A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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REEB STABILITY FOR NONSINGULAR FOLIATIONS DERIVED FROM THAT FOR SINGULAR ONES

In [4] the stability theorem for Stefan foliations ([6]) was proved. There was also proved that this theorem implies the Reeb Stability Theorem ([1], [2], [3], [5]). In the present paper we give a simpler and more elegant proof of the above-mentioned implication.

In [4] the notion of a holonomy of a leaf for an arbitrary Stefan foliation [6] was introduced. It was also asserted that if $\mathcal{F}$ is nonsingular, then the notion of the holonomy is the same as in [1], [2], [3] and [5]. We proved in [4] the stability theorem for Stefan foliations:

THEOREM 1. Let L be a compact leaf of a Stefan foliation $\mathcal{F}$. If the holonomy group of L is trivial, then, for each $x \in L$ there exist an adapted chart $\varphi : D_{\varphi} \to U_{\varphi} \times W_{\varphi}$ around x and an open neighbourhood V of L, such that $D_{\varphi} \subset V$ and the inclusion map $\mathcal{I}: D_{\varphi} \hookrightarrow V$ induces a homeomorphism of the quotient spaces

$$D_{\varphi}/(\mathcal{F}|D_{\varphi}) \cong V/(\mathcal{F}|V).$$

REMARK 1. It is easily seen from the construction of V that we can assume this set to be contained in an arbitrary open neighbourhood U of L.

In the case of nonsingular foliations, Theorem 1 implies the Reeb Stability Theorem. This implication was proved in the last chapter of [4]. The proof was still too complicated and, in fact, based on the construction in the proof of Theorem 1, not on the assertion of this theorem only.

The aim of the present paper is to give a simpler proof of the above-mentioned implication. The proof will be based on the assertion of Theorem 1 only.

We adopt all the notation of [4].

Let $\mathcal{F}$ be a nonsingular foliation. We prove the following version of the Reeb Stability Theorem:

THEOREM 2. If L is a compact leaf of $\mathcal{F}$ with the trivial holonomy group, then, for each open neighbourhood U of L, there exists an open saturated neighbourhood W of L such that $W \subset U$.

P r o o f. Let U be an arbitrary open neighbourhood of L. Let φ and V be an adapted chart, and an open neighbourhood of L, respectively, whose existence is assured by Theorem 1. Assume that $V \subset U$ (Remark 1).

Observe that there exists a natural homeomorphism $h\colon W_\varphi \to D_\varphi/ /(\mathcal{F}|D_\varphi)$ by the nonsingularity of $\mathcal{F}$. Let $g\colon = \hat{\mathcal{J}} \circ h$ where $\hat{\mathcal{J}}\colon D_\varphi/ /(\mathcal{F}|D_\varphi) \to V/(\mathcal{F}|V)$ is induced by $\mathcal{J}$. In virtue of Theorem 1, the mapping g is a homeomorphism.

Let V' be a relatively compact neighbourhood of L such that $\overline{V'} \subset V$. Since the boundary $\mathrm{fr}V'$ of this neighbourhood is compact $\pi_V(\mathrm{fr}V')$ is compact, too. Here $\pi_V\colon V \to V/(\mathcal{F}|V)$ is the canonical projection. Therefore $K\colon = g^{-1}\pi_V(\mathrm{fr}V')$ is a compact subset of W_φ.

Observe that $0 \notin K$. Indeed, if $0 \in K$, then $g(0) = L \in \pi_V(\mathrm{fr}V')$, which means that $L \cap \mathrm{fr}V' \neq \emptyset$ and this contradicts the inclusion $L \subset V'$.

Since π_V is an open mapping, $g^{-1}\pi_V(V')$ is an open neighbourhood of 0 in W_φ. Since W_φ is a normal space, there exist open neighbourhoods G and H of 0 and K, respectively, such that

$$G \subset g^{-1}\pi_V(V'), \qquad G \cap H = \emptyset \tag{1}$$

Let $W\colon = \pi_V^{-1}g(G)$. It is obvious that W is an open neighbourhood of L, saturated by leaves of $\mathcal{F}|V$. Moreover, $W \subset V \subset U$.

It remains to prove that W is saturated by leaves of $\mathcal{F}$.

Let $L' \in \mathcal{F}$ be a leaf intersecting W. Let $y \in L' \cap W$. Then $\pi_V(y) \in g(G)$, i.e. $(L' \cap V)_y = g(w)$ for some $w \in G$ (here A_Y denotes the connected component of the set A, containing y). Observe that $g(w) \in \pi_V(V')$ by (1). Thus there exists $v \in V'$ such that $g(w) = \pi_V(v)$. Since $y \in L'$, we have

$$\pi_V(y) = (L' \cap V)_y = \pi_V(v) \tag{2}$$

Therefore $v \in V' \cap L'$. Assume now that L' is not contained in V. Let $z \in L' \setminus V$. Let $c: \langle 0, 1\rangle \to L'$ be a curve joining v to z. Then c intersects $\mathrm{fr}V'$. Let $t_0 := \inf c^{-1}(\mathrm{fr}V')$. We have $t_0 \in c^{-1}(\mathrm{fr}V')$ and $t_0 > 0$ because $c^{-1}(\mathrm{fr}V')$ is closed in $\langle 0, 1\rangle$ and does not contain 0. Thus

$$x' := c(t_0) \in \mathrm{fr}V' \tag{3}$$

Obviously, $x' \in (L' \cap V)_v$ since it can be joined to v by the curve $c|\langle 0, t_0\rangle$ which lies in $L' \cap \overline{V'} \subset L' \cap V$. By (3),

$$g^{-1}\pi_V(x') \in K \subset H \tag{4}$$

On the other hand,

$$g^{-1}\pi_V(x') = g^{-1}\pi_V(y) \in G \tag{5}$$

by (2). Thus we have

$$g^{-1}\pi_V(x') \in G \cap H$$

by (4) and (5). This contradicts (1). So, $L' \subset V$.

Since W is saturated by leaves of $\mathcal{F}|V$, therefore $L' \subset W$. This completes the proof of our theorem.■

REMARK 2. It is obvious that the chart φ in the above proof can be chosen from some family of coherent charts with transversals given by a tubular neighbourhood (ξ, J) ([4]) of L in U. It can be assumed that if $\xi = (E, p, L)$, then $V \subset J(E) \subset U$.

REMARK 3. If the conditions of the previous remark hold, then we have ([4]):

(i) the set $\overline{L \cap D_\varphi}$ is connected and compact;

(ii) there exists an adapted chart $\tilde{\varphi}$ around x such that $D_{\tilde{\varphi}} \supset \supset D_\varphi$, $\overline{L \cap D_\varphi} \subset D_{\tilde{\varphi}}$, $\tilde{\varphi}|D_\varphi = \varphi$;

(iii) for each $y \in D_{\tilde{\varphi}} \cap L$, the inclusion $\tilde{\varphi}^{-1}(\{\mathrm{pr}_1\tilde{\varphi}(y)\} \times W_{\tilde{\varphi}}) \subset \subset p'^{-1}(y)$ holds, where $p' := JpJ^{-1}$.

REMARK 4. The neighbourhood V' in the above proof can be chosen in such a way that, in addition, for each $y \in L \cap D_\varphi$,

$$V' \cap p'^{-1}(y) \subset D_\varphi$$

holds. Indeed, let V'' be an arbitrary relatively compact neighbourhood of L contained in V. Since $\overline{L \cap D_\varphi}$ is compact by Remark 3, there exists an open neighbourhood G' of 0 in W_φ, such that $\tilde{\varphi}^{-1}(\overline{U_\varphi} \times G') \subset V''$. We can assume that $\overline{G'} \subset W_\varphi$. It is easy to check that the set $V' := (V'' \setminus p'^{-1}(\overline{L \cap D_\varphi})) \cup \tilde{\varphi}^{-1}(\overline{U_\varphi} \times G')$ is an open relatively compact neighbourhood of L, and that V' fulfils (6).

REMARK 5. Of course, we can assume that the set G in the proof of Theorem 2 is contained in G'.

REMARK 6. Since the set W is saturated, each leaf of $\mathcal{F}$ contained in W is a leaf of $\mathcal{F}|V$. Thus, by Theorem 1, for each leaf $L' \subset W$, the intersection $L' \cap D_\varphi$ is a plaque of φ.

Using the assertion of Theorem 1, we can also prove the following version of the Reeb Stability Theorem:

THEOREM 3. Let $\mathcal{F}$ be a nonsingular foliation of codimension q. If L is a compact leaf of $\mathcal{F}$ with the trivial holonomy group, then, for each neighbourhood U of L, there exist a saturated open neighbourhood W of L and a diffeomorphism $f: W \to L \times D$, such that $W \subset U$ and for any leaf $L' \subset W$, we have $f(L') = L \times \{w\}$ for some $w \in D$. Here D is an open neighbourhood of 0 in $\mathbb{R}^q$.

P r o o f. Let U be an arbitrary neighbourhood of L. Let (ξ, J) be a tubular neighbourhood of L such that $J(E) \subset U$ where $\xi = (E, p, L)$. By Theorem 1 and Remark 2, for each $x \in L$, there exist an adapted chart φ_x around x and a neighbourhood V_x of L, such that $D_{\varphi_x} \subset V_x \subset J(E)$ and the first inclusion induces a homeomorphism of the quotient spaces $D_{\varphi_x}/(\mathcal{F}|D_{\varphi_x}) \cong V_x/(\mathcal{F}|V_x)$. We assume that φ_x are chosen from some family of coherent charts with transversals given by (ξ, J) (Remark 2). Choose a finite subset $\{\varphi_1, \ldots, \varphi_r\}$ of $\{\varphi_x\}_{x \in L}$, such that $L \subset \bigcup_{i=1}^{r} D_{\varphi_i}$. Let V_i, $i = 1, \ldots, r$, be the respective neighbourhoods of L and let g_i, $i = 1, \ldots, r$, be the respective homeomorphism $W_{\varphi_i} \to V_i/(\mathcal{F}|V_i)$. For each $i \in \{1, \ldots, r\}$, choose a relatively compact neighbourhood V_i' of L, satisfying the conditions of Remark 4.

In the same way as in the proof of Theorem 2, we construct, for each $i \in \{1, \ldots, r\}$ a neighbourhood G_i of 0 in W_{φ_i} such that the assertion of Remark 5 holds. Let $W_i := \pi_{V_i}^{-1} g_i(G_i)$. The set W_i is a saturated open neighbourhood of L and each leaf contained in W_i intersects D_{φ_i} in the unique plaque by Remark 6.

The set W_i is contained in V_i'. Indeed, let $y \in W_i$. Then $\pi_{V_i}(y) \in g_i(G_i) \subset g_i(G_i')$ by Remark 5. Thus there exists $w \in G_i'$ such that $\pi_{V_i}(y) = \pi_{V_i}(\varphi_i^{-1}(0, w))$. Note that $\varphi_i^{-1}(0, w) \in V_i'$ by Remark 4. Therefore, the leaf through y intersects V_i'. Since the leaf is connected and does not intersect $\mathrm{fr} V_i'$, it must be contained in V_i'. In particular, $y \in V_i'$.

Thus we have the following implication:

$$\text{if } y \in D_{\varphi_i} \cap L, \text{ then } W_i \cap p'^{-1}(y) \subset V_i' \cap p'^{-1}(y) \subset D_{\varphi_i} \quad (7)$$

by Remark 4.

Put $W := \bigcap_{i=1}^{r} W_i$. Then W is a saturated open neighbourhood of L such that each leaf $L' \subset W$ intersects every D_{φ_i} in the unique plaque. Moreover, $W \subset \bigcup_{i=1}^{r} D_{\varphi_i}$. Indeed, let $z \in W$. Set $y := p'(z) \in L$. Since $L \subset \bigcup_{i=1}^{r} D_{\varphi_i}$, there exists $i_0 \in \{1, \ldots, r\}$ such that $y \in L \cap D_{\varphi_i}$. Then $z \in W \cap p'^{-1}(y) \subset W_{i_0} \cap p'^{-1}(y) \subset D_{\varphi_{i_0}}$ by (7).

Since W is saturated, the mapping $\psi_i := \varphi_i | W \cap D_{\varphi_i}$ is an adapted chart. The equality $W = \bigcup_{i=1}^{r} D_{\psi_i}$ holds and each leaf $L' \subset W$ intersects every D_{ψ_i} in the unique plaque.

By the assumption, for any $k, l \in \{1, \ldots, r\}$, there exists exactly one holonomy germ in $\mathcal{A}_{\psi_k, \psi_l} / \equiv$ ([4]). From the above facts it follows that this germ can be represented by a diffeomorphism f_{kl} which is defined on the whole set W_{ψ_k} and maps it into W_{ψ_l}.

Put $D: = g_1^{-1}(W/\mathcal{F}) = W_{\psi_1}$. Then D is an open neighbourhood of 0 in R^q. For each $z \in W$, set

$$f(z) = (p'(z),\ g_1^{-1}\pi_W(z)).$$

Thus $f(z) \in L \times D$.

The mapping f is continuous by its definition; f is also smooth since if $z \in D_{\psi_1}$ for some $l \in \{1, \ldots, r\}$, then

$$f(z) = (\psi_1^{-1}\ pr_1\psi_1(z),\ 0),\ \ f_{11}pr_2\psi_1(z)).$$

It is easy to show that the mapping F which sends $(y, w) \in L \times D$ to the only point z of intersection of $g_1(w)$ with $J(E_y) = p'^{-1}(y)$ is the inverse of f. The mapping F is also smooth since if $y \in D_{\psi_1}$, then

$$F(y, w) = \psi_1^{-1}(pr_1\psi_1(y),\ f_{11}(w)).$$

So, f is a diffeomorphism.

Let $L' \subset W$ be a leaf. It is obvious that $f(L') = L \times \{pr_2\psi_1(L')\}$. This completes the proof of the theorem. ■

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STABILNOŚĆ REEBA DLA FOLIACJI NIEOSOBLIWYCH
WNIOSKIEM ZE STABILNOŚCI DLA FOLIACJI OSOBLIWYCH

W pracy [4] zostało udowodnione twierdzenie o stabilności dla foliacji Stefana ([6]). Pokazano, że twierdzenie to implikuje twierdzenie Reeba o stabilności ([1], [2], [3], [5]). Dowód tego wynikania nie był jednak oparty na samej tezie twierdzenia o stabilności, a wykorzystywał konstrukcje zawarte w dowodzie. W prezentowanym artykule podajemy krótszy i bardziej elegancki, bo oparty na samej tezie, dowód wyżej wspomnianego wynikania.