A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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SOME EXTREMAL PROBLEMS IN THE CLASS OF HOLOMORPHIC UNIVALENT FUNCTIONS

Let $S'(b)$, $0 < b < 1$, denote the class of functions of the form $F(z) = z + \sum_{n=3}^{\infty} a_n z^n$, holomorphic and univalent in the disc $|z| < 1$, satisfying the condition $|F(z)| < b^{-1}$ and let $G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k$ denote the function inverse to F.

In the paper the estimation of some initial coefficients of G as well as the estimation of a_5 in $S'(b)$ and in the odd subclass of $S'(b)$ are given for some b from the interval $(0, 1)$.

1. INTRODUCTION

Let $S(b)$, $0 < b < 1$, denote the class of functions of the form

$$f(z) = b(z + \sum_{n=2}^{\infty} a_n z^n), \tag{1}$$

holomorphic and univalent in the disc $D = \{z: |z| < 1\}$ and satisfying the condition $|f(z)| < 1$.

Denote by $S'(b)$, $0 < b < 1$, the class of holomorphic-univalent functions of the form

$$\omega = F(z) = z + \sum_{n=3}^{\infty} a_n z^n, \quad z \in D,$$

satisfying the condition $|F(z)| < b^{-1}$ and let

$$z = G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k$$

denote the function inverse to F.

Recently many authors (e.g. [3], [4], [9]) considered the problem of coefficient estimations in some classes of functions inverse to classes of meromorphic functions. In [1] L a u n o n e n estimated coefficients of the inverse functions of S(b) applying Fitz-Gerald-Launonen inequality.

In this paper the estimation of some initial coefficients of functions inverse to $S'(b)$ functions will be considered. The Launonen method and the Power-inequality will be used.

2. THE ESTIMATION OF THE COEFFICIENTS c_3, c_4, c_5, c_7, c_9

It follows from the connection between the functions f, F and G that

$$\begin{aligned} c_1 &= 1, \\ c_2 &= 0, \\ c_3 &= -a_3, \\ c_4 &= -a_4, \\ c_5 &= -a_5 + 3a_3^2 \end{aligned} \tag{2}$$

The estimation of c_3 and c_4 follows then immediately from the estimation of a_3 and a_4 in the class S(b) in the case $a_2 = 0$. From [7] (p. 265) we have

$$|c_3| = |a_3| \leqq 1 - b^2 \tag{3}$$

and the equality holds for the function G_o which is inverse to the one satisfying the equation

$$\frac{F_o}{1 + b^2F_o^2} = \frac{z}{1 + z^2} \tag{4}$$

From Grunsky-type inequalities which are sharp in the considered case $a_2 = 0$ ([5]) it follows that

$$\text{re } a_4 \leqq \frac{2}{3}(1 - b^3) - \frac{|a_3|^2}{2(1 - b)} \leqq \frac{2}{3}(1 - b^3).$$

The equality holds for $a_3 = a_2 = 0$ and the extremal function F_1 is defined by

$$\frac{F_1}{(1 - b^3F_1^3)^{2/3}} = \frac{z}{(1 - z^3)^{2/3}} \tag{5}$$

Thus

$$|c_4| \leq \frac{2}{3}(1 - b^3)$$

with the equality for the function inverse to F_1.

In order to estimate next coefficients we apply the Launonen inequality [1]. The inequality for every function $z = G(\omega)$ inverse to $S'(b)$-function has the form

$$\left|\iint_{\gamma\gamma} \mu(\omega)\mu(s) \frac{G(\omega) - G(s)}{\omega - s} d\omega ds\right| \leq$$

$$\leq \iint_{\gamma\gamma} \mu(\omega)\overline{\mu(s)} \frac{G(\omega)}{\omega} \frac{\overline{G(s)}}{\bar{s}} \frac{1 - b^2\omega\bar{s}}{1 - G(\omega)\overline{G(s)}} d\omega d\bar{s}, \tag{6}$$

where γ is a closed analytic curve and μ is a continuous weight function on γ. For $\mu(\omega) = \omega^{-3}$ (6) takes the form

$$|-4\pi^2 c_5| \leq 4\pi^2(|c_3|^2 + 1 - b^2).$$

Thus by (3) we have

$$|c_5| \leq |c_3|^2 + 1 - b^2 \leq 2 - 3b^2 + b^4 \tag{7}$$

The maximum is reached by the same function as in the case $|c_3|$ i.e. by the function inverse to F_o.

For $\mu(\omega) = \omega^{-4}$ the condition (6) yields

$$|-4\pi^2 c_7| \leq 4\pi^2(|c_4|^2 + |c_3|^2(4 - b^2) + 1 - b^2),$$

from where

$$|c_7| \leq |c_4|^2 + |c_3|^2(4 - b^2) + 1 - b^2 \tag{8}$$

In order to estimate this we apply the area inequality for the class S(b) ([7], p. 182):

$$\sum_{\nu=1}^{\infty} \nu|\alpha_\nu - b^2 a_\nu|^2 \leq 1 \tag{9}$$

where

$$\frac{1}{F(z)} = \frac{1}{z} + \sum_{\nu=0}^{\infty} \alpha_\nu z^\nu \tag{10}$$

From (1) and (10) it follows that

$$\alpha_1 = -a_3, \qquad \alpha_2 = -a_4,$$

and as the consequence of (9) we have

$$|-a_3 - b^2|^2 + 2|a_4|^2 \leq 1,$$

and by (2)

$$|c_4|^2 \leq \frac{1}{2}(1 - |c_3 - b^2|^2) \tag{11}$$

The inequality (8) takes then the form

$$|c_7| \leq \frac{1}{2} - \frac{1}{2}|c_3 - b^2|^2 + |c_3|^2(4 - b^2) + 1 - b^2.$$

The rotated function $\tau^{-1}G(\tau\omega) = \omega + \sum_{k=3}^{\infty} \tau^{k-1}c_k\omega^k$, $|\tau| = 1$,

preserves $|c_7|$ and allows the normalization $c_3 \leq 0$. Denoting

$$x = c_3 \in \langle -(1 - b^2);\ 0\rangle \quad \text{and} \quad P(x) = (\frac{7}{2} - b^2)x^2 + b^2x,$$

we have then

$$|c_7| \leq \frac{3}{2} - b^2 - \frac{1}{2}b^4 + P(x).$$

Require

$$P(-(1 - b^2)) = (1 - b^2)(b^4 - \frac{11}{2}b^2 + \frac{7}{2}) \geq P(0) = 0.$$

This yields

$$\max P = P(-(1 - b^2)) \quad \text{for} \quad 0 \leq b \leq b_o,$$

where

$$b_o = \frac{1}{2}(11 - \sqrt{65})^{1/2} = 0{,}856992160\ldots \tag{12}$$

is the root of the equation $b^4 - \frac{11}{2}b^2 + \frac{7}{2} = 0$. Hence for $0 \leq b \leq$ $\leq b_o$ the sharp estimation holds

$$|c_7| \leq \frac{3}{2} - b^2 - \frac{1}{2}b^4 + P(-(1 - b^2)) = 5 - 10b^2 + 6b^4 - b^6.$$

As in the case $|c_5|$ the coefficient $|c_7|$ is maximized with $|c_3|$ i.e. by the function inverse to F_o.

In the case $\mu(\omega) = \omega^{-5}$ we can proceed similarly. From (6) we have

$$|c_9| \leq (9 - 4b^2)|c_3|^2 + (4 - b^2)|c_4|^2 + |c_5|^2 + 1 - b^2,$$

from where by (7), (11) and the fact that again we can assume $c_3 \leq 0$, there holds the inequality

$$|c_9| \leq \frac{1}{2}(1 - b^2)(8 + 3b^2 - 2b^3 - b^4) + xQ(x),$$

where

$$x = c_3 \in \langle -(1 - b^2);\ 0\rangle \quad \text{and}$$

and

$$Q(x) = x^3 + x(9 - \frac{11}{2}b^2) + b^2(4 - b^2).$$

Since $Q'(x) > 0$ for $b \in (0, 1)$ and $x \in \langle -(1 - b^2); 0\rangle$ then the equation $Q(x) = 0$ can have only one root in the interval $\langle -(1 - b^2); 0\rangle$.

Require

$$Q(-(1 - b^2)) = -(1 - b^2)(-b^6 - \frac{19}{2}b^4 + \frac{43}{2}b^2 - 10) \geq 0.$$

This holds for $0 < b \leq b_1$, where $b_1 = 0{,}843285210\ldots$ is the root of the equation

$$-b^6 - \frac{19}{2}b^4 + \frac{43}{2}b^2 - 10 = 0 \tag{13}$$

Hence for $0 < b \leq b_1$ the sharp estimation

$$|c_9| \leq 14 - 35b^2 + 30b^4 - 10b^6 + b^8$$

holds. Again with $|c_3|$ also $|c_9|$ is maximized by the function inverse to F_o.

So we have shown

THEOREM 1. For every function

$$z = G(\omega) = \sum_{c=1}^{\infty} c_k\omega^k,$$

inverse to $S'(b)$-functions, the estimations

$$|c_3| \leq 1 - b^2, \qquad b \in (0, 1),$$

$$|c_4| \leq \frac{2}{3}(1 - b^3), \qquad b \in (0, 1),$$

$$|c_5| \leq 2 - 3b^2 + b^4, \qquad b \in (0, 1),$$

$$|c_7| \leq 5 - 10b^2 + 6b^4 - b^6, \qquad b \in (0, b_o\rangle,$$

$$|c_9| \leq 14 - 35b^2 + 30b^4 - 10b^6 + b^8, \qquad b \in (0, b_1\rangle$$

hold, where b_o is given by (12) and b_1 is the root of equation (13). Except for $|c_4|$ the function inverse to F_o defined by (4) is the extremal one. In the case $|c_4|$ the extremal function is the one inverse to F_1 given by (5).

3. ON THE ESTIMATION OF a_5 IN THE CLASS $S'(b)$

From the Grunsky-type inequality for a_5 for which $a_2 = 0$ (see [2], p. 473) we have

$$2 \operatorname{re} a_5 - (1 - b^4) \leq \left(3 - \frac{2}{\ln b^{-1}}\right)u^2 - 3v^2 \tag{14}$$

where we denoted $a_3 = u + iv$. Provided $3 - \frac{2}{\ln b^{-1}} \leq 0$ what implies two cases

$$\text{a) } 1 - \frac{2}{3 \ln b^{-1}} < 0 \iff e^{-\frac{2}{3}} < b < 1,$$

$$\text{b) } 1 - \frac{2}{3 \ln b^{-1}} = 0 \iff b = e^{-\frac{2}{3}} \tag{15}$$

we obtain

$$\operatorname{re} a_5 \leq \frac{1}{2}(1 - b^4).$$

In the case a) the equality in (14) holds for $u = v = a_3 = 0$. In the case b) it requires $v = 0$ but u is left as a free parameter.

As the rotation $\tau^{-1}F(\tau z)$ preserves the class $S'(b)$ we can assume that $a_5 = |a_5| \geq 0$ and $\operatorname{re} a_3 \leq 0$.

In order to study the equality cases in (15) put $a_2 = 0$ in the inequality (82), p. 472 of [2]. We obtain

$$\operatorname{re}\left(\ln bx_o^2 + a_3x_1^2 + a_5 - \frac{3}{2}a_3^2 + 2a_3x_o + 2a_4x_1\right) \leq$$

$$\leq (1 - b^2)|x_1|^2 + \frac{1}{2}(1 - b^4),$$

where x_o, x_1 are free complex parameters. Since, in the normalized equality case of (14), $a_5 = \frac{1}{2}(1 - b^4)$, $v = 0$, the above inequality takes the form

$$\operatorname{re}\left(\ln bx_o^2 + ux_1^2 - \frac{3}{2}u^2 + 2ux_o + 2a_4x_1\right) \leq (1 - b^2)|x_1|^2 \tag{16}$$

Putting $x_o = 0$, $x_1 = |x_1|e^{i\phi}$ in the case (a) $u = 0$ we have

$$2 \operatorname{re}(e^{i\phi}a_4) \leq (1 - b^2)|x_1|^2$$

what with $0 < |x_1| \to 0$ gives $\operatorname{re}(e^{i\phi}a_4) \leq 0$ for $\phi \in \langle 0, 2\pi\rangle$ which implies $a_4 = 0$.

In order to prove that also in the case (b) $a_4 = 0$ it is sufficient to put $x_o = \frac{3}{2}u$, $x_1 = |x_1|e^{i\phi}$ in the inequality (16) and tends with $|x_1|$ to zero.

So we have shown that in the extremal cale all the coefficients up to a_5 are real. From the Power inequality it then follows that we may use condition (35), p. 488 in [6]:

$$2x_o \ln bF + b^2(b^2F^2 - b^{-2}F^{-2}) = 2x_o \ln z + z^2 - z^{-2},$$

$$2x_o = a_3 = u \leqq 0 \tag{17}$$

In the case (a) in (16) $u = x_o = 0$. In the case (b) the extremal case can be studied by aid of the boundary correspondence. For that purpose let us put in (17) $z = e^{i\phi}$, $F(e^{i\phi}) = r(\phi)e^{i\psi(\phi)}$ and compare the real parts:

$$u \ln br + e^{-\frac{4}{3}} \cos 2\psi(b^2r^2 - b^{-2}r^{-2}) = 0,$$

from where

$$\cos 2\psi\ 0 - e^{4/3} \frac{u \ln br}{b^2r^2 - b^{-2}r^{-2}} \xrightarrow[r \to b^{-1}]{} \frac{-ue^{4/3}}{4}$$

what implies the limitation for u:

$$-4e^{-\frac{4}{3}} \leqq u \leqq 0.$$

So we have proved

THEOREM 2. In the class $S'(b)$ for $b \in \langle e^{-\frac{2}{3}}; 1)$ the estimation

$$|a_5| \leqq \frac{1}{2}(1 - b^4)$$

holds. The equalit holds for the function given by (17) where $x_o = 0$ for $b \in (e^{-\frac{2}{3}}; 1)$ and arbitrary $x_o = u \in \langle -4e^{-\frac{4}{3}}; 0\rangle$ for $b = e^{-\frac{2}{3}}$.

4. ON THE ESTIMATION OF a_5 IN THE ODD SUBCLASS OF $S'(b)$

The problem of estimation a_5 for $b \in (0, e^{-2/3})$ remains open in the class $S'(b)$ but we solve the corresponding question in the odd subclass of $S'(b)$. We will use the well-known fact that if $f(z)$ of form (1) belongs to $S(b)$ then

$$\widetilde{f(z)} = \sqrt{f(z^2)} = \tilde{b}(z + \tilde{a}_3 z^3 + \dots),$$

is an odd function from $S(b^{1/2})$ and the connections

$$b = \tilde{b}^2, \quad a_2 = 2\tilde{a}_3, \quad a_3 = 2\tilde{a}_5 + \tilde{a}_3^2$$

hold.

From the Power inequality in the class $S(b)$, [8] we have

$$\mathrm{re}\,(a_3 - a_2^2) \leqq 1 - b^2 + \frac{U^2}{\ln b}, \quad \text{where } U = \mathrm{re}\, a_2,$$

and the equality can be reached if $|U| \leqq 2b|\ln b|$. In the terms of odd $S(\tilde{b})$ functions it gives

$$2\,\mathrm{re}\,\tilde{a}_5 - (1 - \tilde{b}^4) \leqq 3\,\mathrm{re}\,\tilde{a}_3^2 + 4\frac{(\mathrm{re}\,\tilde{a}_3)^2}{\ln \tilde{b}^2} =$$

$$= (3 + \frac{4}{\ln \tilde{b}^2})\tilde{u}^2 - 3\tilde{v}^2 \leqq (3 + \frac{4}{\ln \tilde{b}^2})\tilde{u}^2 = \tilde{M}(\tilde{u}),$$

where we denote $\tilde{a}_3 = \tilde{u} + i\tilde{v}$. Equality holds for $\tilde{v} = 0$, and if $\tilde{b} \in \langle e^{-\frac{2}{3}}; 1)$ we obtain former estimation

$$\mathrm{re}\,\tilde{a}_5 \leqq \frac{1}{2}(1 - \tilde{b}^4).$$

If $\tilde{b} \in (0, e^{-\frac{2}{3}})$ and $|\tilde{u}| \leqq \tilde{b}^2|\ln \tilde{b}^2|$ we have an estimation

$$\tilde{M}(\tilde{u}) \leqq (3 + \frac{4}{\ln \tilde{b}^2})\tilde{b}^4 \ln^2 \tilde{b}^2 = \tilde{b}^4 \ln \tilde{b}^2 (4 + 3\ln \tilde{b}^2) \tag{18}$$

For $|\tilde{u}| \geqq \tilde{b}^2|\ln \tilde{b}^2|$ what is equivalent to $|\mathrm{re}\, a_2| = |U| \geqq 2b|\ln b|$ from [8], p. 17 we obtain

$$\mathrm{re}\,(a_3 - a_2^2) \leqq 1 - b^2 - 2|U|\sigma + 2(\sigma - b)^2,$$

where $\sigma \in \langle b, 1\rangle$ is the root of the equation

$$\sigma \ln \sigma - \sigma + b + \frac{|U|}{2} = 0.$$

In terms of $S'(\tilde{b})$ it means that

$$2\,\mathrm{re}\,\tilde{a}_5 - (1 - \tilde{b}^4) \leqq 3(\tilde{u}^2 - \tilde{v}^2) - 4|\tilde{u}|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2;$$

$$\tilde{\sigma}^2 = \sigma \in \langle \tilde{b}^2, 1\rangle,$$

$$|\tilde{u}| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2).$$

So, for $\tilde{b}^2 |\ln \tilde{b}^2| \leq |\tilde{u}| \leq 1 - \tilde{b}^2$,

$$2 \operatorname{re} \tilde{a}_5 - (1 - \tilde{b}^4) \leq 3\tilde{u}^2 - 4|\tilde{u}|\sigma^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2,$$

$$|\tilde{u}| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2),$$

with the equality for $\tilde{v} = 0$. To estimate the upper bound, return for brevity to the variable σ:

$$2 \operatorname{re} \tilde{a}_5 - (1 - \tilde{b}^4) \leq 3(\sigma \ln \sigma - \sigma + b)^2 + 4\sigma(\sigma \ln \sigma - \sigma + b) + +2(\sigma - b)^2 = M(\sigma), \quad \tilde{b}^2 = b \leq \sigma \leq 1. \tag{19}$$

Since $\sigma = \sigma(u) \in \langle b, 1\rangle$ is uniquely determined by

$$|\tilde{u}| \in \langle \tilde{b}^2 |\ln \tilde{b}^2|, 1 - \tilde{b}^2\rangle$$

([8], p. 15), then it is sufficient to maximize $M(\sigma)$ for $\sigma \in \langle b, 1\rangle$. Since

$$\frac{dM(\sigma)}{d\sigma} = 2 \ln \sigma (3\sigma \ln \sigma + \sigma + 3b)$$

the considered maximum is reached for the root of the equation

$$3\sigma \ln \sigma + \sigma + 3b = 0$$

which belongs to the interval $\langle e^{-\frac{4}{3}}, 1\rangle$, what by (19) gives

$$\operatorname{re} \tilde{a}_5 \leq \frac{1}{2}(1 - b^4) + (\tilde{\sigma}^2 - \tilde{b}^2)^2.$$

Since the maximum of $\tilde{M}(\tilde{u})$ in (18) equals to $M(\sigma)$ we have maximized $\operatorname{re} \tilde{a}_5$ and hence $|\tilde{a}_5|$:

THEOREM 3. In the odd subclass of $S'(b)$, $b \in (0, e^{-\frac{2}{3}})$,

$$|a_5| \leq \frac{1}{2}(1 - b^4) + (\sigma^2 - b^2),$$

where $\sigma \in \langle e^{-\frac{2}{3}}, 1\rangle$ is the root of the equation

$$3\sigma^2 \ln \sigma^2 + \sigma^2 + 3b^2 = 0.$$

REFERENCES

[1] L a u n o n e n E., On exponentiated Grunsky inequalities for bounded univalent functions, Ann. Sci. Fenn. Ser. AI Math., Dissert., 1 (1975), 1-34.

[2] S c h i f f e r M., T a m m i O., On the coefficient problem for bounded univalent functions, Trans. Amer. Math. Soc., 140 (1969), 461-474.

[3] S c h o b e r G., Coefficients of inverse of meromorphic univalent functions, Proc. Amer. Math. Soc., 67/1, (1977), 111-116.

[4] S i e j k a H., T a m m i O., On estimationg the inverse coefficients for meromorphic univalent functions omitting a disc, Ann. Acad. Sci. Fenn., Ser. AI Math., 12 (1987), 85-93.

[5] T a m m i O., On optimizing parameters of the Power inequality for a_4 in the class of bounded univalent functions, Ann. Acad. Sci. Fenn., Ser. AI, 560 (1973), 1-24.

[6] T a m m i O., Inequalities for the fifth coefficient for bounded real univalent functions, I. D'Anal. Math., 30 (1976), 481-497.

[7] T a m m i O., Extremum problems for bounded univalent functions, Lecture Notes in Mathematics, Vol. 646, Berlin-Heidelberg-New York 1978.

[8] T a m m i O., Extremum problems for bounded univalent functions II, Lecture Notes in Mathematics, Vol. 913, Berlin-Heidelberg-New York 1982.

[9] T a m m i O., Some coefficient estimations in the class Σ_b' of meromorphic univalent functions (in print).

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O PEWNYCH PROBLEMACH EKSTREMALNYCH

W KLASIE FUNKCJI HOLOMORFICZNYCH I JEDNOLISTNYCH

Niech $S'(b)$, $0 < b < 1$, oznacza klasę funkcji postaci

$$\omega = F(z) = z + \sum_{n=3}^{\infty} a_n z^n,$$

holomorficznych i jednolistnych w kole $|z| < 1$, spełniających tam $|F(z)| < b^{-1}$ i niech funkcja

$$z = G(\omega) = \sum_{k=1}^{\infty} c_k \omega^k$$

będzie funkcją odwrotną od F.

W prezentowanym artykule otrzymano oszacowania współczynników c_3, c_4, c_5, c_7, c_9 dla pewnych b z przedziału (0, 1). Ponadto otrzymano oszacowanie współczynnika a_5 w klasie $S'(b)$ dla $b \in \langle e^{-2/3}, 1\rangle$ oraz w podklasie funkcji nieparzystych tej klasy dla $b \in (0, e^{-2/3})$. Stosowano metodę Launonena oraz pewne nierówności potęgowe (Power-inequality).