ACTA UNIVERSITATIS LODZIENSIS
FOLIA MATHEMATICA 4, 1991

Andrzej Biś

ENTROPY OF TRANSVERSE FOLIATIONS

A new definition of the topological entropy of a foliation is introduced in this paper. This definition is slightly different from the definition of the topological entropy of a foliation given by E. Ghys, R. Langevin, P. Walczak in [1]. However, for any foliation F, the topological entropy of F defined in [1] is less or equal to the topological entropy of F defined here. For transverse foliations F_1 and F_2, the topological entropy of $F_1 \cap F_2$ is estimated.

1. ENTROPY OF A FINITELY GENERATED PSEUDOGROUP OF MAPS OF A COMPACT METRIC SPACE

R. Bowen defined the topological entropy of uniformly continuous maps $T: X \to X$ of a compact metric space (X, d) (see [4]). Using a similar method, one can define the topological entropy of a finitely generated pseudogroup of maps of a compact metric space.

Let (X, d) be a compact metric space with the metric d, G - a finitely generated pseudogroup of maps of X, G_1 - a finite set of generators of G. We assume that $id_X \in G_1$ and $G_1^{-1} \subset G_1$.

Let

$$G_n = \{g_1 \circ \ldots \circ g_n : g_i \in G_1\}$$

and

$$\tilde{g}(x) = \begin{cases} g(x), & x \in D_g \\ x, & x \notin D_g. \end{cases}$$

Define a sequence of maps $d_x\colon X \times X \to \mathbb{R}$ in the following way:

$$d_n(x, y) = \max\{d(\tilde{g}(x), \tilde{g}(y)) : g \in G_n\}.$$

These maps define metrics in the space X. Indeed, $d_n(x, y) = 0$ iff $x = y$ and $d_n(x, y) = d_n(y, x)$. For arbitrary $x, y, z \in X$, there exist $g_1, g_2, g_3 \in G_n$ such that $d_n(x, y) = d(\tilde{g}_1(x), \tilde{g}_1(y))$, $d_n(y, z) = d(\tilde{g}_2(y), \tilde{g}_2(z))$ and $d_n(x, z) = d(\tilde{g}_3(x), \tilde{g}_3(z))$. Then $d(\tilde{g}_1(x), \tilde{g}_1(y)) \geq d(\tilde{g}_3(x), \tilde{g}_3(y))$ and $d(\tilde{g}_2(y), \tilde{g}_2(z)) \geq (d(\tilde{g}_3(y), \tilde{g}_3(z))$. Hence the inequality $d_n(x, y) + d_n(y, z) \geq d_n(x,z)$ holds

DEFINITION 1. Let $n \in N$ and let $\varepsilon > 0$. A subset A of X is said to be (n, ε)-separated if, for arbitrary $x, y \in A$, $x \neq y$, $d_n(x, y) \geq \varepsilon$. Let $s(G, G_1, n, \varepsilon)$ denote the largest cardinality of an (n, ε)-separated subset of X. Put

$$s(G, G_1, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log s(G, G_1, n, \varepsilon).$$

DEFINITION 2. A subset B of X is said to be (n, ε)-spanning if, for any $x \in X$, there exists $y \in B$ such that $d_n(x, y) < \varepsilon$. Let $r(G, G_1, n, \varepsilon)$ denote the smallest cardinality of an (n, ε)--spanning subset of X. Put

$$r(G, G_1, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log r(G, G_1, n, \varepsilon).$$

PROPOSITION 1. We have $\lim_{\varepsilon\to 0^+} r(G, G_1, \varepsilon) = \lim_{\varepsilon\to 0^+} s(G, G_1, \varepsilon)$.

P r o o f. Very similar to that in Remark 5 ([4], p. 169).

DEFINITION 3. The topological entropy of a pseudogroup G with respect to G_1 equals $h_d(G, G_1) = \lim_{\varepsilon\to 0^+} r(G, G_1, \varepsilon)$.

REMARK 1. If metrics d and d′ on X are equivalent, then $h_d(G, G_1) = h_{d'}(G, G_1)$.

PROPOSITION 2. Let $X_1 = \bigsqcup_{i=1}^{k} U_i$ (respectively, $X_2 = \bigsqcup_{i=1}^{k} V_i$ and $X = \bigsqcup_{i=1}^{k} (U_i \times V_i))$ be a compact metric space with a metric d′ (respectively d″ and d) and let F (respectively G and H) be a finitely generated pseudogroup of maps of the space X_1 (respectively X_2 and X) generated by F_1 (respectively G_1 and H_1). If the metric d is defined by $d((x_1, y_1), (x_2, y_2)) = \max\{d'(x_1, x_2),$

$d''(y_1, y_2))$ and, for any $h \in H_1$, there exist $f \in F_1$ and $g \in G_1$ such that $h = f \times g$, then

$$h_d(H, H_1) \le h_{d'}(F, F_1) + h_{d''}(G, G_1).$$

P r o o f. Let $n \in N$ and let $\varepsilon > 0$. Let $A = \bigcup_{i=1}^{k} A_i$, $A_i \subset U_i$, be an (n, ε)-spanning subset of (X_1, d') such that card $A = r(F, F_1, n, \varepsilon)$ and similarly, let $B = \bigcup_{i=1}^{k} B_i$, $B_i \subset V_i$, be an (n, ε)-spanning subset of (X_2, d'') such that card $B = r(G, G_1, n, \varepsilon)$. For any $(x, y) \in X$, there exist $x_1 \in A$ and $y_1 \in B$ such that $d'_n(x, x_1) < \varepsilon$ and $d''_n(y, y_1) < \varepsilon$. We have $d_n((x, y), (x_1, y_1)) = \max\{d(\tilde{h}(x, y), \tilde{h}(x_1, y_1)) : h \in H_n\} = \max\{d(\widetilde{h_1 \circ \ldots \circ h_n}(x, y), \widetilde{h_1 \circ \ldots \circ h_n}(x_1, y_1)) : h_i \in H_1\} \le \max\{d(\widetilde{(f_1 \circ \ldots \circ f_n) \times (g_1 \circ \ldots \circ g_n)}(x, y), \widetilde{(f_1 \circ \ldots \circ f_n) \times (g_1 \circ \ldots \circ g_n)}(x_1, y_1)) : f_i \in F_1, g_i \in G_1\} = \max\{d((\widetilde{f_1 \circ \ldots \circ f_n})(x), (g_1 \circ \ldots \circ g_n)(y)), ((\widetilde{f_1 \circ \ldots \circ f_n})(x_1), (\widetilde{g_1 \circ \ldots \circ g_n})(y_1))) : f_i \in F_1, g_i \in G_1\} = \max\{\max\{d'(\tilde{f}(x), \tilde{f}(x_1)), d''(g(y), g(y_1))\} : f \in F_n, g \in G_n\} < \varepsilon.$

So, we can see that the subset $C = \bigcup_{i=1}^{k} (A_i \times B_i)$ is (n, ε)-spanning in the space (X, d), hence the minimal cardinality of an (n, ε)-spanning subset of (X, d) is less or equal to card $A \cdot$ card B, i.e.

$$r(H, H_1, n, \varepsilon) \le r(F, F_1, n, \varepsilon)\ s(G, G_1, n, \varepsilon).$$

So,

$$r(H, H_1, \varepsilon) \le r(F, F_1, \varepsilon) + r(G, G_1, \varepsilon)$$

and, finally,

$$h_d(H, H_1) \le h_{d'}(F, F_1) + h_{d''}(G, G_1).$$

2. ENTROPY OF A FOLIATION

The basic definitions and properties concerning the geometry of foliations can be found in [2]. The notion of the entropy of a finitely generated pseudogroup can be applied to the theory of foliations, namely, to the space of plaques of a foliation and its holonomy pseudogroup.

Consider a compact Riemannian manifold $(M, \langle .,. \rangle)$ with the metric d obtained from the Riemannian structure $\langle .,. \rangle$, and a foliation F of M, $\dim F = p$, $\operatorname{codim} F = q$, $p + q = n = \dim M$.

DEFINITION 4. Let F be a foliation of M. We say that a finite covering $\mathcal{U}$ of M by closed distinguished sets is nice if the following conditions hold:

1) for any $U \in \mathcal{U}$, there exists a distinguished chart $\phi: U' \to D^p(1) \times D^q(1)$ such that U' is open and $U \subset U'$;

2) for arbitrary distinguished charts $\phi_1: U_1' \to \mathbb{R}^n$, $\phi_2: U_2' \to \mathbb{R}^n$ each plaque of U_1' intersects at most one plaque of U_2';

3) a plaque of U_1' intersects a plaque of U_2' iff the corresponding plaque in U_1 intersects the corresponding plaque in U_2;

4) if $U_1, U_2 \in \mathcal{U}$ and $U_1 \cap U_2 \neq \emptyset$, then $\operatorname{int} U_1 \cap \operatorname{int} U_2$ is non-empty and connected;

here, $D^k(r)$ denotes the open ball of radius r and centre 0 in $\mathbb{R}^k$.

G. Reeb in [3] proved the existence of nice coverings of foliated manifolds, he showed that, for any locally finite covering $\mathcal{U}$ of M, there exists a nice covering ω of M subordinated to $\mathcal{U}$.

Let X be the space of plaques of a nice covering $\mathcal{U} = \{U_1, \ldots, U_k\}$, that is,

$$X = \bigsqcup_{i=1}^{k} U_i/\sim$$

where points $x, y \in M$ are equivalent $(x \sim y)$ iff there exists $i \in \{1, \ldots, k\}$ such that $x, y \in U_i$ and x, y belong to the same plaque.

Let ρ be the Hausdorff metric in the space X:

$$\rho(P, Q) = \sup_{x \in P} \inf_{y \in Q} d(x, y) + \sup_{y \in Q} \inf_{x \in P} d(x, y)$$

for arbitrary plaques $P, Q \in X$. In the compact metric space (X, ρ), we can consider the holonomy pseudogroup $H_{\mathcal{U}}$ of the foliation F. This pseudogroup is generated by the set $H_{\mathcal{U},1} = \{h_{ij} : i, j \in \{1, \ldots, k\}, U_i \cap U_j \neq \emptyset\}$, where, for plaques $P, Q \in X$, $h_{ij}(P) = Q$ iff $P \subset U_i$, $Q \subset U_j$ and $P \cap Q \neq \emptyset$.

DEFINITION 5. The topological entropy of F with respect to a nice covering $\mathcal{U}$ is defined as $h_\rho(H, H_{\mathcal{U},1})$ and denoted by $h_\rho(F,\mathcal{U})$.

If metrics ρ_1 and ρ_2 are determined by Riemannian structures, then they are equivalent, so $h_{\rho 1}(F, \mathcal{U}) = h_{\rho 2}(F,\mathcal{U})$ and that is why we can write $h(F,\mathcal{U})$ instead of $h_\rho(F,\mathcal{U})$.

DEFINITION 6. We say that foliations F_1 and F_2 of a manifold M are transverse if, for any point $p \in M$, we have

$$T_pF_1 + T_pF_2 = T_pM.$$

DEFINITION 7. Let F_1 and F_2 be transverse foliations of a manifold M, $\dim M = n$, $\operatorname{codim} F_1 = k$, $\operatorname{codim} F_2 = 1$. A chart ϕ on M

$$\phi = (\phi_1, \phi_2, \phi_3): U \to \mathbb{R}^{n-1-k} \times \mathbb{R}^1 \times \mathbb{R}^k$$

is bidistinguished with respect to F_1 and F_2 if each plaque of F_1 contained in U is described by the equation $\phi_3 = $ const., while each plaque of F_2 contained in U - by the equation $\phi_2 =$ = const.

THEOREM. Let $(M, \langle .,. \rangle)$ be a compact Riemannian manifold, F_1 and F_2 - transverse foliations of M. The family $F_1 \cap F_2$ of connected components of intersections of leaves of foliations F_1 and F_2 is a foliation of M such that

$$(*)\ h(F_1 \cap F_2, \mathcal{Z}) \leq h(F_1, \mathcal{Z}) + h(F_2, \mathcal{Z})$$

for a nice covering $\mathcal{Z}$ of M which consists of the domains of charts bidistinguished with respect to the foliations F_1 and F_2.

P r o o f. Let F_1 and F_2 be transverse foliations of M, $\dim M = n$, $\dim F_i = p_i$, $\operatorname{codim} F_i = q_i$, $i = 1, 2$. Using Frobenius theorem, we immediately obtain that the family $F_3 = F_1 \cap F_2$ is a foliation of M.

Consider a pair of charts $\phi_i: U_i \to \mathbb{R}^{p_1} \times \mathbb{R}^{q_1}$ and $\psi_j: V_j \to \mathbb{R}^{p_2} \times \mathbb{R}^{q_2}$, $p \in U_i \cap V_j$, that are distinguished for F_1 and F_2, respectively. The map $x \to (p_2\phi_i(x), \tilde{p}_2\psi_j(x))$, where the maps $p_2: \mathbb{R}^{p_1} \times \mathbb{R}^{q_1} \to \mathbb{R}^{q_1}$, $\tilde{p}_2: \mathbb{R}^{p_2} \times \mathbb{R}^{q_2} \to \mathbb{R}^{q_2}$ are projections, is a submersion. Each fibre of this submersion is contained in a leaf of $F_1 \cap F_2$. Take a chart $\lambda: W_{ij} \to \mathbb{R}^{n-q_1-q_2} \times \mathbb{R}^{q_1+q_2}$, $p \in W_{ij} \subset U_i \cap V_j$, which flattens the fibres of that submersion. Then the map $x \to (p_1\lambda(x), p_2\phi_i(x), \tilde{p}_2\psi_j(x))$ is a chart distinguished with respect to the foliations F_1, F_2 and F_3 and is defined in a neighbourhood of the point p, the map $p_1: \mathbb{R}^{n-q_1-q_2} \times \mathbb{R}^{q_1+q_2} \to \mathbb{R}^{n-q_1-q_2}$ being a projection.

Therefore, we can consider the family of maps $\Theta_k: W_k \to \mathbb{R}^{n-q_1-q_2} \times \mathbb{R}^{p_2} \times \mathbb{R}^{q_2}$, $k = 1, \ldots, m_1$, such that Θ_k is a chart distinguished with respect to the foliations F_1, F_2 and F_3, while the sets W_k cover M. To the covering $\omega = \{W_1, \ldots, W_{m_1}\}$ we can subordinate a covering $\mathcal{T} = \{T_1, \ldots, T_{m_2}\}$ of M nice with respect to the foliation F_1. To the covering $\mathcal{T}$ we can subordinate a covering $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ nice with respect to the foliation F_2. The covering $\mathcal{Z}$ is nice with respect to the foliations F_1, F_2 and F_3. The maps Θ_k restricted to the sets of the covering $\mathcal{Z}$ are charts distinguished with respect to the foliations F_1, F_2 and F_3.

Consider the spaces X_1, X_2 and X_3 of the nice covering $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$, determined by the foliations F_1, F_2 and F_3 respectively.

So, $X_j = \bigcup_{i=1}^{m} Z_i/R_j$ where xR_jy iff there exists $i \in \{1, \ldots, m\}$ such that $x, y \in Z_i$ and x, y belong to the same plaque of the foliation F_j, $j = 1, 2, 3$.

Take plaques $P_1 \in Z_i/R_1$ and $P_2 \in Z_i/R_2$. Then, using the form of the chart Θ_i, we obtain

$P_1 = \{x \in Z_i: p_2\phi_i(x) = a\}$,

$P_2 = \{x \in Z_i: \tilde{p}_2\psi_j(x) = b\}$,

$$P_1 \cap P_2 = \{x \in Z_i: p_2\phi_i(x) = a \text{ and } \tilde{p}_2\psi_j(x) = b\} = p_3$$

for some $a \in \mathbb{R}^{q_1}$ and $b \in \mathbb{R}^{q_2}$.

So, the set $P_3 = P_1 \cap P_2$ is a plaque of the foliation F_3. Conversely, with a plaque $P_3 \in Z_i/R_3$ given by $P_3 = \{x \in Z_i : p_2\phi_i(x) = a$ and $\tilde{p}_2\psi_j(x) = b\}$ we can associate the plaque $P_1 = \{x \in Z_i: p_2\phi_i(x) = a\}$ of the foliation F_1 and the plaque $P_2 = \{x \in Z_i: \tilde{p}_2\psi_j(x) = b\}$ of the foliation F_2. In this way we can identify the plaque $P_3 \in Z_i/R_3$ with the pair of plaques $(P_1, P_2) \in Z_i/R_1 \times Z_i/R_2$. Therefore,

$$X_3 = \bigcup_{i=1}^{m} Z_i/R_3 = \bigcup_{i=1}^{m} (Z_i/R_1 \times Z_i/R_2).$$

Denote by $F_{Z,1}$ (respectively, $G_{Z,1}$ and $H_{Z,1}$) the finite set of generators of the holonomy pseudogroup F (respectively, G and H) of the foliation F_1 (respectively, F_2 and F_3) with respect to the nice covering $\mathcal{Z}$.

Take $h_{ij} \in H_{Z,1}$. Then $h_{ij}(P) = Q$ iff $P \subset Z_i$, $Q \subset Z_j$ and $P \cap Q \neq \emptyset$. Remembering that $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$, where $P_1 \in Z_i/R_1$, $P_2 \in Z_i/R_2$, $Q_1 \in Z_j/R_1$, $Q_2 \in Z_j/R_2$, we obtain

$$h_{ij}((P_1, P_2)) = (Q_1, Q_2) = (f_{ij}(P_1), g_{ij}(P_2)) = (f_{ij} \times g_{ij})(P_1, P_2)$$

where $f_{ij} \in F_{Z,1}$, $g_{ij} \in G_{Z,1}$.

Take in the spaces X_1 and X_2 the Hausdorff metrics ρ_1 and ρ_2 determined by the Riemannian structure of M and, in the space X_3, the metric ρ_3 defined by the following formula:

$$\rho_3((x_1, x_2), (y_1, y_2)) = \max\{\rho_1(x_1, y_1), \rho_2(x_2, y_2)\}.$$

Then, using Proposition 2, we obtain

$$h_{\rho_3}(H, H_{Z,1}) \leqslant h_{\rho_1}(F, F_{Z,1}) + h_{\rho_2}(G, G_{Z,1}).$$

So

$$h(F_1 \cap F_2, \mathcal{Z}) \leq h(F_1, \mathcal{Z}) + h(F_2, \mathcal{Z}).$$

REMARK 2. The equality in (*) need not hold. The following example shows such a situation.

E x a m p l e. Let $T = \bar{D}^2 \times S^1 = \{z_1 \in \mathbb{C}: |z_1| \leq 1\} \times \{z_2 \in \mathbb{C}: |z_2| = 1\}$. Take the map $j: T \to T$ given by the formula

$$j(\rho e^{2\pi i\theta}, e^{2\pi i\phi}) = (\frac{1}{2} e^{2\pi i\phi} + \frac{1}{4} \rho e^{2\pi i\theta}, e^{4\pi i\phi}).$$

The compact manifold $T \setminus \widehat{\overset{\circ}{j}(T)}$ is foliated by the surfaces given by the equation $z_2 = \text{const}$. The components of the boundaries ∂T and $j(\partial T)$ can be identified by $j|\partial T$. In this way, we obtain a foliation F_1 of a compact manifold M^3 - Hirsch's foliation.

Let F_2 be a foliation transverse to F_1, $\dim F_2 = 1$. Take a covering $\mathcal{U}$ nice for the foliations F_1 and F_2. Since the foliation $F_1 \cap F_2$ consists of points, therefore $h(F_1 \cap F_2, \mathcal{U}) = 0$, while $h(F_1, \mathcal{U}) > 0$ (see Example 4.2 in [1]). Thus

$$h(F_1 \cap F_2, \mathcal{U}) < h(F_1, \mathcal{U}) + h(F_2, \mathcal{U})$$

in this case.

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Institute of Mathematics
University of Łódź

Andrzej Biś

ENTROPIA TRANSWERSALNYCH FOLIACJI

W prezentowanym artykule została wprowadzona nowa definicja topologicznej entropii. Definicja ta różni się trochę od definicji topologicznej entropii foliacji podanej przez E. Ghys, R. Langevin i P. Walczaka [1]. Jednakże, dla dowolnej foliacji F, topologiczna entropia foliacji F zdefiniowana w [1] jest mniejsza bądź równa topologicznej entropii foliacji F zdefiniowanej w tej pracy. Dla transwersalnych foliacji F_1 i F_2 szacowana jest entropia foliacji $F_1 \cap F_2$.