A C T A   U N I V E R S I T A T I S   L O D Z I E N S I S
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ON GENERALIZATION OF SOME THEOREM OF DINI

The main aims of the paper are to give some conditions implying the monotonicity in the class of functions with the Darboux property and to generalize the following theorem:

THEOREM (DINI). If a function $f: I \to R$, where $I = \langle a, b \rangle \subset R$, is continuous, then the following conditions are satisfied:

$$(*) \begin{cases} \sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \quad x_1 \neq x_2\} = \\ \quad = \sup \{D^+f(x); \ x \in I\} = \sup \{D_+f(x); \ x \in I\} = \\ \quad = \sup \{D^-f(x); \ x \in I\} = \sup \{D_-f(x); \ x \in I\} \\ \inf \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \quad x_1 \neq x_2\} = \\ \quad = \inf \{D^+f(x); \ x \in I\} = \inf \{D_+f(x); \ x \in I\} = \\ \quad = \inf \{D^-f(x); \ x \in I\} = \inf \{D_-f(x); \ x \in I\}, \end{cases}$$

where

$D^+f(x)$ - the right-hand upper derivative of the function f at a point x,
$D_+f(x)$ - the right-hand lower derivative of the function f at a point x,
$D^-f(x)$ - the left-hand upper derivative of the function f at a point x,
$D_-f(x)$ - the left-hand lower derivative of the function f at a point x.

In the paper we shall use the following notations:

$\overline{D}f(x)$ - the upper derivative of the function f at a point x,
$\underline{D}f(x)$ - the lower derivative of the function f at a point x,
Q - the set of rational numbers,
$I = \langle a, b \rangle$,

$|I|$ - the length of the interval I,

D - the class of functions having the Darboux property in I,

B_1 - the family of functions of the first class of Baire in I,

DB_1 - the family of functions of the first class of Baire the Darboux property in I.

LEMMA 1. Let $f: I \to R$ satisfies conditions

(i) $\forall x \in \langle a, b) \limsup_{h \to 0^+} f(x + h) \leq f(x)$,

(ii) $\forall x \in (a, b\rangle\ D_-f(x) < 0$.

Then f is decreasing.

P r o o f. Assume that f is not decreasing in I. Then there exist points $c, d \in I$ such that $c < d$ and $f(c) \leq f(d)$. Two cases are possible: $f(c) < f(d)$ or $f(c) = f(d)$. Consider now the first case, i.e. $f(c) < f(d)$. Let

$$x_o = \inf \{x \in \langle c, d\rangle: f(x) \geq f(d)\}.$$

Then $c < x_o \leq d$ and $f(x_o) \geq f(d)$, which follows from the condition (i). Consequently,

$$\frac{f(x) - f(x_o)}{x - x_o} \geq 0 \quad \text{for} \quad x \in (c, x_o),$$

whence it follows that $D_-f(x_o) \geq 0$, which contradicts the assumption.

Let now $f(d) = f(c)$. The function f cannot be constant on the interval $\langle c, d\rangle$ since, by assumption, $D_-f(x) < 0$ for $x \in I$. So, there exists a point $x_1 \in (c, d)$ such that $f(x_1) \neq f(c) = f(d)$. But then we have two possibilities: $f(x_1) < f(d)$ or $f(c) < f(x_1)$ which reduce the question to the first case. Consequently, in both cases we come to contradicting the assumption which ends the proof of the lemma.

COROLLARY 1. Let $f: I \to R$ satisfies conditions

(i) $\forall x \in \langle a, b) \limsup_{h \to 0^+} f(x + h) \leq f(x)$,

(ii)' $\forall x \in (a, b\rangle\ D_-f(x) \leq 0$.

Then f is non-increasing.

P r o o f. Let $f_n(x) = f(x) - \frac{x}{n}$ for all $x \in I$. Then we have $D_-f_n(x) = D_-f(x) - \frac{1}{n} < 0$ for all $x \in (a, b\rangle$. By the Lemma 1 we have that for each $n \in N$, f_n is a decreasing function. Thus $f = \lim_{n\to\infty} f_n$ is non-increasing functions.

It is easy to prove the following theorem.

THEOREM 1. Let $f: R \to R$. Then, for any interval $I \subset R$, we have

$$\sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}, \ x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \sup \{\overline{D}f(x): x \in I\},$$

$$\inf \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}, \ x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \inf \{\underline{D}f(x): x \in I\}.$$

REMARK 1. Not for every function f,

$$\sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \sup \{\underline{D}f(x); \ x \in I\}.$$

Similarly, not for every function f,

$$\inf \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \inf \{\overline{D}f(x): x \in I\}.$$

This is testified by the following example: let

$$f(x) = \begin{cases} 0 & \text{for } x \in Q \\ 1 & \text{for } x \notin Q. \end{cases}$$

For this function,

$$\sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \sup \{\overline{D}f(x); \ x \in I\} = +\infty \neq \sup \{\underline{D}f(x), \ x \in I\} = -\infty$$

and

$$\inf \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\} =$$

$$= \inf \{\underline{D}f(x), x \in I\} = -\infty \neq \inf \{\overline{D}f(x), \ x \in I\} = +\infty.$$

THEOREM 2. Let $f: I \to R$ satisfies conditions

(iii) $\liminf_{h\to 0^+} f(x + h) \leq f(x) \leq \limsup_{h\to 0^+} f(x + h)$,

(iv) $\liminf_{h\to 0^+} f(x-h) \leqq f(x) \leqq \limsup_{h\to 0^+} f(x-h)$. Then

$$\sup\left\{\frac{f(x_1)-f(x_2)}{x_1-x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\right\} =$$

$$= \sup\{D^+f(x): x \in I\} = \sup\{D^-f(x): x \in I\}$$

and

$$\inf\left\{\frac{f(x_1)-f(x_2)}{x_1-x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\right\} =$$

$$= \inf\{D_+f(x): x \in I\} = \inf\{D_-f(x): x \in I\}.$$

P r o o f. We shall confine ourselves to proving the equality

$$\sup\left\{\frac{f(x_1)-f(x_2)}{x_1-x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\right\} =$$

$$= \sup\{D^+f(x): x \in I\}. \tag{1}$$

The remaining equalities of the assertion of the theorem can be proved in a similar way.

In order to prove (1), it suffices to show that

$$\sup\left\{\frac{f(x_1)-f(x_2)}{x_1-x_2}: x_1, x_2 \in I, \ x_1 \neq x_2\right\} \leqq$$

$$\leqq \sup\{D^+f(x): x \in I\} \tag{2}$$

since the opposite inequality is self-evident. So, let us take a number M such that, for some $x_1, x_2 \in I$ then is $\frac{f(x_1)-f(x_2)}{x_1-x_2} = M$

We shall show the existence of $x_o \in \langle x_1, x_2)$ such that $D^+f(x_o) \geqq M$. Let $g(x) = f(x) - Mx$. Then g satisfies the condition (iii), (iv). Suppose that, at each point of the interval $\langle x_1, x_2\rangle$, there is $D^+g(x) < 0$. Then by (55.9) [4], we have that the function g is non-increasing in the interval $\langle x_1, x_2\rangle$. Considering the fact that $g(x_1) = g(x_2)$, it follows that g is a constant function on $\langle x_1, x_2\rangle$. But then, $g'(x) = 0$ for $x \in (x_1, x_2)$, which contradicts the assumption that $D^+g(x) < 0$ for $x \in \langle x_1, x_2)$. Consequently, there exists $x_o \in \langle x_1, x_2)$ such that $D^+g(x_o) \geqq 0$, and whence we get $D^+f(x_o) \geqq M$, which yields (2) and completes the proof of the theorem.

DEFINITION 1. We shall say that, in some class $\mathcal{L}$ of functions defined in the interval I, the Dini theorem is trus if, for each function $f \in \mathcal{L}$ and each interval $I' \subset I$, relations (*) are satisfied by the function $f|I'$.

REMARK 2. Note that in the class D, and even in the class of approximatively continuous functions, the Dini theorem is not true because equalities (*) do not hold for the approximatively continuous function f, mentioned earlier, considered in the proof of Theorem 1 in paper [2].

THEOREM 3. If a real function f defined on I satisfied a condition

(i) $\limsup_{h\to 0^+} f(x + h) \leq f(x)$

or condition

$\liminf_{h\to 0^+} f(x + h) \geq f(x)$ then

$$\sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\} =$$
$$= \sup \{D_-f(x): x \in I\} = \sup \{D^-f(x): x \in I\}$$

or

$$\inf \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\} =$$
$$= \inf \{D_-f(x): x \in I\} = \inf \{D^-f(x): x \in I\},$$

respectively.

P r o o f. We shall limit ourselves to proving only the equality:

$$\sup \{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\} =$$
$$= \sup \{D_-f(x): x \in I\}.$$

For the purpose, it is enough to show that, for any number $M = \frac{f(x) - f(y)}{x - y}$, $x \neq y$, $x, y \in I$, there exists a point $\xi \in (x, y\rangle$ such that $D_-f(\xi) \geq M$. Suppose it is not so. Let $g(t) = f(t) - Mt$ for $t \in \langle x, y\rangle$. Then $D_-g(t) = D_-f(t) - M < 0$ for $t \in (x, y\rangle$. Since the function g satisfies the condition (i), therefore, in virtue of Lemma 1, it is monotone on $\langle x, y\rangle$; but, on account of the equality $g(x) = g(y)$, the function is constant on $\langle x\ y\rangle$, which is impossible because $D_-g(t) < 0$ for $t \in (x, y\rangle$, and this ends the proof of the theorem.

COROLLARY 2. If a real function f is defined on closed interval I and f is a right-hand continuous function on I then

$$\sup\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \sup\{D_-f(x): x \in I\} = \sup\{D^-f(x): x \in I\}$$

$$\inf\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \inf\{D_-f(x): x \in I\} = \inf\{D^-f(x): x \in I\}.$$

In analogously way we can prove the following theorem.

THEOREM 4. Let $f: I \to R$ satisfies a condition

$$\limsup_{h\to 0^+} f(x - h) \leq f(x) \quad \text{or} \quad \liminf_{h\to 0^+} f(x - h) \geq f(x)$$

then

$$\sup\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \sup\{D^+f(x): x \in I\} = \sup\{D_+f(x): x \in I\}$$

or

$$\inf\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \inf\{D^+f(x): x \in I\} = \inf\{D_+f(x): x \in I\},$$

respectively.

COROLLARY 3. Let $f: I \to R$ and f is a left-hand continuous on I then

$$\sup\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \sup\{D^+f(x): x \in I\} = \sup\{D_+f(x): x \in I\},$$

$$\inf\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} =$$

$$= \inf\{D^+f(x): x \in I\} = \inf\{D_+f(x): x \in I\}.$$

Note that the Dini theorem is a conclusion from the above corollaries.

DEFINITION 2. If a function f is defined in I, then $\xi \in I$ will be called the point at which f attains its right-hand local minimum, if there exists $\delta > 0$ such that $f(x) \geqq f(\xi)$ for $x \in (\xi, \xi + \delta)$. The set of all points of the interval I at which the function f attains its right-hand local minimum will be denoted by $E^+(f)$. In an analogous manner we define the points at which the function attains its right-hand local maximum, left-hand local maximum, and the sets of all such points will be denoted by $E_+(f)$, $E^-(f)$, $E_-(f)$, respectively.

THEOREM 5. A function f defined on the interval I satisfies the Dini theorem if and only if, for each closed interval $I' = \langle x, y\rangle \subset I$, $x < y$, and for a function $g(t) = f(t) - \frac{f(x) - f(y)}{x - y} t$, $t \in I'$, each of the sets $E^+(g)$, $E_+(g)$, $E^-(g)$ and $E_-(g)$ is non-empty.

P r o o f. To prove sufficiency, it is enough to confine oneself to showing the equality:

$$\sup \{\frac{f(x) - f(y)}{x - y}, x, y \in I, \ x \neq y\} = \sup \{D_+f(x): x \in I\} \quad (3)$$

In order to show (5), it is sufficient to prove that, for any number $M = \frac{f(x) - f(y)}{x - y}$, $x, y \in I$, $x < y$, there exists a point $\xi \in \langle x, y)$ such that $D_+f(\xi) \geqq M$. Suppose it is not so. Then, for the function $g(t) = f(t) - Mt$ defined for $t \in \langle x, y\rangle$, we have $D_+g(\xi) < 0$ for $\xi \in \langle x, y)$. On the other hand, by assumption, there exists a point $\xi_1 \in E^+(g)$ at which the function g attains its right-hand local minimum. But then $D_+g(\xi_1) \geqq 0$, which yields a contradiction and concludes the proof of (3).

The proof of necessity will also be confined to proving that, for the interval $I' = \langle x, y\rangle \subset I$, $x < y$, and the function $g(t) = f(t) - \frac{f(x) - f(y)}{x - y} t$, $(t \in I')$, the set $E^+(g)$ is non-empty.

By assumption, we have (3). So, let us take $I^* = \langle x, y\rangle$ and $M = \frac{f(x) - f(y)}{x - y}$, $x < y$. Then, either $M = \tau \overset{df}{=} \sup \{\frac{f(z) - f(u)}{z - u}$, $z, u \in I'$, $z \neq u\}$ or $M < \tau$. If $M = \tau$, then f is a linear function on I' and $f(t) = M(t - x) + f(x)$ for $t \in I'$, and then, the function $g(t) = f(t) - Mt$ is a constant function, and thus, $E^{+}(g) = \langle x, y\rangle \neq \emptyset$. Whereas if $M < \tau$, there exists $M' = \frac{f(x_1) - f(y_1)}{x_1 - y_1}$, $x_1, y_1 \in I'$, such that $M' > M$, but then, from the analogue of (5) for I' it follows that there exists a point $\eta \in I$ such that $D_{+}f(\eta) > M$; but then, the point $\eta \in E^{+}(g)$, which ends the proof.

COROLLARY 4. If a function f satisfies the Dini theorem on I, then, on each interval $I' \subset I$, the sets $E^{+}(f)$ and $E_{-}(f)$ are non-empty or the sets $E_{+}(f)$ and $E^{-}(f)$ are non-empty.

P r o o f. To begin with, let us consider the case when there exist point $x, y \in I$, $x < y$, such that

$$M = \frac{f(y) - f(x)}{y - x} > 0.$$

From assumption of Corollary 4 as well as from Theorem 5 we have that there exists a point $\xi \in E^{+}(g)$ where $g(t) = f(t) - Mt$ for $t \in \langle x, y\rangle$. Then $D_{+}g(\xi) \geq 0$, and so, $D_{+}f(\xi) > 0$. But then, $\xi \in E^{+}(f)$. In an analogous manner one can prove that, in this case, the set $E_{-}(f)$ is non-empty.

If now, for any two points $x, y \in I$, $x < y$, the difference quotient $M = \frac{f(y) - f(x)}{y - x} \leq 0$, then the function f is non-increasing in I, and thus, $E_{+}(f) = \langle a, b) \neq \emptyset$ and $E^{-}(f) = (a, b\rangle \neq \emptyset$, which completes the proof.

Similarly, by making use of Corollaries 2 and 3 as well as Theorem 5, one can prove:

COROLLARY 5. For each function f right-hand continuous in I, the sets $E^{-}(f)$ and $E_{-}(f)$ are non-empty.

COROLLARY 6. For each function f left-hand continuous in I, the sets $E^{+}(f)$ and $E_{+}(f)$ are non-empty.

REMARK 3. Note that continuous functions satisfy the assumptions of the sufficient condition from Theorem 5. However, there exist discontinuous functions that also satisfy the above assumptions of Theorem 5, and thus, they satisfy the Dini theorem. An example of such a function in, for instance, the function

$$f(x) = \begin{cases} 0 & \text{for } x = 0 \\ \sin\frac{1}{x} & \text{for } x \in \langle -1, 0) \cup (0, 1\rangle. \end{cases}$$

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O UOGÓLNIENIU PEWNEGO TWIERDZENIA DINIEGO

W prezentowanym artykule jest podane uogólnienie następującego twierdzenia Diniego: jeśli funkcja $f: I \to R$, gdzie $I = \langle a, b) \subset R$, jest ciągła, wtedy są spełnione następujące warunki:

$$\sup\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} = \sup\{D^+f(x): x \in I\} =$$

$$= \sup\{D_+f(x): x \in I\} = \sup\{D^-f(x): x \in I\} = \sup\{D_-f(x): x \in I\}$$

$$\inf\left\{\frac{f(x_1) - f(x_2)}{x_1 - x_2}: x_1, x_2 \in I, x_1 \neq x_2\right\} = \inf\{D^+f(x): x \in I\} =$$

$$= \inf\{D_+f(x): x \in I\} = \inf\{D^-f(x): x \in I\} = \inf\{D_-f(x): x \in I\}.$$