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# ON THE GENERALIZED ZINK CLASSIFICATION

We study the generalized Zink classification for systems $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$ where $(X, \mathcal{T})$ is a topological space and $\mathcal{J}$ is a $\sigma$-ideal in a $\sigma$-algebra $\mathcal{S} \subset \mathcal{P}(X)$, such that $\mathcal{T} \setminus \{\emptyset\} \subset \mathcal{S} \setminus \mathcal{J}$. We obtain a characterization analogous to Zink's one. Some new examples are given.

Zink in [8] introduced and explored a classification of topological measure spaces. A quadruple $(X, \mathcal{T}, \mathcal{S}, \mu)$ is called a topological measure space if and only if $(X, \mathcal{T})$ is a topological space and $(X, \mathcal{S}, \mu)$ is a measure space, such that $\mathcal{T} \subset \mathcal{S}$ and $\mu(U) > 0$ for all $U \in \mathcal{T} \setminus \{\emptyset\}$.

We observed that the notion of measure is not essential in the proofs of Zink's theorems and it suffices only to use the $\sigma$-ideal of sets on which the measure is zero. Thus, we consider here a classification, analogous to Zink's, for quadruples $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$ where $(X, \mathcal{T})$ is a topological space and $\mathcal{J}$ is a $\sigma$-ideal in a $\sigma$-algebra $\mathcal{S} \subset \mathcal{P}(X)$, such that $\mathcal{T} \setminus \{\emptyset\} \subset \mathcal{S} \setminus \mathcal{J}$.

In the sequel, let a fixed system $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$ be given.

We say that two sets $A, B \in \mathcal{S}$ (respectively, two real-valued functions $f, g$ defined on $X$, measurable with respect to $\mathcal{S}$) are equivalent if and only if their symmetric difference $A \triangle B$ (respectively, the set $\{x \in X : f(x) \neq g(x)\}$) belongs to $\mathcal{J}$.

Throughout the paper, we consider continuous and semicontinuous functions mapping the space $(X, \mathcal{T})$ into the real line $R$ with the natural topology.

Recall the notation of Zink. The classes $\mathcal{L}_\alpha$, $\mathcal{U}_\alpha$, $\alpha < \omega_1$, are defined as follows: $\mathcal{L}_1$ $(\mathcal{U}_1)$ is the class of all lower - (respectively, upper -) semicontinuous functions; if $1 < \alpha < \omega_1$, and $\mathcal{L}_\beta$ (resp. $\mathcal{U}_\beta$) have been defined for $\beta < \alpha$, then $\mathcal{L}_\alpha$ (respectively, $\mathcal{U}_\alpha$) is the class of all limits of pointwise convergent sequences of elements of $\bigcup_{\beta < \alpha} \mathcal{L}_\beta$ (respectively, $\bigcup_{\beta < \alpha} \mathcal{U}_\beta$). Moreover, let $\mathcal{L}_0$ and $\mathcal{U}_0$ be equal to the class of all continuous functions.

A system $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$ will be called an $\alpha$-space (where $0 \leq \alpha < \omega_1$) if and only if $\alpha$ is the first ordinal $\gamma$ such that each bounded real-valued $\mathcal{S}$-measurable function on X is equivalent to an element of $\mathcal{L}_\gamma$. In particular, we obtain Zink's classification by considering $\mathcal{J} = \{A \in \mathcal{S} : \mu(A) = 0\}$ in our scheme where $\mu$ denotes a measure on $\mathcal{S}$ which does not vanish on non-empty open sets.

Let $F_\alpha$, $G_\alpha$, $\alpha < \omega_1$, denote the classes of Borel (with respect to $\mathcal{T}$) subsets of X, defined as in [2], p. 251-252. Moreover, let $F_{-1}$ and $G_{-1}$ be equal to the class of all closed-and-open subsets of X.

The following theorem will be the main tool in establishing places of various systems in the classification described above.

THEOREM 1. Let $\alpha$ be a finite ordinal number. In order that each bounded $\mathcal{S}$-measurable function be equivalent to an element of $\mathcal{L}_\alpha$, it is both necessary and sufficient that each $\mathcal{S}$-measurable set be equivalent to a set of type $G_{\alpha-1}$.

REMARK. As in [8], one can observe that each bounded $\mathcal{S}$-measurable function (respectively, each $\mathcal{S}$-measurable set) is equivalent to an element of $\mathcal{L}_\gamma$ (respectively, to a set of type $G_\gamma$) if and only if the analogous condition with $\mathcal{L}_\gamma$ replaced by $\mathcal{U}_\gamma$ (respectively, $G_\gamma$ replaced by $F_\gamma$) holds.

The proof of Theorem 1 is similar to that from [8]. Most of modifications are needed in the proof of sufficiency for $\alpha = 0$, thus we provide that part with details and omit the rest. Note that in the case $\alpha = 0$, the condition that each $\mathcal{S}$-measurable set is equivalent to a closed-and-open set implies that the closure of each open set is again open, i.e. the topological space $(X, \mathcal{T})$ is extremally disconnected (see the preliminary remark preceding Theorem 6 in [8]).

Proof of sufficiency for $\alpha = 0$. If $f = \chi_E$ (the characteristic function of E) with $E \in \mathcal{S}$, let $g = \chi_U$, where U is a closed-

-and-open set that is equivalent to E. Then, g is continuous and equivalent to f. Thus, it easily follows that each simple function is equivalent to a continuous function.

Let f be a non-negative bounded $\mathcal{S}$-measurable function and let $\{f_n\}$ denote a non-decreasing sequence of simple functions converging to f. For each natural number n, let $g_n$ be a continuous function equivalent to $f_n$. Since

$$\{x: g_n(x) > \sup \{f(y): y \in X\}\} \subset \{x: g_n(x) > f(x)\} \subset \{x: g_n(x) > f_n(x)\},$$

and since the first of these sets is open while the last belongs to $\mathfrak{I}$, the first one must be empty, and so, the functions $g_n$ are uniformly bounded above. According to a theorem of Stone [7], if $(X, \mathcal{T})$ is an extremally disconnected topological space and if $(\mathcal{L}_o; \leqslant)$ is the lattice of continuous real-valued functions associated with $(X, \mathcal{T})$, then a non-void subset of $\mathcal{L}_o$ that has an upper bound in $(\mathcal{L}_o; \leqslant)$ has also a least upper bound there. Thus, $\{g_n\}$ has a least upper bound g in $(\mathcal{L}_o, \leqslant)$. From the method of choice of $g_n$ it follows that $\{x: g(x) < f(x)\} \in \mathfrak{I}$. We shall show that also $\{x: g(x) > f(x)\} \in \mathfrak{I}$. Thus, f and g will be equivalent. Let $\varepsilon > 0$ and let

$$E = \{x: g(x) \geqslant f(x) + \varepsilon\},$$

$$F_k = \{x: g(x) \geqslant g_k(x) + \varepsilon\}, \ k = 1, 2, \ldots,$$

$$F = \bigcap_{k=1}^{\infty} F_k.$$

We then have

$$E \setminus F = \bigcup_{k=1}^{\infty} (E \setminus F_k) \subset \bigcup_{k=1}^{\infty} \{x: f(x) < g_k(x)\} \subset \bigcup_{k=1}^{\infty} \{x: f_k(x) < g_k(x)\} \in \mathfrak{I}.$$

Let U be a closed-and-open set which is equivalent to F. Since F is closed, the set $U \setminus F$ is open. $U \setminus F$ belongs to $\mathfrak{I}$, so it must be empty. Consequently, $U \subset F$. Thus, the continuous function

$$h = g - \varepsilon \cdot \chi_U$$

is an upper bound of $\{g_n\}$ in $\{\mathcal{L}_o; \leqslant\}$, whence, for every x, we have $h(x) \geqslant g(x)$. Thus $h(x) = g(x)$ for all $x \in X$, and so, U is

REMARK. Probably, it is still not known whether, for each finite ordinal number $\alpha$, there is a topological measure space $(X, \mathcal{T}, \mathcal{S}, \mu)$ such that if $\mathcal{J} = \{A: \mu(A) = 0\}$, then $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$ is an $\alpha$-space (that problem was mentioned in [8]).

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### O UOGÓLNIONEJ KLASYFIKACJI ZINKA

**W artykule jest badana uogólniona klasyfikacja Zinka dla systemów $(X, \mathcal{T}, \mathcal{S}, \mathcal{J})$, gdzie $(X, \mathcal{T})$ jest przestrzenią topologiczną, zaś $\mathcal{J}$ jest $\sigma$-ideałem w $\sigma$-algebrze $\mathcal{S} \subset \mathcal{P}(X)$ takim, że $\mathcal{T} \setminus \{\emptyset\} \subset \mathcal{S} \setminus \mathcal{J}$. Uzyskano charakteryzację analogiczną do tej, którą podał Zink oraz omówiono kilka przykładów.**