, functions and operators to describe multi-period criteria function of process realization.

$(W, \leqslant, \circ)$ – the structure in which $(W, \leqslant)$ is the partially ordered set, and $(W, \circ)$ is semigroup satisfies following condition

$$a \leqslant b \Rightarrow (a \circ c \leqslant b \circ c \quad \text{and} \quad c \circ a \leqslant c \circ b); \quad a, b, c \in W \text{ (monotonicity condition)} \tag{1}$$

For each finite subset $A \subset W$ we define

$$\max(A) \cong \{a^* \in A: \ \sim \exists_{a \in A} a^* \leqslant a \quad \text{and} \quad a^* \neq a\} \tag{2}$$

Values of the criteria function are given by the structure $(W, \leqslant, \circ)$

$f_t: R_t \to W$ – the period criteria functions with returns in W.

$F_t: R_t(S_t) \to W$ – the functions defined in the following way

$$F_t \cong f_t \circ (f_{t-1} \circ (... (f_{T-1} \circ f_T))), \quad t = T, ..., 1 \tag{3}$$

$F = F_1$ – the multi-period criteria function,

(P, F) – denotes discrete dynamic decision process. It is given, if there are discrete dynamic process P and multi-period criteria function F defined.

Realization $d^* \in D$ is said to be efficient, if

$$F(d^*) \in \max \ F(D) \tag{4}$$

Theorem 1. Let (P, F) be decision dynamic process.

(i)

For all $t = T - 1, ..., 1$ and all $y_t \in Y_t$ holds

$$\max \ \{F_t(R_t(s_t)\} = \max \ \{f_t(s_t, s_{t+1}) \circ \max \ F_{t+1}(R_{t+1}(s_t + 1)) : s_{t+1} \in D_t(s_t)\} \tag{5}$$

(ii)

$$\max \ \{F(R)\} = \max\{\max \ F_1(R_1(s_1)) : s_1 \in S_1\} \tag{6}$$

3. PROCEDURE

We now present an algorithm for determination of the set of all maximal returns of the process, which is based on Th. 1. This procedure is stated as follows:

step 1

Calculate the set: $\max\{F_T(R_T(s_T)\}$ for all states $s_T \in S_T$.

    step $(T + 1 - t)$, for $t = T - 3, T - 4, ..., 1$.

Calculate the set: $\max\{F_t(R_t(s_t))$ using Th. 1 (i).

    step $T + 1$

Calculate the set: $\max\{F(R)\}$ using Th. 1 (ii).

4. EXAMPLE

The algorithm presented in the third section is now applied to solve dynamic problem. We use different orders to compare values of the criteria

function i.e. random variables. The notation below agrees with symbols previously used.

The process (P, F) is defined as follows:
We consider a process, which consists of 3 periods $[T = 3]$, in which:

$$S_t = \{0,1\}, \text{ for } t = 1, 2, 3, 4; \ D_t(0) = D_t(1) = \{0, 1\}, \text{ for } t = 1, 2, 3 \quad (7)$$

The terms connected with the criteria function are defined as follows. Set W is described as the set of discrete random variables

$$W = \{(p_0, p_1, p_2, ..., p_n): n \in N, \ p_n > 0, \ p_i \geqslant 0, \ \sum_{i=0}^{n} p_i = 1\} \quad (8)$$

where p_i denotes probability of number i.
Operator ∘ is defined as a sum of random variables.
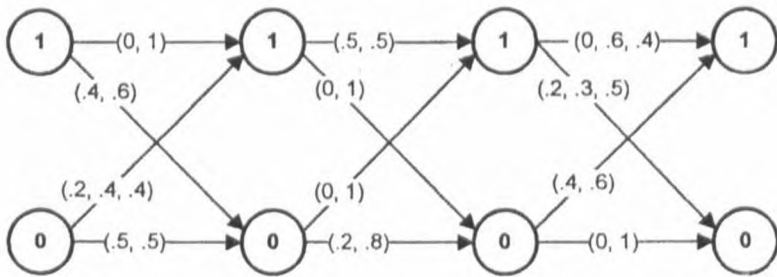The values of the criteria function are presented in the Fig. 1.

Fig. 1. The values of the period criteria function

5. THE ORDERS USED IN THE EXAMPLE

To compare such values of the criteria function we use known orders generated by:

1) first order stochastic dominance FSD,
2) second order stochastic dominance SSD,
3) second order inverse stochastic dominance SISD,
4) mean-variance model (as two-criteria: maximizing mean and minimizing variance).

Those classical definition can be found among others in M. S h a k e d and J. G. S h a n t h i k u m a r (1993) – stochastic orders; and mean-variance model in H. M. M a r k o v i t z (1989). The relations used in this example

are not antisymmetric, which is one of the conditions of partial order, but one can consider the equivalence relation $(x \leqslant 'y \Leftrightarrow x \leqslant y$ and $y \leqslant x)$ G. B i r k h o f f (1973). The monotonicity conditions of these structures are shown in M. S h a k e d and J. G. S h a n t h i k u m a r (1993). Moreover the rest of the conditions which are needed to hold the Th. 1 are easy to check.

6. RESULTS

Below, there are results of using algorithm presented. The tables show maximal values obtained in each step.

Table 1

The computation in the case of first order stochastic dominance (FSD) and second order stochastic dominance (SSD). The values connected with SSD case are bold

t	$\max\{F_t(R_t(0))$	$\max\{F_t(R_t(1))$
3	**(0,1)**	(.2, .3, .5), **(0, .6, .4)**
2	(0, .2, .3, .5), **(0, 0, .6, .4)**	**(0, 0, 1)**, (.1, .25, .4, .25), (0, .3, .5, .2)
1	**(0, 0, .2, .4, .4)**, (.02, .09, .22, .31, .26, .1), (0, .06, .22, .36, .28, .08)	**(0, 0, 0, 1)**, (0,0, .24, .52, .24), (0, .08, .24, .38, .3)
	max $F(R)$	
	(0, 0, 0, 1), **(0, 0,. 2, .4, .4)**, (.02, .09, .22, .31, .26, .1), (0, .06, .22, .36, .28, .08)	

Table 2

The computation in the case of second order inverse stochastic dominance (SISD)

t	$\max\{F_t(R_t(0))$	$\max\{F_t(R_t(1))$
3	(0, 1)	(.2, .3, .5), (0, .6, .4)
2	(0, .2, .3, .5), (0, 0, .6, .4)	(0, 0, 1), (.1, .25, .4, .25), (0, .3, .5, .2)
1	(0, 0, .2, .4, .4) (.02, .09, .22, .31, .26, .1) (0, .06, .22, .36, .28, .08)	(0, 0, .24, .52, .24), (0, .08, .24, .38, .3)
	max $F(R)$	
	(0, 0, .2, .4, .4), (.02, .09, .22, .31, .26, .1), (0, .06, .22, .36, .28, .08)	

Table 3

The computation in the case of mean-variance model. There are values of the mean and variance, instead of elements of W, in the following form: (mean, variance)

t	(mean, variance) of $\max\{F_t(R_t(0))$	(mean, variance) of $\max\{F_t(R_t(1))$
3	(1, 0)	(1.4, .24)
2	(1.8, .16), (2.4, .24)	(2, 0)
1	(3.2, .56), (2.9, .49), (2.3, .41)	(3, 0)
(mean, variance) of max $F(R)$		
(3.2, .56), (3, 0)		
max $F(R) = (0, 0, .2, .4, .4), (0, 0, 0, 1)$		

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ZMIENNE LOSOWE W DYSKRETNYM PROGRAMOWANIU DYNAMICZNYM

W artykule opisano dyskretny model programowania dynamicznego z wartościami funkcji kryterium z przestrzeni zmiennych losowych wyposażonej w częściowy porządek. Opisany proces dynamiczny ma charakter deterministyczny. Porównując zmienne losowe stosowane są różne rodzaje relacji porządkujących. Własności struktur zmiennych losowych pozwalają stosować uogólnioną metodę programowania dynamicznego – tzw. zasadę Bellmana. Efektem tej procedury jest uzyskanie pełnego zbioru wartości optymalnych (w sensie relacji częściowego porządku). Analogicznie, jak w programowaniu wielokryterialnym, tak i tu rozwiązaniem problemu optymalizacyjnego może być duży zbiór wartości optymalnych. Przedstawione są metody zawężające ten zbiór, wykorzystujące dynamiczną postać zadania oraz własności zmiennych losowych.