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CHARACTERIZATION OF POLYNOMIALS IN ALGEBRAIC ELEMENTS WITH COMMUTATIVE COEFFICIENTS AND ITS APPLICATIONS

The present paper is a continuation of the autor's paper [7], in which we have defined and studied characteristic polynomials for polynomials in algebraic elements in a linear commutative ring. We also have given examples of applications for singular integral operators with rotation.

1. ALGEBRAIC AND ALMOST ALGEBRAIC ELEMENTS OVER A COMMUTATIVE LINEAR RING

Let X be a linear ring over the complex scalar field with unit I. Throughout this paper, X_o will stand for a commutative linear ring in X and $I \in X_o$.

DEFINITION 1.1. An element $S \in X$ is said to be an algebraic element over X_o if there is a polynomial

$$P(t) = \sum_{k=0}^{m} p_k t^k, \quad p_o \neq 0 \tag{1.0}$$

in variable t with the coefficients in X_o such that

$$P(S) = 0, \quad S_{p_k} = p_k S; \; k = 0, 1, \ldots, m.$$

DEFINITION 1.2. If there is a polynomial P(t) of the form (1.0) satisfying the conditions $p_k \in \mathcal{J} \cup X_o$ $(k = 0, 1, \ldots, m)$ and $P(S) = T \in \mathcal{J}$, where $\mathcal{J}$ is a two-sides ideal in X, then we say that S is an almost algebraic element with respect to the ideal

$\mathcal{T}$ over X_o. If there is a polynomial $P(t)$ with the smallest degree m for which the identity $P(S) = 0$ $(P(S) = T \in \mathcal{T})$ holds, we say that S is an algebraic (almost algebraic) element of order m.

It is easy to see that each element $S \in X_o$ is an algebraic element over X_o with the characteristic polynomial of the form $P_S(t) = t - S$. Notice that all algebraic elements (over a field of scalars [1] - [2]) are the ones over X_o.

We denote by $\mathcal{A}(X_o)$ the set of all algebraic elements over X_o. Similarly, by $\mathcal{A}(X_o/\mathcal{T})$ we denote the set of all almost algebraic elements over X_o with respect to an ideal $\mathcal{T}$. The characteristic polynomials of S will be denoted by $P_S(t)$. Evidently, if an element S is almost algebraic with respect to an ideal $\mathcal{T} \subset X$ then the corresponding coset $[S]$ in the quotient ring $[X] = X/\mathcal{T}$ is algebraic and if $P_S(t) = p_o t^m + p_1 t^{m-1} + \dots + p_m$ then $P(t)_{[S]} = [p_o]t^m + [p_1]t^{m-1} + \dots + [p_m]$.

The following examples show that an algebraic (almost algebraic) element over X_o is not necessarily an algebraic (almost algebraic) over a field of scalars.

E x a m p l e 1.1. Let $X_o = \varphi[0, 1]$ and let $(S\varphi)(t) = \varphi(1 - t)$ $(V\varphi)(t) = a(t)\varphi(t) + b(t)(S\varphi)(t)$ where $a(t), b(t) \in \varphi[0, 1]$. It is easy to verify that $S^2 = I$; $SA = AS$; $SB = BS$; $V^2 - AV + B = 0$ where $A = [a(t) + a(1 - t)]I$; $B = [a(t)a(1 - t) - b(t)b(1 - t)]I$. From these relations we obtain the following results: V is an algebraic element over X_o with characteristic polynomials $P_V(t) = t^2 - At + B$. It is an algebraic element over a ring of scalars if and only if $a(t) + a(1 - t) = const$; $a(t)a(1 - t) - b(t)b(1 - t) = const$.

E x a m p l e 1.2. Let Γ be a simple closed contour of Liaponour type. Denote by $L_o(L_p(\Gamma))$ $(1 < p < \infty)$ the set of all linear operators A with domains $D_A = L_p(\Gamma)$ and with values in $L_p(\Gamma)$.

Let $X_o = \varphi(\Gamma)$; $X = L_o(L_p(\Gamma))$

$$(S\varphi)(u) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - u} \tag{1.1}$$

$$(V\varphi)(u) = a(u)\varphi(u) + b(u)(S\varphi)(u); \; a(u), b(u) \in \varphi(\Gamma). \tag{1.2}$$

The following well-known result was stated in [1]-[4]: $S^2 = I$; $Sa - aS \in \mathcal{T}$ for $a \in \varphi(\Gamma)$ where $\mathcal{T}$ is an ideal of compact continuous operators. This result permits us to obtain the following theorem.

THEOREM 1.1. Suppose that V is given by the formula (1.2) then $V \in \mathcal{A}(X_o/\mathcal{T})$ and $P_V(t) = t^2 - 2at + a^2 - b^2$.

E x a m p l e 1.3. Suppose that Γ, X and $\mathcal{T}$ are defined as in the Example 1.2. Denote by X_o the linear ring generated by all operators of the following form $V = aI + bS + D$; $a, b \in \varphi(\Gamma)$, $D \in \mathcal{T}$. Observe that $[X_o] = X_o/\mathcal{T}$ is the commutative ring. Let $(W\varphi)(u) = \varphi[\alpha(u)]$, $u \in \Gamma$, where $\alpha(u)$ is a Carleman function ([1]-[4]). The operator W defined by means of a Carleman function of order 2 is a multiplicative involution $W^2 = I$.

By straightforward calculations we can prove the following.

THEOREM 1.2. Let $K = aI + bS + (cI + dS)W$ where $a, b, c, d \in \varphi(\Gamma)$, S and W are defined by the formulas

$$(S\varphi)(u) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - u}; \quad (W\varphi)(u) = \varphi[\alpha(u)]; \quad W^2 = I.$$

Then $K \in \mathcal{A}(X_o/\mathcal{T})$ with the characteristic polynomial

$$P_K(t) = t^2 - (A + A_1)t + AA_1 - CC_1$$

where

$$A = aI + bS; \quad A_1 = a(\alpha)I + \gamma b(\alpha)S;$$

$$C = cI + dS; \quad C_1 = c(\alpha)I + \gamma d(\alpha)S$$

($\gamma = 1$ when the shift does not change the orientation of the contour Γ, $\gamma = -1$ for the contrary case).

Similar examples can easily be extended (see [1]-[4]).

2. CHARACTERIZATION OF THE POLYNOMIALS IN ALGEBRAIC ELEMENTS WITH COMMUTATIVE COEFFICIENTS

In this section we consider the polynomial

$$V := V(S) = \sum_{j=1}^{m} A_j S^{m-j} \tag{2.1}$$

where S is an algebraic element (over a field of scalars) with the characteristic polynomial

$$P_S(t) = \prod_{j=1}^{n} (t - t_j)^{r_j}; \quad t_j \in \mathbb{C} \quad t_i \neq t_j \quad \text{for } i \neq j,$$

$$r_o + r_1 + \dots + r_n = N \tag{2.2}$$

and $A_j \in X_o$, $j = 0, 1, \dots, m$.

DEFINITION 2.1. We recall that an element $S \in X$ is X_o-stationary if $SA = AS$ for all $A \in X_o$.

For stationary elements we can formulate the following result.

THEOREM 2.1. Let S be an algebraic element with the characteristic polynomial (2.2). Suppose that S is X_o-stationary. Then V of the form (2.1) is an algebraic element over X_o.

P r o o f. It is easy to verify that

$$V(S) - V(t_j)I = (S - t_j)\, V(S, t_j) \tag{2.3}$$

where

$$V(S, t_j) = A_o \delta_{m-1}(S, t_j) + A_1 \delta_{m-2}(S, t_j) + \dots + A_{m-1}$$

$$\delta_k(S, t_j) = S^k + t_j S^{k-1} + t_j^2 S^{k-2} + \dots + t_j^k I.$$

Put $P(t) = \prod_{j=1}^{n} (t - V(t_j))^{r_j}$. From (2.3) we get

$$P(V) = \prod_{j=1}^{n} (V - V(t_j))^{r_j} = \prod_{j=1}^{n} (S - t_j I)^{r_j} \prod_{j=1}^{n} [V(S, t_j)]^{r_j} = 0$$

which proves the V is an algebraic element over X_o.

To determine the characteristic polynomial of the element V over a commutative ring we have to introduce some necessary notions.

First we consider the case of simple roots.

LEMMA 2.1. Suppose that the algebraic element S has simple roots $t_1, t_2, \dots, t_n$ only and that

$$V(t_1) = \dots = V(t_{n_1}) = B_1$$

$$V(t_{n_1+1}) = \dots = V(t_{n_2}) = B_2$$

$$\dots$$

$$V(t_{n_{s-1}+1}) = \dots = V(t_n) = B_s \tag{2.4}$$

$(B_i \neq B_j$ for $i \neq j$, $n_1 + n_2 + \dots + n_s = n)$.

Moreover, we assume that S is X_o-stationary and

$$\prod_{\nu \neq j} (B_j - B_\nu) \sum_{\nu=n+1}^{n_j} P_\nu \neq 0; \quad j = 1, 2, \dots, s$$

where $P_1, P_2, \dots, P_n$ are projectors associated with S.

Then $P_V(t) = \prod_{\nu=1}^{s} (t - B_\nu)$.

P r o o f. Denote $\prod_{\nu=1}^{s} (t - B_\nu)$ by $Q(t)$, then $Q(V) = \prod_{\nu=1}^{s} (V - B_\nu)$.

It is easy to see that

$$V(S) - B_{\nu+1} = \prod_{j=n_\nu+1}^{n_{\nu+1}} (S - t_j I) \cdot Q_{\nu+1}(S); \quad \nu = 0, 1, \dots, s-1$$

where $Q_{\nu+1}(S)$ are polynomials in variable S with coefficients in X_o. Thus $Q(V) = \prod_{j=1}^{n} (S - t_j I) \prod_{j=1}^{n} Q_j(S) = 0$.

Suppose that $\tilde{Q}(t) = \prod_{\nu \neq \nu_o} (t - B_\nu)$ $(1 \leq \nu_o \leq s;\ \nu_o = \text{fix})$ then by the assumptions

$$\tilde{Q}(V) = \sum_{\nu=1}^{n} Q(v(t_\nu))P_\nu = \prod_{\nu \neq \nu_o} (B_{\nu_o} - B_\nu) \sum_{\nu=n_{\nu_o-1}+1}^{n_{\nu_o}} P_\nu \neq 0$$

From these we get $P_V(t) = Q(t)$.

Lemma 2.1 permits us to introduce.

DEFINITION 2.2. An algebraic element S (over a field of scalars) of order m is said to be X_o-linearly independent, if the condition

$$A_o + A_1 S + \ldots + A_m S^{m-1} = 0; \quad A_j \in X_o, \quad j = 0, 1, \ldots, m-1$$

implies $A_o = A_1 = \ldots = A_{m-1} = 0$.

LEMMA 2.2. Suppose that S is an algebraic element with single roots $t_1, t_2, \ldots, t_m$ only. Then S is X_o-linearly independent if and only if the projectors $P_1, P_2, \ldots, P_m$ associated with S are X_o-linearly independent.

P r o o f. Sufficiency. Suppose that $P_1, P_2, \ldots, P_m$ are X_o-linearly independent and $A_o + A_1 S + \ldots + A_{m-1} S^{m-1} = 0$; $A_j \in X_o$. This equality can be rewritten as follows:

$$\sum_{j=0}^{m-1} A_j S^j = \sum_{j=0}^{m-1} A_j \sum_{k=1}^{m} t_k^j P_k = \sum_{k=1}^{m} \Big(\sum_{j=0}^{m-1} A_j t_k^j\Big) P_k = 0.$$

thus, by our assumptions, we get $\sum_{j=0}^{m-1} A_j t_k^j = 0$. It is easy to verify that the determinant of this system with respect to the unknows A_j is the Vandermonde determinant of the numbers $t_1, t_2, \ldots, t_m$. This implies $A_j = 0$; $j = 0, 1, \ldots, m-1$. Thus, S is X_o-linearly independent.

Necessity. Suppose that S in X_o-linearly independent and

$$\sum_{j=1}^{m} A_j P_j = 0; \quad A_j \in X_o, \quad j = 1, 2, \ldots, m.$$

Acting on both sides of this equality by the elements P_k we obtain $A_k P_k = 0$ $(k = 1, 2, \ldots, m)$ where $P_k = \prod_{j \neq k} \frac{S - t_j I}{t_k - t_j}$ (see [1]). Since $\deg P_k \leq m - 1$ we get $A_k \prod_{j \neq 1} \frac{1}{(t_k - t_j)} = 0$. Thus, $A_k = 0$.

With the aid of Lemma 2.2 we can formulate the result of Lemma 2.1 as follows.

THEOREM 2.2. Let S be an algebraic element with single roots

$t_1, t_2, \ldots, t_m$ only. Suppose that S is X_o-linearly independent and that $\prod_{\nu\neq j} (B_j - B_\nu) \neq 0$ for $j = 1, 2, \ldots, s$ where B_j ($j = 1, 2, \ldots, s$) are defined by (2.4). Then $P_V(t) = \prod_{\nu=1}^{s} (t - B_\nu)$.

Consider now the case of multiple roots.

LEMMA 2.3. Suppose that S is an algebraic element with the characteristic polynomial of the form (2.2). Then S is X_o-linearly independent if and only if the elements P_j; $(S - t_j I)^{\nu_j} P_j$ ($j = 1, 2, \ldots, n$; $\nu_j = 1, 2, \ldots, \nu_j - 1$) associated with S are X_o-linearly independent.

P r o o f. Necessity. Suppose that S is X_o-linearly independent and

$$\sum_{j=1}^{n} \sum_{\nu=0}^{j-1} A_{j\nu} Q_j^\nu = 0 \quad \text{where } A_{j\nu} \in X_o, \quad Q_j^\nu = (S - t_j I)^\nu P_j.$$

Applying the element Q_k^μ to both sides of this equality, we obtain the following relations

$$\sum_{\nu=0}^{r_k-1} A_{k\nu} Q_k^{\nu+\mu} = 0, \quad k = 1, 2, \ldots, n; \quad \mu = 0, 1, \ldots, r_{k-1}.$$

Since $Q_k^{r_k} = 0$ (see [1]-[2]) we can rewrite these relations as follows

$$A_{k0} P + A_{k1} Q_k + \ldots + A_{kr_k-1} Q_k^{r_k-1} = 0$$

$$A_{k0} Q_k + \ldots + A_{kr_k-2} Q_k^{r_k-1} = 0$$

$$\ldots$$

$$A_{k0} Q_k^{r_k-1} = 0 \tag{2.5}$$

By our assumptions, from the last equality of (2.5) we have $A_{k0} = 0$. This and equalities (2.5) together imply that $A_{k\nu} = 0 \ \forall \ k, \nu$

Sufficiency. Suppose that Q_k^ν ($k = 1, 2, \ldots, n$; $\nu = 0, 1, r_{k-1}$) are X_o-linearly independent and

$$\sum_{j=0}^{N-1} A_j S_j = 0, \qquad A_j \in X_o \tag{2.6}$$

Using the equality $S^k P_j = \sum_{\nu=0}^{k} \binom{k}{\nu} t_j^{k-\nu} Q_j^{\nu}$ we can write (2.6) as follows

$$\sum_{j=1}^{n} \sum_{\nu=0}^{r_j-1} \left(\sum_{k=\nu}^{N-1} \binom{k}{\nu} A_k t_j^{k-\nu} \right) Q_j^{\nu} = 0$$

By our assumptions, we get $\sum_{k=\nu}^{N-1} \binom{k}{\nu} t_j^{k-\nu} A_k = 0$; $j = 1, 2, \ldots, n$; $\nu = 0, 1, \ldots, r_j - 1$.

It is easy to verify that the determinant of this system with respect to unknowns A_k is invertible. This implies that $A_k = 0$, for every k.

REMARK 2.1. All algebraic elements over X_o are X_o-linearly independent. For instance, all algebraic element are $\mathbb{C}$-linearly independent.

LEMMA 2.4. Suppose that S is an algebraic element with the characteristic polynomial of (2.2). Let

$$V(S) = \sum_{k=1}^{n} \sum_{\nu=0}^{r_k-1} A_{k\nu} Q_k^{\nu}; \quad A_{k\nu} \in X_o; \ A_{k0} \neq 0, \quad k = 1, 2, \ldots, n$$

$$P(S) = \sum_{k=1}^{n} \sum_{\nu=0}^{r_k-1} \alpha_{k\nu} Q_k^{\nu}; \quad \alpha_{k\nu} \in \mathbb{C}$$

and suppose that S is X_o-linearly independent and X_o-stationary. Then $P(S)\,V(S) = 0$ implies $P(S) = 0$.

P r o o f. It is easy to verify that

$$P(S)\,V(S) = \sum_{k=1}^{n} \sum_{i=1}^{r_k-1} \sum_{\nu+\mu=i} \alpha_{k\nu} A_{k\mu} = Q_k^{i} = 0.$$

This implies $\sum_{\nu+\mu=i} \alpha_{k\nu} A_{k\mu} = 0$; $k = 1, 2, \ldots, n$; $i = 0, 1, \ldots, r_k - 1$. We can rewrite these relations as follows

$$\begin{aligned}
&\alpha_{k0} A_{k0} = 0 \\
&\alpha_{k1} A_{k0} + \alpha_{k0} A_{k1} = 0 \\
&\alpha_{kr_k-1} A_{k0} + \ldots + \alpha_{k0} A_{kr_k-1} = 0
\end{aligned} \tag{2.7}$$

By our assumptions, from the first equality of (2.7) we get $\alpha_{k0} = 0$. This, and equalities (2.7), together imply that $\alpha_k = 0$ $\forall\, k, \nu$.

Now we can formulate the main result in our investigations:

THEOREM 2.3. Let S be an algebraic element with the characteristic polynomial

$$P_S(t) = \prod_{i=1}^{n} \prod_{j=1}^{n_i} (t - t_{ij})^{r_{ij}}; \quad t_{ij} \neq t_{\nu\mu}$$

for $(i, j) \neq (\nu, \mu)$ (2.8)

and V(t) - a polynomial in variable t with coefficients belong to X_o. Suppose that

(i) $V(t_{kj}) = R_k$; $k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n_k$

(ii) $V'(t_{kj}) = \ldots = V^{(s_{kj})}(t_{kj}) = 0$; $V^{(s_{kj}+1)}(t_{kj}) \neq 0$.

If S is X_o-linearly independent and X_o-stationary, then

$$P_V(t) = \prod_{i=1}^{n} (t - R_i)^{\delta_i}; \quad V := V(S) \tag{2.9}$$

where

$$\delta_i = \begin{cases} \alpha_i & \text{when } \alpha_i \text{ is an integer,} \\ [\alpha_i] + 1 & \text{otherwise,} \end{cases}$$

$$\alpha_i = \max\left\{\frac{s_{i1}}{s_{i1}+1}, \frac{r_{i2}}{s_{i2}+1}, \ldots, \frac{r_{in_i}}{s_{in}+1}\right\}; \quad i = 1, 2, \ldots, n.$$

We base the proof Theorem 2.3 on three additional lemmas.

LEMMA 2.5. Let S be an algebraic element with the characteristic polynomial of the form (2.2). Suppose that V(t) is a polynomial with coefficients belonging to X_o such that

$$V(t_i) \neq V(t_j) \quad \text{for } i \neq j; \quad V'(t_i) \neq 0; \quad i = 1, 2, \ldots, n.$$

If S is X_o-linearly independent and X_o-stationary, then

$$P_V(t) = \prod_{j=1}^{n} (t - V(t_j))^{r_j}; \quad V = V(S).$$

P r o o f. Denote $\prod_{j=1}^{n} (t - V(t_j))^{r_j}$ by P(t). According to Theorem 2.1, P(V) = 0. Put

$$Q_1(t) = (t - V(t_1))^{\alpha_1} \prod_{j=2}^{n} (t - V(t_j))^{r_j} \quad \text{for} \quad \alpha_1 < r_1.$$

Observe that

$$Q_1(V) = (S - t, I)^{\alpha_1} \prod_{j=2}^{n} [S - t_j)]^{r_j} \cdot [V(S, t_1)]^{\alpha_1} \cdot$$

$$\cdot \prod_{j=2}^{n} [V(S, t_j)]^{r_j}$$

where $V(S, t_j)$ are defined by the formula (2.3).

Put $Q(t) = \prod_{j=2}^{n} [V(t, t_j)]^{r_j} \cdot [V(t, t_1)]^{\alpha_1}$.

According to Theorem 2.1, the element Q(S) has characteristic roots belonging to the set

$\{Q(t_j); \ j = 1, 2, \ldots, n\}$.

On the other hand, by our assumptions, we have

$$V(t_i, t_j) = \frac{V(t_i) - V(t_j)}{t_i - t_j} \neq 0 \quad \text{for} \quad i \neq j$$

$$V(t_j, t_j) = V'(t_j) \neq 0.$$

Hence, the element Q(S) has the same properties as V(S) in Lemma 2.4. According to Lemma 2.4, $Q_1(V) = 0$ if and only if

$$Q_2(V) = \prod_{j=1}^{n} (S - t_j I)^{r_j} \cdot (S - t_1 I)^{\alpha_1} = 0$$

Thus $Q_1(V) \neq 0$ and we get $P_V(t) = P(t)$.

LEMMA 2.6. Suppose that S is an algebraic element satisfying all assumptions of Lemma 2.5 and that

$$V(t) = \sum_{j=0}^{s} A_j t^{s-j}$$

is a polynomial with coefficients in X_o satisfying the following conditions

$$V(t_1) = V(t_n); \quad V(t_1) \neq V(t_i) \quad \text{for} \quad i = 2, 3, \ldots, n-1$$

$$V(t_i) \neq 0; \quad i = 1, 2, \ldots, n. \tag{2.10}$$

If $V = V(S)$ then $P_V(t) = [t - V(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} t - V(t_j)^{r_j}$

where $\alpha_1 = \max(r_1, r_n)$.

P r o o f. From (2.10) we can write:

$$V(t) - V(t_1) = (t - t_1)(t - t_n) V(t, t_1, t_n) \tag{2.11}$$

where $V(t, t_1, t_n) = A_o \delta_{s-2} + A_1 \delta_{s-3} + \ldots + A_{s-2}$

$$\delta_k = \frac{\delta_k(t, t_1)}{t_1 - t_n} - \frac{\delta_k(t, t_n)}{t_1 - t_n}; \quad \delta_k(t, t_j) \text{ are given by (2.3).}$$

Denote $(t - V(t_1))^{\alpha_1} \prod_{j=2}^{n-1} (t - V(t_j))^{r_j}$ by $P(t)$. From (2.11) we get

$$P(V) = [V(S) - V(t_1)]^{\alpha_1} \prod_{j=2}^{n-2} [V(S) - V(t_j)]^{r_j} =$$

$$= P_S(S) \cdot [(S - t_1 I)(S - t_n I)]^{\alpha_1 - r_1} .$$

$$\cdot [V(S, t_1, t_n)]^{\alpha_1} \cdot \prod_{j=2}^{n-1} [V(S, t_j)]^{r_j} = 0.$$

To end the proof it is enough to show α_1 is the smallest positive integer possessing the above property.

By Theorem 2.1, without loss of generality, we consider the polynomial

$$P_1(t) = [t - V(t_1)]^{\alpha} \prod_{j=2}^{n-1} [t - V(t_j)^{r_j}; \quad \alpha < \alpha_1$$

and suppose that $P_1(V) = 0$ i.e:

$$[V(S) - V(t_1)]^{\alpha} \prod_{j=2}^{n-1} [V(S) - V(t_j)]^{r_j} = 0$$

$$(S - t_1 I)^{\alpha} \cdot (S - t_n I)^{\alpha} \prod_{j=2}^{n-1} (S - t_j I)^{r_j} \cdot G(S) = 0$$

where

$$G(S) = [V(S, t_1, t_n)]^{\alpha} \prod_{j=2}^{n-1} [V(S, t_j)]^{r_j}$$

By assumptions

$$V(t_j, t_1, t_n) = \frac{V(t_j) - V(t_1)}{(t_j - t_1)(t_j - t_n)} \neq 0$$

for $j = 2, 3, \ldots, n - 1$,

$$V(t_1, t_1, t_n) = \frac{V'(t_1)}{t_1 - t_n} \neq 0; \quad V(t_n, t_1, t_n) = \frac{V'(t_n)}{t_1 - t_n} \neq 0.$$

Hence $V(S, t_1, t_n)$ possesses the same properties as $V(S)$ in Lemma 2.4. From this and $P_1(V) = 0$ we get

$$(S - t_1 I)^{\alpha} (S - t_n I)^{\alpha} \prod_{j=2}^{n-1} (S - t_j I)^{r_j} = 0$$

which is a contradiction and the proof is complete.

LEMMA 2.7. Suppose that S satisfies all assumptions of Lemma 2.4 and that

$$V(S) = \sum_{i=0}^{m} A_j S^{m-j}$$

is a polynomial satisfying the following conditions:

(i) $V(t_1) = V(t_n)$

(ii) $V(t_1) \neq V(t_i) \neq V(t_j)$ for $i \neq j$, $i, j = 2, 3, \ldots, n - 1$

(iii) $V'(t_j) = \ldots = V^{(s_j)}(t_j) = 0; \quad V^{(s_j+1)}(t_j) \neq 0;$

$j = 1, 2, \ldots, n.$

Then

$$P_V(t) = [t - V(t_1)]^{\alpha_1} \prod_{j=2}^{n-1} [t - V(t_j)]^{\alpha_j} \tag{2.12}$$

where

$$\alpha_i = \begin{cases} \beta_i & \text{when } \beta_i \text{ is an integer} \\ [\beta_i] + 1 & \text{otherwise.} \end{cases}$$

$$\beta_1 = \max \left\{\frac{r_1}{s_1 + 1}; \frac{r_n}{s_n + 1}\right\}; \quad \beta_j = \frac{r_j}{s_j + 1} \quad \text{for } j = 2, 3, \ldots, n - 1.$$

P r o o f. From the conditions (i) - (iii) we obtain

$$V(S, t_j) = (S - t_j I)^{s_j} V_j(S, t_j) \quad \text{for } j = 2, 3, \ldots, n - 1$$

$$V(S, t_1, t_n) = (S - t_1 I)^{s_1} (S - t_n I)^{s_n} V_1(S, t_1, t_n)$$

where $V_j(S, t_j)$ and $V_1(S, t_1, t_n)$ possesses the same properties as V(S) in Lemma 2.4

Suppose that $P(t) = \prod_{j=1}^{n-1} [t - V(t_j)]^{\lambda_j}$. From the above argument $P(V) = 0$ if and only if

$$P_1(V) = [(S - t_1 I)(S - t_n I)]^{\lambda_1} (S - t_1 I)^{\lambda_1 s_1} (S - t_n I)^{\lambda_n s_n} \cdot Q(S) = 0$$

with $Q(S) = \prod_{j=2}^{n-1} (S - t_j I)^{(1+s_j)\lambda_j}$.

Hence λ_j satisfy the conditions

$$\lambda_1 + \lambda_1 s_1 \geqq r_1$$

$$\lambda_1 + \lambda_1 s_n \geqq r_n$$

$$\lambda_j + \lambda_j s_j \geqq r_j, \quad j = 2, 3, \ldots, n - 1.$$

From these inequalities it follows that the formula (2.12) is proved.

We proceed to prove Theorem 2.3.

By hypothesis we obtain the characteristic roots of the element $V(S)$: $R_1, R_2, \ldots, R_m$. Hence, the characteristic polynomial of V is a polynomial of the form

$$P_V(t) = \prod_{i=1}^{m} (t - R_i)^{\beta_i}.$$

According to Lemmas 2.5-2.7 we get $\beta_i + \beta_i s_{ij} \geqq r_{ij}$. From these inequalities it follows that the formula (2.9) is valid. The proof of Theorem 2.3 is complete.

3. SINGULAR INTEGRAL EQUATIONS WITH ROTATION

Let Γ be an oriented system. Suppose that Γ is invariant with respect to rotation through an angle $2\pi/n$, where n is an arbitrary positive integer.

Now consider the following operators

$$(M\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{P(\tau, t)}{\tau^n - t^n}\varphi(\tau)d\tau \tag{3.1}$$

where $P(\tau, t) = \sum_{j=0}^{n-1} a_j \tau^j t^{n-1-j}$; $a_j \in C$

$$(M_j\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{\tau^j t^{n-1-j}}{\tau^n - t^n}\varphi(\tau)\,di \tag{3.2}$$

and

$$K = \sum_{j=0}^{m} a_j(t) M^j; \tag{3.3}$$

where $a_j(t)$ are invariant with respect to rotation:

$$a_j(\varepsilon_1 t) = a_j(t);\quad j = 0, 1, \ldots, m;\quad \varepsilon_1 = \exp({}^{2\pi i}/_n).$$

THEOREM 3.1. If Γ is an oriented system and invariant with respect to rotation through an angle $2\pi/n$, then M_j are algebraic operators with characteristic polynomials:

$$P_{M_j}(t) = t^3 - t.$$

P r o o f. Observe that

$$\frac{\tau^k t^{n-1-k}}{\tau^n - t^n} = \sum_{j=1}^{n}\frac{\varepsilon_j^k}{\omega'(\varepsilon_j)}\cdot\frac{1}{\tau - \varepsilon_j t}$$

where $\varepsilon_j = \varepsilon_1^j$; $\varepsilon_1 = \exp({}^{2\pi i}/_n)$; $\omega(t) = t^n - 1$;

and $\sum_{k=1}^{n}\frac{\varepsilon_k^j}{\omega'(\varepsilon_k)} = \begin{cases}1 & \text{when } j = 1 - n\\ 0 & \text{when } j = 0, 1, \ldots, n-2.\end{cases}$

We can write

$$M_j = \sum_{k=1}^{n}\frac{\varepsilon_k^j}{\omega'(\varepsilon_k)}\delta W^k = S P_{n-1-k}$$

where

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - t}; \quad P_{n-i-k} = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_{n-1-k}^{j+1} W^{n-1-j}$$

$$(W\varphi)(t) = \varphi(\varepsilon_1 t).$$

It is easy to see that $P_{n-1-k}^2 = P_{n-1-k}$; $S P_{n-1-k} = P_{n-1-k} S$. Thus, $M_j^2 = S^2 P_{n-1-j}^2 = P_{n-1-j}$ and $M_j^3 = M_j$ which was to be proved.

Suppose that $X_o = \{a I + b S,\ a, b \in C\}$. Then X_o is a commutative linear ring with unit I. It is easy to verify that W is X_o-stationary and X_o-linearly independent. Hence, from Lemma 2.1 we can formulate the following

THEOREM 3.2. Let M be of the form (3.1) and $X_o = \{a I + b S\}$. Then M is an algebraic element over X_o with characteristic roots belonging to $\{a_o, a_1, a_2, \ldots, a_{n-1}\}$. Suppose that:

$a_1 = a_2 = \ldots = a_{n_1} = b_1$

$a_{n_1+1} = \ldots = a_{n_2} = b_2$

...

$a_{n_{s-1}+1} = \ldots = a_{n_s} = b_s$

where $b_i \neq b_j$ if $i \neq j$. Then the characteristic polynomial of the operator M over X_o is of the form $P_M(t) = \prod_{j=1}^{s} (t - b_j)$.

COROLLARY 3.1. Put $P(t) = \prod_{j=0}^{n-1} (t^2 - a_j^2)$. Then

1) $P(M) = 0$

2) M is invertible if and only if $a_j \neq 0$, $\forall_j$ and

$$M^{-1} = \sum_{j=0}^{n-1} a_j^{-1} S P_{n-1-j} \qquad (P_n \equiv P_o).$$

P r o o f. It is easy to verify that $P(M) = Q(W, S)$ is even divisible by $\prod_{j=1}^{n}(W - \varepsilon_j I)$. This implies $P(M) = 0$. On the other hand,

$$\sum_{j=0}^{n-1} a_j S P_{n-1-j} \sum_{j=0}^{n-1} a_j^{-1} S P_{n-1-j} = \sum_{j=0}^{n-1} a_j a_j^{-1} S^2 P_{n-1-j}^2 =$$

$$= \sum_{j=0}^{n-1} P_{n-1-j} = I$$

which was to be proved.

COROLLARY 3.2. M is an algebraic operator with characteristic roots belonging to $\{\pm a_o, \pm a_1, \ldots, \pm a_{n-1}\}$.

Now we consider the operator K of the form (3.3). Suppose that $a_j(t) \in H^\lambda(\Gamma)$ $(0 < \lambda < 1)$ and $X_o = H^\lambda(\Gamma) I = \{a(t) I; a \in H^\lambda(\Gamma)\}$. If $a_j(t)$ are invariant with respect to rotation: $a_j(\varepsilon_1 t) = a_j(t)$ then W is X_o-stationary and S is almost X_o--stationary with respect to an ideal of compact operators [1]-[4]:

$$S A - A S \in \mathcal{T}, \qquad \forall A \in X_o$$

where $\mathcal{T}$ is an ideal of compact operators. These imply that M is almost X_o-stationary with respect to $\mathcal{T}$.

As a simple consequence of Theorem 2.3 we obtain the following

THEOREM 3.3. Let K be of the form (3.3) and let $a_j(t)$ be invariant with respect to rotation $a_j(\varepsilon_1 t) = a_j(t)$. If $X_o = H^\lambda(\Gamma)I$ then K is an almost algebraic element over X_o with respect to an ideal of compact operators $\mathcal{T}$. Moreover, the characteristic roots of K belong to the set

$$\{A_k = \sum_{j=0}^{m} a_j(t) a_k^j;\ k = 0, 1, \ldots, 2n - 1\} \quad (a_{n+j} := -a_j;$$

$$j = 0, 1, \ldots, n - 1).$$

Suppose that

$$A_1 = A_2 = \ldots = A_{n_1} = B_1$$

$$A_{n_1+1} = \ldots = A_{n_2} = B_2$$

$$\ldots$$

$$A_{n_s+1} = \ldots = A_{n_{s+1}} = B_{s+1}$$

where $B_i \neq B_j$ if $i \neq j$. Then the characteristic polynomial of K is of the form

$$P_K(\lambda) = \prod_{j=1}^{s+1} (\lambda - B_j).$$

COROLLARY 3.3. Let K satisfy the conditions of Theorem 3.3. Suppose that B_j (j = 1, 2, ..., s + 1) are invertible. Then there exists a simple regularizez of the element K to the ideal $\mathcal{J}$ which is given by the formula

$$R = Q(K) \left[(-1)^s \prod_{j=1}^{s+1} B_j\right]^{-1} \quad \text{where} \quad Q(\lambda) = \frac{P_K(\lambda) - P_K(0)}{\lambda}$$

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CHARAKTERYSTYKA WIELOMIANÓW DLA ELEMENTÓW ALGEBRAICZNYCH Z PRZEMIENNYMI WSPÓŁCZYNNIKAMI I ICH ZASTOSOWANIA

Ten artykuł jest uogólnieniem pracy autora [7], w której określone i opisane są wielomiany charakterystyczne dla wielomianów z elementami algebraicznymi w liniowym pierścieniu przemiennym. Także przedstawione są przykłady zastosowania dla całkowych operatorów osobliwych z obrotem.