A C T A U N I V E R S I T A T I S L O D Z I E N S I S
FOLIA MATHEMATICA 4, 1991

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REMARKS ON CONVERGENCE OF SEQUENCES OF MEASURABLE FUNCTIONS

Let (X, S) be a measurable space and let $I \subset S$ be a proper σ-ideal of sets. In this note there is considered a notion of the sequence of functions $\{f_n\}$ which satisfies the vanishing restriction with respect to the function f. This condition is equivalent to the convergence of the sequence $\{f_n\}$ to f I-a.e. (Theorem 1). There is proved (Theorem 2) that if $\phi_n(x) = \sup_{i \geq n} |f_i(x) - f(x)|$, then the sequence $\{\phi_n\}$ converges to zero with respect to the σ-ideal I if and only if the sequence $\{f_n\}$ satisfies the vanishing restriction with respect to f.

Let $(X, \mathcal{S})$ be a measurable space and let $\mathcal{J} \subset \mathcal{S}$ be a proper σ-ideal of sets. Let f, f_n, $n \in N$, be $\mathcal{S}$-measurable functions on X. Put

$$E_n(\alpha) = \bigcup_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| > \alpha\}$$

for $\alpha > 0$ and $n \in N$. It is evident that the sets $E_n(\alpha)$ belong to $\mathcal{S}$ and if $m \leq n$ then $E_n(\alpha) \subset E_m(\alpha)$. Obviously, if $0 < \alpha < \beta$ then $E_n(\beta) \subset E_n(\alpha)$.

DEFINITION 1 (see [1]). We shall say that the sequence $\{f_n\}_{n \in N}$ of $\mathcal{S}$-measurable functions satisfies the vanishing restriction with respect to the $\mathcal{S}$-measurable function f if and only if

$$\bigcap_{n=1}^{\infty} E_n(\alpha) \in \mathcal{J}$$

for all $\alpha > 0$.

Clearly, the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f if and only if $\lim_n \sup\{x \in X: |f_n(x) - f(x)| > \alpha\} \in \mathcal{J}$ for all $\alpha > 0$.

DEFINITION 2 (see [4]). We shall say that the sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions converges to the $\mathcal{S}$-measurable function f in the sense of Egoroff if and only if there exists a sequence $\{E_m\}_{m\in N}$ of sets belonging to $\mathcal{S}$ such that $X - \bigcup_{m=1}^{\infty} E_m \in \mathcal{J}$ and for every $m \in N$ the sequence $\{f_n|E_m\}_{n\in N}$ converges uniformly to $f|E_m$.

We shall say that some property holds $\mathcal{J}$-almost everywhere (in abbr. $\mathcal{J}$-a.e.) if the set of points which do not have this property belongs to $\mathcal{J}$.

PROPOSITION 1. If the sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions converges to the $\mathcal{S}$-measurable function f in the sense of Egoroff, then $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f.

P r o o f. From the assumption it follows that there exists a sequence $\{E_m\}_{m\in N}$ of $\mathcal{S}$-measurable sets such that $X - \bigcup_{m=1}^{\infty} E_m \in$ $\in \mathcal{J}$ and for every $m \in N$ the sequence $\{f_n|E_m\}_{n\in N}$ converges uniformly to $f|E_m$. Then for every $m \in N$ and for every $\alpha > 0$ there exists a natural number $n(\alpha, m)$ such that $|f_n(x) - f(x)| \leq \alpha$ for every $n \geq n(\alpha, m)$ and for all $x \in E_m$. Hence for every $m \in N$ and for every $\alpha > 0$ there exists $n(\alpha, m) \in N$ such that $\bigcup_{n=n(\alpha,m)}^{\infty} \{x \in$ $\in X: |f_n(x) - f(x)| > \alpha\} \subset X - E_m$. Consequently, for every $m \in N$ and for every $\alpha > 0$ there exists $n(\alpha, m) \in N$ such that

(*) $E_m \subset X - E_{n(\alpha,m)}(\alpha)$.

Let $\alpha > 0$. We shall prove that $\lim_n \sup \{x \in X: |f_n(x) - f(x)| >$ $> \alpha\} \subset X - \bigcup_{m=1}^{\infty} E_m$. Let $x \in \bigcup_{m=1}^{\infty} E_m$. Then there exists $m_o \in N$ such that $x \in E_{m_o}$. From condition (*) it follows that there exists a natural number $n(\alpha, m_o)$ such that $x \notin E_{n(\alpha,m_o)}(\alpha)$. But the sequence $\{E_n(\alpha)\}_{n\in N}$ is a nonincreasing sequence of sets.

Thus $x \notin \limsup_n \{x \in X: |f_n(x) - f(x)| > \alpha\}$. Consequently, $\limsup_n \{x \in X: |f_n(x) - f(x)| > \alpha\} \in \mathcal{I}$ for all $\alpha > 0$.

THEOREM 1. The sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions converges to the $\mathcal{S}$-measurable function f $\mathcal{I}$-a.e. if and only if the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f.

P r o o f. Necessity. Let $C = \{x \in X: f(x) = \lim_{k\to\infty} f_k(x)\}$. Then $X - C \in \mathcal{I}$. Put $C_n(\alpha) = X - E_n(\alpha)$ for $n \in N$ and for all $\alpha > 0$. Observe that $C = \bigcap_{\alpha>0} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| \leq \alpha\} = \bigcap_{\alpha>0} \bigcup_{n=1}^{\infty} C_n(\alpha)$. Therefore $C \subset \bigcup_{n=1}^{\infty} C_n(\alpha)$ for all $\alpha > 0$, so $X - \bigcup_{n=1}^{\infty} C_n(\alpha) \subset X - C \in \mathcal{I}$. We have $X - \bigcup_{n=1}^{\infty} C_n(\alpha) = \bigcap_{n=1}^{\infty} E_n(\alpha) \in \mathcal{I}$ for all $\alpha > 0$. Consequently, the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f.

Sufficiency. Let $C = \{x \in X: f(x) = \lim_{k\to\infty} f_k(x)\}$. We have $C = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} (X - E_n(1/k))$. Hence $X - C = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} E_n(1/k)$. From the assumption it follows that $\bigcap_{n=1}^{\infty} E_n(\alpha) \in \mathcal{I}$ for all $\alpha > 0$. Consequently, $X - C \in \mathcal{I}$.

Obviously, the convergence in the sense of Egoroff implies convergence $\mathcal{I}$-a.e. If the pair $(\mathcal{S}, \mathcal{I})$ fulfils (C.C.C) and the condition (E) (see [3], [4]), then the inverse implication holds. Let $\mathcal{B}$ denote the σ-algebra of sets having the Baire property and let $\mathcal{K}$ be the σ-ideal of meager sets. It is known (see [3]) that the pair $(\mathcal{B}, \mathcal{K})$ does not fulfil the condition (E). The example from [2], p. 38 shows the sequence of continuous functions which is convergent to the function $f \equiv 0$ on a real line, so it satisfies the vanishing restriction with respect to f, but it is not convergent in the sense of Egoroff because it is uniformly convergent only on nowhere dense sets.

DEFINITION 3 (see [3]). We shall say that the sequence

$\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions converges with respect to the σ-ideal $\mathcal{J}$ to the $\mathcal{S}$-measurable function f if and only if every subsequence $\{f_{m_n}\}_{n\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{m_{p_n}}\}_{n\in N}$ which converges to f $\mathcal{J}$-a.e. We shall use the denotation $f_n \overset{\mathcal{J}}{\to} f$.

Put

$$\varphi_n(x) = \sup\{|f_i(x) - f(x)|: i \in N;\ i \geq n\}.$$

Obviously, the functions φ_n, $n \in N$ are $\mathcal{S}$-measurable and if $m \leq n$, then $\varphi_n(x) \leq \varphi_m(x)$ for all $x \in X$.

REMARK 1. If $\alpha > 0$ and $n \in N$, then

$$E_n(\alpha) = \{x \in X: \varphi_n(x) > \alpha\}.$$

For the proof see [1].

COROLLARY 1. The sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions satisfies the vanishing restriction with respect to the $\mathcal{S}$-measurable function f if and only if

$$\bigcap_{n=1}^{\infty} \{x \in X: \varphi_n(x) > \alpha\} \in \mathcal{J}$$

for all $\alpha > 0$.

LEMMA 1. The sequence $\{\varphi_n\}_{n\in N}$ converges with respect to the σ-ideal $\mathcal{J}$ to a function $\varphi \equiv 0$ if and only if

$$\bigcap_{n=1}^{\infty} \{x \in X: \varphi_n(x) > \alpha\} \in \mathcal{J}$$

for all $\alpha > 0$.

P r o o f. Suppose that there exists a positive number α such that $\bigcap_{n=1}^{\infty} \{x\in X: \varphi_n(x) > \alpha\} \notin \mathcal{J}$. Put $B = \bigcap_{n=1}^{\infty} \{x\in X: \varphi_n(x) > \alpha\}$. Then the sequence $\{\varphi_n\}_{n\in N}$ cannot contain any subsequence, which is convergent to φ $\mathcal{J}$-a.e. on X, because $B = \{x \in X: \varphi_n(x) > \alpha$ for $n \in N\} \notin \mathcal{J}$. Consequently, $\varphi_n \overset{\mathcal{J}}{\nrightarrow} \varphi$.

Suppose now that the sequence $\{\varphi_n\}_{n\in N}$ is not convergent with respect to the σ-ideal $\mathcal{J}$ to the function $\varphi \equiv 0$. From Lemma 4 in [3] it follows that there exist a subsequence $\{\varphi_{m_n}\}_{n\in N}$ of

$\{\varphi_n\}_{n\in N}$, a set $A \in \mathcal{S} - \mathcal{I}$ and a natural number k_o such that $\lim_n \sup \varphi_{m_n}(x) > 1/k_o$ $\mathcal{I}$-a.e. on A_o. Hence $\{x \in X: \lim_n \sup \varphi_n(x) > 1/k_o\} \notin \mathcal{I}$. It is easy to see that $\{x \in X: \lim_n \sup \varphi_n(x) > 1/k_o\} = \bigcap_{n=1}^{\infty} \{x \in X: \varphi_n(x) > 1/k_o\}$, because the sequence $\{\varphi_n\}_{n\in N}$ is nonicreasing. Consequently, we have $\bigcap_{n=1}^{\infty} \{x \in X: \varphi_n(x) > 1/k_o\} \notin \mathcal{I}$.

THEOREM 2. Let $\{f_n\}_{n\in N}$ be a sequence of $\mathcal{S}$-measurable functions and let f be an $\mathcal{S}$-measurable function. Then the following conditions are equivalent:

(i) the sequence $\{f_n\}_{n\in N}$ converges to f $\mathcal{I}$-a.e. on X;

(ii) the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f;

(iii) the sequence $\{\varphi_n\}_{n\in N}$ converges to zero with respect to the σ-ideal $\mathcal{I}$.

The proof follows immediately from Theorem 1, Remark 1 and Lemma 1.

If the sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions converges with respect to the σ-ideal $\mathcal{I}$ to an $\mathcal{S}$-measurable function f, then $\{f_n\}_{n\in N}$ need not satisfy the vanishing restriction with respect to f. In the case when $\mathcal{I}$ is the σ-ideal of sets of Lebesgue measure zero then for $\{f_n\}_{n\in N}$ we can take an arbitrary sequence of measurable functions defined on $[0, 1]$, which is convergent in measure but is not convergent a.e., for example the sequence of characteristic functions of the intervals $[0, 1]$, $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, $[0, \frac{1}{4}]$, $[\frac{1}{4}, \frac{1}{2}]$, $[\frac{1}{2}, \frac{3}{4}]$, ... This sequence does not satisfy the vanishing restriction with respect to $f \equiv 0$ because $E_n(\alpha) = \bigcup_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| > \alpha\} = [0, 1]$ for $0 < \alpha < 1$ and for every $n \in N$. We can take the same example for the σ-ideal of meager sets.

DEFINITION 4 (see [1]). We shall say that the sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable functions is M-convergent to an $\mathcal{S}$-measurable function f if and only if for all $\alpha > 0$ we have

$$\{x \in X: |f_i(x) - f(x)| > \alpha\} \subset \{x \in X: |f_j(x) - f(x)| > \alpha\}$$

for $i \geq j$, $i, j \in N$.

PROPOSITION 2. If the sequence $\{f_n\}_{n\in N}$ of S-measurable functions converges with respect to the σ-ideal $\mathcal{J}$ to an S-measurable function f and $\{f_n\}_{n\in N}$ is M-convergent to f, then the sequence $\{f_n\}_{n\in N}$ satisfies the vanishing restriction with respect to f and $\{f_n\}_{n\in N}$ converges to f $\mathcal{J}$-a.e.

P r o o f. Suppose that the sequence $\{f_n\}_{n\in N}$ does not satisfy the vanishing restriction with respect to f. Then there exists a number $\alpha > 0$ such that $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| > \alpha\} \notin \mathcal{J}$. From the assumption it follows that $\bigcup_{i=n}^{\infty} \{x \in X: |f_i(x) - f(x)| > \alpha\} = \{x \in X: |f_n(x) - f(x)| > \alpha\}$. Hence $\bigcap_{n=1}^{\infty} \{x \in X: |f_n(x) - f(x)| > \alpha\} \notin \mathcal{J}$. Put $B = \bigcap_{n=1}^{\infty} \{x \in X: |f_n(x) - f(x)| > \alpha\}$. The sequence $\{f_n\}_{n\in N}$ does not converge to f with respect to the σ-ideal $\mathcal{J}$ because it has not subsequence convergent to f $\mathcal{J}$-a.e. on B which gives a contradiction.

DEFINITION 5. We shall say that the sequence $\{f_n\}_{n\in N}$ of S-measurable functions is bounded with respect to the σ-ideal $\mathcal{J}$ if and only if the sequence $\{a_n f_n\}_{n\in N}$ converges with respect to the σ-ideal $\mathcal{J}$ to zero for every sequence $\{a_n\}_{n\in N}$ of real numbers tending to zero.

Obviously, if the sequence $\{f_n\}_{n\in N}$ is bounded, then it is also bounded with respect to the σ-ideal $\mathcal{J}$.

REMARK 2. If $f_n \overset{\mathcal{J}}{\to} f$ and $g_n \overset{\mathcal{J}}{\to} g$, then $f_n g_n \overset{\mathcal{J}}{\to} fg$.

The proof is obvious.

REMARK 3. Every sequence of S-measurable functions which is convergent with respect to the σ-ideal $\mathcal{J}$ is bounded with respect to the σ-ideal $\mathcal{J}$.

The proof follows immediately from previous remark because every constant function is S-measurable.

The analog of Bolzano-Weierstrass's theorem does not hold.

Put $f_n(x) = \chi_{A_n}(x)$, where $A_n = \bigcup_{i=0}^{2^{n-1}-1} [\frac{2i}{2^n}, \frac{2i+1}{2^n}]$, for $n \in N$. This sequence is bounded but none of its subsequence is convergent in measure.

REFERENCES

[1] B a r t l e R. G., An extension of Egorov's theorem, Amer. Math. Month., 87/8 (1980), 628-633.

[2] O x t o b y J. C., Measure and category New York-Heidelberg-Berlin 1971.

[3] W a g n e r E., Sequences of measurable functions, Fund. Math., 112 (1981), 89-102.

[4] W a g n e r E., W i l c z y ń s k i W., Convergence almost everywhere of sequences of measurable functions, Colloq. Math., 45 (1981), 119-124.

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UWAGI O ZBIEŻNOŚCI CIĄGÓW FUNKCJI MIERZALNYCH

Niech $(X, \mathcal{S})$ będzie przestrzenią mierzalną i niech $\mathcal{I} \subset \mathcal{S}$ będzie właściwym σ-ideałem. W artykule rozważane jest pojęcie ciągu funkcji $\{f_n\}$ mającego znikające obcięcie względem funkcji f. Pojęcie to jest równoważne zbieżności ciągu $\{f_n\}$ I-p.w. do funkcji f (twierdzenie 1). Udowodniono (twierdzenie 2), że jeśli $\phi_n(x) = \sup_{i \geq n} |f_i(x) - f(x)|$, to ciąg $\{\phi_n\}$ jest zbieżny do zera wg σ-ideału $\mathcal{I}$ wtedy i tylko wtedy, gdy ciąg $\{f_n\}$ ma znikające obcięcie względem funkcji f.