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ON THE PROBABILISTIC PROPERTIES OF THE SOLUTION OF RANDOM INTEGRAL EQUATIONS

In this work, random variables ξ_N, $\xi_N(x)$ on the probability space generated by trajectories of the Markov chain are defined; it is shown that expected values $E\xi_N$, $E\xi_N(x)$ are asymptotically unbiased estimators of the product $(f, E\xi)$ and of the solution $f(x)$ of a random integral equation of the form: $f(x)=E\eta(x)+\int E\gamma(x,y)f(x-y)\mu(dy)$ in space $L^p(0,\infty)$ $(p\geq 1)$ respectively. Unbiased estimators and some related problems are also investigated.

1. Introduction

Let $\xi(x) = \xi(x; u)$; $\eta(x) = \eta(x; u)$, $\gamma(x, y) = \gamma(x, y; u)$ be random real processes on a probability space (U, Σ, P) with $x \in (0, \infty)$ and $y \in (0, \infty)$. Suppose that:

(A) For each (p, q) satisfying $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

$$E|\eta(\cdot)| = \int_U |\eta(\cdot; u)| \, P(du) \in L^p(0, \infty) \tag{1.1}$$

$$E|\xi(\cdot)| = \int_U |\xi(\cdot; u)| \, P(du) \in L^q(0,\infty) \tag{1.2}$$

(B) The applications τ and $\overline{\tau}$ defined by the formulas:

$$[\tau f](x) = \int_0^x E\gamma(x, y) \, f(x - y) \, \mu(dy) \tag{1.3}$$

$$[\overline{\tau} f](x) = \int_0^x E|\gamma(x, y)| \, f(x - y) \, \mu(dy) \tag{1.4}$$

are integral operators in $L^p(0, \infty)$, where μ - the Lebesgue measure on R^1.

(C) For each $f = L^p(0,\infty)$, the Neumann series $\sum_{n=0}^{\infty} \bar{\tau}^{\eta} f$ converges in $L^p(0,\infty)$.

In the work, we shall solve the following integral equation:

$$(1.5) \qquad f(x) = E\eta(x) + \int_0^x E\gamma(x, y)\, f(x - y)\,\mu(dy),$$

$$\forall x \in (0, \infty) \pmod{\mu}$$

and compute the value of the scalar product

$$(1.6) \qquad (f, E\xi) = \int_0^x f(x) E\xi(x)\,\mu(dx)$$

where f is the solution of equation (1.5) in $L^p(0,\infty)$.

It is well known that if for each $(x, y) \in (0,\infty) \times (0, \infty) \equiv R_+^2$. The mathematical means $E\eta(x)$, $E\gamma(x, y)$ and $E\xi(x)$ are given, these problems have been solved in the very general forms (see [3], [4], [10], [11]. However, in fact (for example, in many problems of the renewal theory) the above mentioned mathematical means are not known in general because the probability distribution P is not known. On the other hand, we can only observe the independent samples of random variables $\eta(x)$, $\gamma(x, y)$ and $\xi(x)$. We shall use these samples to estimate the solution $f(x)$ and the value of functional (1.6) and it is just the basic difference between the results in this work and the published results for example in [3],[4], [10], [11], [12].

In numeral analysis, we usually consider the integral equation of Wiener-Hopf form (see [12], pp. 249-273)

$$(1.5^*) \qquad f(x) = g(x) + \int_0^x k(x, y)\, f(x - y)\,\mu(dy), \quad \forall x \in (0, \infty)$$

where $g(x)$, $k(x, y)$ are given. However, in many cases, we know only the approximate values $\eta(x)$ and $\gamma(x, y)$ of functions $g(x)$ and $k(x, y)$ and do not know errors .

$$\varepsilon_\eta(x) = g(x) - \eta(x)$$

$$\varepsilon_\gamma(x, y) = k(x, y) - \gamma(x, y)$$

that can be considered as unsystematic errors, i.e. $E\varepsilon_\eta = E\varepsilon_\gamma = 0$. Then, equation (1.5^*) has the form (1.5). Hence, the results in the paper can be used to solve problems of numeral analysis, in which initial data have unsystematic random errors.

In fact, we also need estimate values $x^* \in (0, \infty)$ optimal in the following sense

$$(1.7) \qquad f(x^* + \delta) - f(x^*) = \max_{0<x<\infty} [f(x + \delta) - f(x)]$$

or

$$(1.8) \qquad \int_0^{x^*} E\xi(x^*, y) f(y) \mu(dy) = \max_{x>0} \int_0^{x} E\xi(x, y) f(y) \mu(dy)$$

where f is the solution of equation (1.5).

The probability models are also constructed to solve these problems in the work.

2. Probability models for the solution of integral equation (1.5) and for the evaluation of function (1.6)

Suppose that $\{\varphi_0, \varphi\}$ is a homogeneous Markov chain in the phase space R^1_+, in which $\varphi_0(x)$ is the initial probability density, $\varphi(x, y)$ is the transition probability density, such that

$$(2.1) \qquad \varphi_0(x) > 0, \quad \forall x \in [0,\infty), \quad \int_0^\infty \varphi_0(x)\mu(dx) = 1$$

$$(2.2) \qquad \varphi(x, y) > 0, \quad \forall(x, y) \in [0,\infty) \times [0,\infty),$$

$$\int_0^\infty \varphi(x, y)\ \mu(dy) = 1,$$

$$\forall x \in [0, \infty)$$

Let $(R_+^{N+1}, \mathcal{B}_+^{N+1}, P_{N+1})$ be the probability space generated by the first segments with fixed length N of all trajectories of the Markov chain, i.e. $\mathcal{B}_+^{N+1}$ is a σ-Borel field on R_+^{N+1} and for each $A \in \mathcal{B}_+^{N+1}$, then

(2.3) $$P_{N+1}(A) =$$

$$= \int_A \varphi_0(x_0)\varphi(x_0, x_1) \dots \varphi(x_{N-1}, x_N)\ \mu(dx_0) \dots \mu(dx_N) =$$

$$= \int_A \varphi_0(x_0)\ \varphi(x_0, x_1) \dots \varphi(x_{N-1}, x_N)\mu^{N+1}\left(\prod_{i=0}^{N} dx_i\right)$$

Put

(2.4) $$\Omega_N = R_+^{N+1} \times U^{N+2}, \quad \mathcal{C}_N = \mathcal{B}_+^{N+1} \times \Sigma^{N+2},$$

$$\lambda_N = P_{N+1} \times P^{N+2}$$

where (U^N, Σ^N, P^N) is the product of n initial probability spaces (U, Σ, P) and we define a random value d_N on the probability space $(R_+^{N+1}, \mathcal{B}_+^{N+1}, P_{N+1})$ as follows:

(2.5) $$d_N = \begin{cases} i, \text{ if } x_0 \geq x_1 \geq \dots \geq x_i\ (0 \leq i < N) \text{ and } x_i < x_{i+1} \\ N, \text{ if } x_0 \geq x_1 \geq \dots \geq x_N \end{cases}$$

Suppose that.

(D) For each trajectory of the Markov chain $x_0 \to x_1 \to \dots$ and for each $i = 1, 2, \dots d_N$, $\xi(x_0; u_0)$, $\{\gamma(x_k, x_k - x_{k+1}; u_{k+1})\}_{k=0}^{i-1}$, $\eta(x_i; u_{N+1})$ are independent samples of random values $\xi(x_0)$, $\{\gamma(x_k, x_k - x_{k+1})\}_{k=0}^{i-1}$, $\eta(x_i)$.

We define the following random variables on $(\Omega_N, \mathcal{C}_N, \lambda_N)$:

(2.6) $$\xi_N \equiv \xi_N(x_0, x_1, \dots, x_N; u_0, u_1, \dots, u_{N+1}) \equiv$$

$$\equiv \sum_{i=0}^{d_N} \frac{\xi(x_0; u_0)\, \gamma(x_0, x_0 - x_1; u_1), \dots, \gamma(x_{i-1}, x_{i-1} - x_i; u_i)}{\varphi_0(x_0)\ \ \varphi(x_0, x_1) \dots \varphi(x_{i-1}, x_i)} \times$$

$$\times\, \eta(x_i; u_{N+1})$$

(2.7) $$\xi_N(x) \equiv \xi_N(x; x_0, x_1, \dots, x_N; u_0, u_1, \dots, u_{N+1}) \equiv$$

$$\equiv \sum_{i=0}^{d_N} \frac{\bar{\gamma}(x, x_0; u_0)\ \bar{\gamma}(x_0, x_1; u_1), \dots, \bar{\gamma}(x_{i-1}, x_i; u_i)}{\varphi_0(x_0)\, \varphi(x_0, x_1) \dots \varphi(x_{i-1}, x_i)} \times$$

$$\times\, \eta(x_i; u_{N+1})$$

with $x \in [0,\infty)$,
where

$$\bar{\gamma}(x, t; u) = \begin{cases} \gamma(x, x-t; u), & \text{if } 0 \leq t \leq x, \\ 0, & \text{if } 0 \leq x < t \end{cases}$$

(2.8)

$$\varphi(x_{-1}, x_0) = \gamma(x_{-1}, x_{-1} - x_0; u_0) \equiv 1$$

<u>Theorem 2.1.</u> If conditions (A)-(D) are satisfied, we have

1. $\xi_N \in L^1(\Omega_N, \lambda_N)$, $\xi_N(x) \in L^1(\Omega_N, \lambda_N)$ for each $x \in [0,\infty)$ and

(2.9) $$\lim_{N\to\infty} E_{\lambda_N} \xi_N = (f, E\xi)$$

(2.10) $$\lim_{N\to\infty} E_{\lambda_N} \xi_N(x) = f(x); \quad \forall x \in [0,\infty)$$

where f is the unique solution of equation (1.5) and

$$E_{\lambda}\xi \equiv \int_{\Omega} \xi(\omega)\lambda(d\omega), \qquad \Theta_N(x) = \eta(x) + \xi_N(x)$$

2. There exists a natural number n_o and a positive number δ_o, such that

(2.11) $$\|C^n\| \leq \delta_o < 1, \qquad \forall n \geq n_o$$

(2.12) $$|(f, E\xi) - E_{\lambda_N}\xi_N| \leq$$

$$\leq \frac{\delta_o^{N+1}}{1-\delta_o} \|E\eta\|_{L^p} \|E\xi\|_{L^q},$$

$$\forall N \geq n_o$$

(2.13) $$|f(x) - E_{\lambda_N}\theta_N(x)| \leq$$

$$\leq \frac{\delta_o^{N+1}}{1-\delta_o} \|E\eta\|_{L^p} \|E\xi_x\|_{L^q},$$

$$\forall N \geq n_o$$

where

(2.14) $$\xi_x(t; u) = \tilde{\gamma}(x, t; u)$$

P r o o f. Let P_o and $P_n^{(x)}$ be the probability measures on R_+ and R_+ defined by the formulas:

(2.15)
$$\begin{cases} P_o(A) = \int_A \varphi_o(x)\,\mu(dx), & \forall A \in B_+ \\ P_n^{(x)}(B) = \int_B \varphi(x; x_1)\,\varphi(x_1, x_2)\ldots\varphi(x_{n-1}, x_n)\,\mu^n\left(\prod_{i=1}^{n} dx_i\right), \end{cases}$$

$$B \in \mathcal{B}_+^n$$

It is easy to see that $P_0 \ll \mu$, $P_n^{(x)} \ll \mu^n$. First consider $i = 0$. From (A), (2.15) and by Fubini's theorem, we have

$$(2.16)\quad \int_{\Omega_N} \left| \frac{\xi(x_0;\, u_0)\,\eta(x_0;\, u_{N+1})}{\varphi_0(x_0)} \right| \lambda_N \left(\prod_{i=0}^{N} dx_i \times \prod_{i=0}^{N+1} du_i \right) =$$

$$= \int_{R_+ \times U^2} \left\{ \left| \frac{\xi(x_0;\, u_0)\,\eta(x_0;\, u_{+1})}{\varphi_0(x_0)} \right| \int_{R_+^N \times U^N} P_N^{(x_0)} \times \right.$$

$$\left. \times P^N \left(\prod_{i=1}^{N} dx_i \times \prod_{i=0}^{N} du_i \right) \right\} P_0 \times$$

$$\times P^2(dx_0 \times du_0 \times du_{N+1}) =$$

$$= \int_{R_+ \times U^2} |\xi(x_0;\, u_0)\,\eta(x_0;\, u_{N+1})|\, \mu \times P^2(dx_0 \times du_0 \times du_{N+1}) =$$

$$= \int_{R_+} E\, |\xi\,(x_0)|\, E\, |\eta\,(x_0)|\, \mu\,(dx_0) < \infty$$

Consider now the case $i \geq 1$. Since μ is the σ-finite measure, from (A), (B) and by Tonelli's theorem (see [2], pp. 341-342) we obtain

$$(2.17)\quad \int_{R_+^{i+1}} E|\xi(x_0)| \prod_{j=1}^{i} E\, |\gamma(x_{j-1},\, x_{j-1} - x_j) \times$$

$$\times |E|\eta(x_i)|\, \mu^{i+1} \left(\prod_{j=0}^{i} dx_j \right) =$$

$$= (\bar{\tau}^i E|\eta|,\ E|\xi|) < \infty$$

From (2.3), (2.4), (2.15), (2.17) and (D) we have

$$(2.18)\quad \int_{\lambda_N} \left| \frac{\xi(x_0;\, u_0)\,\bar{\gamma}(x_0,\, x_1;\, u_1) \ldots \bar{\gamma}(x_{i-1},\, x_i;\, u_i)}{\varphi_0(x_0)\, \varphi(x_0,\, x_1) \ldots \varphi(x_{i-1},\, x_i)} \times \right.$$

$$\times\ \eta(x_i;\ u_{N+1})|\ \lambda_N\ (\prod_{j=0}^{N} dx_j \times \prod_{j=0}^{N+1} du_j) =$$

$$= \int_{R_+^{i+1}} \Big\{ \int_{U^{i+2}} \Big| \frac{\xi(x_0;\ u_0)\ \bar{\gamma}(x_0,\ x_1;\ u_1)\ \ldots\ \bar{\gamma}(x_{i-1},\ x_i;\ u_i)}{\varphi_0(x_0)\ \varphi(x_0,\ x_1)\ \ldots\ \varphi(x_{i-1},\ x_i)} \times$$

$$\times\ \eta(x_i;\ u_{N+1})|\ \times$$

$$\times\ p^{i+2}\ (\prod_{j=0}^{i} du_j \times du_{N+1})\Big\}\ P_{i+1}\ (\prod_{j=0}^{i} dx_j) =$$

$$= \int_{R_+^{i+1}} \frac{E|\xi(x_0)|E|\bar{\gamma}(x_0,\ x_1)|\ \ldots\ E|\bar{\gamma}(x_{i-1},\ x_i)|E|\eta(x_i)|}{\varphi_0(x_0)\ \varphi(x_0,\ x_1)\ \ldots\ \varphi(x_{i-1},\ x_i)} \times$$

$$\times\ P_{i+1}(\prod_{j=0}^{i} dx_j) =$$

$$= \int_{R_+^{i+1}} E|\xi(x_0)|E|\bar{\gamma}(x_0,\ x_1)|\ldots E|\bar{\gamma}(x_{i-1},\ x_i)\ |E|\eta(x_i)|\mu^{i+1}(\prod_{j=0}^{i} dx_j) < \infty$$

Thus $\xi_N \in L^1(\Omega_N,\ \lambda_N)$. Applying Fubini's theorem we obtain

(2.19) $$E_{\lambda_N} \xi_N =$$

$$= \sum_{i=0}^{N} \int_{R_+^{N+1}} \frac{E\xi(x_0)E\ \bar{\gamma}\ (x_0, x_1)\ldots\ E\bar{\gamma}(x_{i-1}, x_i)}{\varphi_0(x_0)\ \varphi(x_0,\ x_1)\ldots\varphi(x_{i-1}, x_i)} \times$$

$$\times\ E\eta\ (x_i)P_{i+1}(\prod_{j=0}^{i} dx_j) =$$

$$= \sum_{i=0}^{N} \int_{R_+^{i+1}} E\xi(x_0)\ E\bar{\gamma}(x_0,\ x_1)\ \ldots\ E\bar{\gamma}(x_{i-1},\ x_i)\ E\eta(x_i)\mu^{i+1}(\prod_{j=0}^{i} dx_j) =$$

$$= \sum_{i=0}^{N} (\tau^i E\eta, E\xi) = (\sum_{i=0}^{N} \tau^i E\eta, E\xi)$$

From (C) it follows that equation (1.5) has the unique solution $f = \sum_{i=0}^{\infty} \tau^i E\eta \in L^p(0, \infty)$. So (2.19) implies (2.9).

From (C) we also deduce that (see [8], p. 153-154)there exists a natural number n_0 such that, for each $n \geqq n_0$, we have

$$\sqrt[n]{\|\tau^n\|} \leqq \delta_0 \equiv \sqrt[k]{\|\tau^k\|} < 1, \qquad k \leqq n_0$$

it is (2.11). Hence, for each $n \geqq n_0$, $\|\tau^n\| \leq \delta_0^n$ and by (2.19) we have

$$|(f, E\xi) - E_{\lambda_N}\xi_N| = |(\sum_{i=N+1}^{\infty} \tau E\eta, E\xi)| \leqq$$

$$\leqq \sum_{i=N+1}^{\infty} \|\tau^i\| \, \|E\eta\|_{L^p} \|E\xi\|_{L^q} \leqq \frac{\delta_0^{N+1}}{1-\delta_0} \|E\eta\|_{L^p} \|E\xi\|_{L^q}$$

for all $N \geqq n$; it is (2.12).

Notice that, from (2.14), (1.4) and condiction (B), we have (see [1], p. 380).

(2.20) $$E\xi_x(t) \in L^q(0,\infty) \text{ for each } x \in [0, \infty) (\text{mod}\,\mu)$$

and equation (1.5) can be written in the form

(2.21) $$f(x) = E\eta(x) + (f(\cdot), E\xi_x(\cdot))$$

It is clear that $\theta_N(x) \in L^1(\Omega_N, \lambda_N)$. Moreover

(2.22) $$E_{\lambda_N}\theta_N(x) = E_{\lambda_N}\eta(x) + E_{\lambda_N}\xi_N(x) = E\eta(x) + E_{\lambda_N}\xi_N(x)$$

From (2.9), (2.21), (2.22) we obtain

$$(2.23)\qquad \lim_{N\to\infty} E_{\lambda_N}\theta_N(x) = E\eta(x) + \lim_{N\to\infty} E_{\lambda_N}\xi_N(x) =$$

$$= E\eta(x) + (f(\cdot), E\xi_x(\cdot)) = f(x)$$

i.e. we obtain (2.10) and

$$|f(x) - E_{\lambda_N}\theta_N(x)| = |(f(\cdot), E\xi_x(\cdot)) - E_{\lambda_N}\xi_N(x)|$$

Applying (2.12), we complete the proof.

We now present unbiased estimators of the solution of random integral equation (1.5) and of the value of scalar product (1.6).

Denote by $(R_+^\infty, \mathcal{B}_+^\infty, P_\infty)$ the probability space generated by all the trajectories of the Markov chain $\{\varphi_0, \varphi\}$ i.e. if

$$(2.24)\qquad A_{B^{(n+1)}} = \{\tilde{x} = (x_0, x_1, \ldots) \in R_+^\infty : (x_0, x_1, \ldots, x_n) \in$$

$$\in B^{(n+1)}, B^{(n+1)} \in \mathcal{B}_+^{n+1}\}$$

then

$$(2.25)\qquad P_\infty(A_{B^{(n+1)}}) = \int_{B^{(n+1)}} \varphi_0(t_0)\varphi(t_0, t_1)\ldots\varphi(t_{n-1}, t_n) \times$$

$$\times \mu^{(n+1)}\left(\prod_{j=0}^{n} dt_j\right) \equiv P_{n+1}(B^{(n+1)})$$

We define the following functions on the space $(R_+^\infty, \mathcal{B}_+^\infty, P_\infty)$:

$$(2.26)\qquad d(\tilde{x}) = \begin{cases} n, & \text{if } x_0 \geq \ldots \geq x_n \text{ and } x_n < x_{n+1} \\ \infty, & \text{if } x_n \geq x_{n+1},\quad n = 0, 1, \ldots, \end{cases}$$

$$(2.27)\qquad n_x(\tilde{x}) = \begin{cases} 0, & \text{if } x_0 > x \\ n, & \text{if } x_i \leq x,\quad i = 0, 1, \ldots, n-1 \text{ and } x_n > x \\ \infty, & \text{if } x_i \leq x,\quad i = 0, 1, 2, \ldots \end{cases}$$

<u>Lemma 2.1</u> Suppose that φ_0, φ satisfy (2.1), (2.2) and

$$(2.28)\qquad \operatorname*{Vrai\,sup}_{0\leq t\leq x} \int_0^x \varphi(t, y)(dy) < 1, \quad \forall x \in R_+$$

Then, for each $x > 0$, we have

$$P_{\infty}\left\{\tilde{x} \in R_{+}^{\infty} : n_x(\tilde{x}) = \infty\right\} = 0 \tag{2.29}$$

i.e. $n_x(\cdot)$ is a random variable defined on $(R_{+}^{\infty}, \mathcal{B}_{+}^{\infty}, P_{\infty})$.

Moreover

$$E_{\infty}\left\{n_x(\cdot)\right\} \equiv \int_{R_{+}^{\infty}} n_x(\tilde{x}) P_{\infty}(d\tilde{x}) < \infty \tag{2.30}$$

P r o o f. Denote by Φ_x the integral operator defined by the formula:

$$[\Phi_x g](t) = \int_0^x \varphi(y, t) g(y) \mu(dy), \quad g \in L^1(0, x) \tag{2.31}$$

From (2.28) and (2.2) it follows that Φ_x is the integral operator in the space $L^1(0, x)$ and (see [4])

$$\|\Phi_x\| \equiv \operatorname{Vrai\,sup}_{0 \leq t \leq x} \int_0^x \varphi(t, y) \mu(dy) < 1 \tag{2.32}$$

so the Neumann series of the equation

$$g = \varphi_0 + \Phi_x g \tag{2.33}$$

converges in $L^1(0, x)$ and we obtain the unique solution of equation (2.33)

$$g = \sum_{n=0}^{\infty} \Phi_x^n \varphi_0 \tag{2.34}$$

For each $n \geq 2$, from (2.24) and (2.25) we have

$$P_{\infty}\left\{\tilde{x} \in R_{+}^{\infty} : n_x(\tilde{x}) = n\right\} = P_{\infty}\left\{\tilde{x} \in R_{+}^{\infty} : x_0 \leq x, \ldots, x_{n-1} \leq\right.$$

$$\left.\leq x,\ x_n > x\right\} \leq P_{\infty}\left\{\tilde{x} \in R_{+}^{\infty} : x_0 \leq x, \ldots, x_{n-1} \leq x\right\} =$$

$$= \int_0^x \cdots \int_0^x \varphi_0(t_0) \varphi(t_0, t_1) \ldots \varphi(t_{n-2}, t_{n-1}) \mu^n\left(\prod_{i=0}^{n-1} dt_i\right) =$$

$$= (\Phi_x^{n-1} \varphi_0, 1)$$

Therefore, for each $N \geqq 2$,

$$P_\infty\left\{\tilde{x} \in R_+^\infty : n_x(\tilde{x}) \geqslant N\right\} = \sum_{n=N}^\infty P_\infty\left\{\tilde{x} \in R_+^\infty : n_x(\tilde{x}) = n\right\} \leqq$$

$$\leqq \sum_{n=N}^\infty (\Phi_x^{n-1} \varphi_0, 1) = (\sum_{n=N}^\infty \Phi_x^{n-1} \varphi_0, 1).$$

By (2.33) we have $\lim\limits_{N\to\infty} \sum\limits_{n=N}^\infty (\Phi_x^{n-1} \varphi_0, 1) = 0$, so

$$\lim_{N\to\infty} P_\infty\left\{n_x(\tilde{x}) \geqq N\right\} = 0 \tag{2.35}$$

On the other hand, since

$$\left\{\tilde{x} \in R_+^\infty : n_x(\tilde{x}) = \infty\right\} \subset \left\{x \in R_+^\infty : n_x(\tilde{x}) \geqq N\right\}, \quad N = 2, 3, \ldots,$$

(2.35) we deduce (2.29). Moreover,

$$E_\infty\left\{n_x(\cdot)\right\} = \sum_{n=1}^\infty n P_\infty\left\{n_x(\tilde{x}) = n\right\} \leq$$

$$\leq \sum_{n=1}^\infty n(\Phi_x^{n-1} \varphi_0, 1) = (\sum_{n=1}^\infty n\Phi_x^{n-1} \varphi_0, 1) =$$

$$= (\sum_{n=1}^\infty \sum_{k=n-1}^\infty \Phi_x^k \varphi_0, 1) = (\sum_{n=1}^\infty \Phi_x^{n-1} g, 1) < \infty$$

because $\|\Phi_x\| < 1$ and $\sum\limits_{n=1}^\infty \Phi_x^{n-1} g \in L^1(0, x)$. This completes the proof.

Lemma 2.2. Under the assumptions of Lemma 2.1, we have

$$P_\infty\left\{\tilde{x} \in R_+^\infty : d(\tilde{x}) = \infty\right\} = 0 \tag{2.36}$$

i.e. $d(\cdot)$ is a random variable defined on $(R_+^\infty, \mathcal{B}_+^\infty, P_\infty)$.

P r o o f. From (2.29) we have

$$P_\infty\left\{\tilde{x} \in R_+^\infty : n_k(\tilde{x}) = \infty\right\} = 0, \quad k = 1, 2, \ldots$$

Denote

$$\Omega_\infty = \bigcup_{k=1}^\infty \left\{\tilde{x} \in R_+^\infty : n_k(\tilde{x}) = \infty\right\} \tag{2.37}$$

$$\Omega_d = R_+^\infty \setminus \Omega_\infty \tag{2.38}$$

It is easy to see that

(2.39) $$P_\infty(\Omega_\infty) = 0$$

(2.40) $$P_\infty(\Omega_d) = 1$$

Suppose that $\tilde{x} = (x_0, x_1, \ldots) \in \Omega_d$. If $d(\tilde{x}) = \infty$, then $x_n \geq x_{n+1}$, $n = 0, 1, \ldots$, and it follows that $x_0 \geq x_n$, $n = 0, 1, \ldots$ On the other hand, there exists a positive number k_0 such that $x_0 < k_0$. Since $\tilde{x} \notin \Omega_\infty$, so $n_{k_0}(\tilde{x}) < \infty$, and from (2.27) we have $x_{n_{k_0}}(\tilde{x}) > k_0$. Thus $x_0 \geq x_{n_{k_0}}(\tilde{x}) > k_0$. It is imposible. It shows that $d(\tilde{x}) < \infty$ and therefore, $\Omega_d \subset \{x \in R_+^\infty : d(\tilde{x}) < \infty\}$. From (2.40) we obtain (2.36). This completes the proof.

For the solution of equation (1.5) and the evaluation of functional (1.6), we denote

(2.41) $$\Omega = R_+^\infty \times U^\infty, \quad C = \mathcal{B}_+^\infty \times \Sigma^\infty, \quad \lambda = P_\infty \times P^\infty$$

where $(U^\infty, \Sigma^\infty, P^\infty)$ is the infinite product of the space (U, Σ, P).

By (2.20) we can define the following random variables on the space (Ω, C, λ):

(2.42) $$\xi_\infty = \sum_{i=0}^{d(\tilde{x})} \frac{\xi(x_0; u_0)\,\bar{\gamma}(x_0, x_1; u_1) \ldots \bar{\gamma}(x_{i-1}, x_i; u_i)}{\varphi_0(x_0)\ \varphi(x_0, x_1) \ldots \varphi(x_{i-1}, x_i)} \times$$
$$\times\ \eta(x_i; u_{i+1})$$

(2.43) $$\xi_\infty(x) = \sum_{i=0}^{d(\tilde{x})} \frac{\bar{\gamma}(x, x_0; u_0)\,\bar{\gamma}(x_0\ x_1; u_1) \ldots \bar{\gamma}(x_{i-1}\ x_i; u_i)}{\varphi_0(x_0)\,\varphi(x_0, x_1) \ldots \varphi(x_{i-1}, x_i)} \times$$
$$\times\ \eta(x_i; u_{i+1})$$
$$\forall x \in [0, \infty)$$

(2.44) $$\theta(x) = \eta(x) + \xi_\infty(x), \quad \forall x \in [0, \infty)$$

Suppose that:

(D*) For each $i = 1, 2, \ldots, d(\tilde{x})$, $\xi(x_0; u_0)$ (or $\bar{\gamma}(x, x_0; u_0)$), $\{\bar{\gamma}(x_{k-1}, x_k; u_k)\}_{k=0}^{i=1}$, $\eta(x_i; u_{i+1})$ are independent samples of random variables $\xi(x_0)$ (or $\bar{\gamma}(x, x_0)$), $\{\bar{\gamma}(x_{k-1}, x_k)\}_{k=0}^{i=1}$ and $\eta(x_i)$.

Theorem 2.2. Under the assumptions of Lemma 2.1 and supposing that conditions (A), (B), (C), (D*) are satisfied, we have

$$E_\lambda \xi_\infty \equiv \int_\Omega \xi_\infty(\omega)\lambda(d\omega) = (f, E\xi) \tag{2.45}$$

$$E_\lambda \theta(x) = f(x), \quad \forall x \in [0,\infty) \pmod{\mu} \tag{2.46}$$

where f is the unique solution of equation (1.5).

P r o o f. We first notice that

$$E_\lambda g \equiv \int_\Omega g(x_0, x_1, \ldots, x_n; u_0, \ldots, u_{n+1}) \times \tag{2.47}$$

$$\times\ \lambda\left(\prod_{i=0}^{\infty} dx_i \times \prod_{i=0}^{\infty} du_i\right) =$$

$$= \int_{\Omega_n} g(x_0, x_1, \ldots, x_n; u_0, \ldots, u_{n+1}) \times$$

$$\times\ \lambda_n\left(\prod_{i=0}^{n} dx_i \times \prod_{i=0}^{n+1} du_i\right) \equiv$$

$$\equiv E_{\lambda_n} g, \quad \forall g \in L^1(\Omega_n, \lambda_n)$$

and, for each $N = 2, 3, \ldots$, from (D*) we deduce (D).

From (2.6) it can be deduced that

$$|\xi_N| \le \sum_{i=0}^{N} \frac{|\xi(x_0; u_0)\,\bar{\gamma}(x_0, x_1; u_1) \ldots \bar{\gamma}(x_{i-1}, x_i; u_i)|}{\varphi_0(x_0)\,\varphi(x_0, x_1) \ldots \varphi(x_{i-1}, x_i)} \times \tag{2.48}$$

$$\times\ |\eta(x_i; u_{i+1})| \equiv \bar{\xi}_N$$

By (2.8) and Lemma 2.2 it follows that there exist finite limits.

$$\lim_{N\to\infty} \bar{\xi}_N = \tag{2.49}$$

$$= \sum_{i=0}^{\infty} \frac{|\xi(x_0;\, u_0)\, \bar{\gamma}(x_0,\, x_1;\, u_1)\, \ldots\, \bar{\gamma}(x_{i-1},\, x_i;\, u_i)|}{\varphi_0(x_0)\, \varphi(x_0,\, x_1)\, \ldots\, \varphi(x_{i-1},\, x_i)} \times$$

$$\times\, |\eta(x_i;\, u_{i+1})| \equiv \bar{\xi}_\infty \pmod{\lambda}$$

(2.50)
$$\lim_{N\to\infty} \xi_N =$$

$$= \sum_{i=0}^{\infty} \frac{\xi(x_0;\, u_0)\, \bar{\gamma}(x_0, x_1;\, u_1)\, \ldots\, \bar{\gamma}(x_{i-1},\, x_i;\, u_i)}{\varphi_0(x_0)\, \varphi(x_0,\, x_1)\, \ldots\, \varphi(x_{i-1},\, x_i)} \times$$

$$\times\, \eta(x_i;\, u_{i+1}) \equiv \xi_\infty \pmod{\lambda}$$

Since $P_{i+1} \ll \mu^{i+1}$, the terms of $\bar{\xi}_\infty$ are non-negative, from (2.47), (D), by Fubini's theorem and by the argument used in the proof of Theorem 2.1. we have

$$E_\lambda \bar{\xi}_\infty = \sum_{i=0}^{\infty} E_{\lambda_i} \frac{|\xi(x_0;\, u_0)\bar{\gamma}(x_0,\, x_1;\, u_1)\, \ldots\, \bar{\gamma}(x_{i-1},\, x_i;\, u_i)|}{\varphi_0(x_0)\varphi(x_0,\, x_1)\, \ldots\, \varphi(x_{i-1},\, x_i)}$$

$$|\eta(x_i;\, u_{i+1})| =$$

$$= \sum_{i=0}^{\infty} \int_{R_+^{i+1}} E|\xi(x_0)|E|\bar{\gamma}(x_0,\, x_1)| \ldots E|\bar{\gamma}(x_{i-1},\, x_i)|E|\eta(x_i)|\mu^{i+1} \left(\prod_{k=0}^{i} dx_k\right) = \sum_{i=0}^{\infty} (\bar{\tau}^i E|\eta|, E|\xi|) = \sum_{i=0}^{\infty} \bar{\tau}^i E|\eta|, E|\xi|)$$

From (A), (C) it is deduced that

(2.51)
$$\bar{\xi}_\infty \in L^1(\Omega, \lambda)$$

and by (2.48), (2.49), (2.50), (2.51) it follows that

$$\xi_\infty \in L^1(\Omega, \lambda)$$

Applying Lebesgue's Theorem, we have

$$\lim_{N\to\infty} E_\lambda \xi_N = E_\lambda \left(\lim_{N\to\infty} \xi_N\right) = E_\lambda \xi_\infty$$

and by Theorem 2.1 we obtain

$$E_\lambda \xi_\infty = (f, E\xi)$$

and deduce that $E_\lambda \xi_\infty(x) = (f(\cdot), E\bar{\gamma}(x,\cdot))$. Then,

$$E_\lambda \theta(x) = E_\lambda \eta(x) + E_\lambda \xi_\infty(x) = E\eta(x) + (f(\cdot), E\bar{\gamma}(x,\cdot)) =$$

$$= E\eta(x) + \int_0^x E\gamma(x, y) f(x - y)\mu(dy)$$

Since f is the unique solution of equation (1.5) in $L^p(0,\infty)$, so from this it implies (2.46) and the proof is completed.

3. The probability models solving problems (1.7) and (1.8)

To solve problems (1.8), (1.7), besides equation (1.5) we also consider the following equation:

$$(3.1) \qquad g(y) = \frac{E^2\eta(y)}{\varphi_0(y)} + \int_0^\infty \frac{E^2\bar{\gamma}(x, y)}{\varphi(x,y)} \mu(dy), \qquad 0 \leq y < \infty$$

Suppose that

(E) $\frac{E^2\eta(y)}{\varphi_0(y)} \in L^1(0,\infty)$ and equation (3.1) has the unique solution in $L^1(0,\infty)$

Theorem 3.1. Suppose that conditions (A), (B), (C), (D*) and (E) are satisfied for $p = 1$. In addition, suppose that

1. The unique solution of equation (1.5) is continuous on the interval $(0,\infty)$.

2. The random Hilbert variable $\xi(x, y)$ (see [7] is differentiable with respect to x and $\xi(x, y)$, $\frac{\partial\xi(x, y)}{\partial x}$ are continuous with respect to (x, y) in the sense of square mean and

$$(3.2) \qquad \xi(x, y) = 0 \quad 0 \leq x \leq y < \infty$$

$$(3.3) \qquad \left|\frac{\partial}{\partial x} E\xi(x, y)\right| \leq Kx + D, \text{ where } K, D \text{ are constants,}$$

$$0 \leq y \leq x < \infty$$

3. The function

(3.4) $$V_\xi(x) = \int_0^x E\xi(x, y) f(y)\mu(dy), \quad x \geq 0$$

has the unique stationary point $x^* \in (0,\infty)$, moreover $x^* > a > 0$ and x^* is the maximum point of $V(x)$.

Then

(3.5) $$\lim E_\lambda |b_k - x^*|^2 = 0$$

(3.6) $$\lambda\left\{\lim_{k\to\infty} b_k = x^*\right\} = 1$$

where

(3.7) $$b_{k+1} = b_k + \gamma_k \tilde{\theta}_\xi(b_k), \; b_0 \text{ is an arbitrary number,}$$

(3.8) $$\gamma_k > 0, \quad \sum_{k=0}^{\infty} \gamma_k = \infty, \quad \sum_{k=0}^{\infty} \gamma_k^2 < \infty$$

(3.9) $$\tilde{\theta}_\xi(x) = \begin{cases} \theta_\xi(x), & \text{if } x > a \\ 1, & \text{if } x \leq a \end{cases}$$

(3.10) $$\theta_\xi(x) = \sum_{i=0}^{d(\bar{x})} \frac{\bar{\xi}(x, x_0; u_0)\,\bar{\gamma}(x_0, x_1; u_1) \ldots \bar{\gamma}(x_{i-1}, x_i; u_i)}{\varphi_0(x_0)\varphi(x_0, x_1) \ldots \varphi(x_{i-1}, x_i)} \times$$
$$\times\, \eta(x_i; u_{i+1}), \qquad x > 0$$

(3.11) $$\bar{\xi}(x, y) = \begin{cases} \frac{\partial}{\partial x}\xi(x, y), & \text{if } 0 \leq y \leq x < 0 \\ 0, & \text{if } 0 \leq x < y \end{cases}$$

P r o o f. Clearly that $V_\xi(x)$ is finite for each $x \in [0,\infty)$. Moreover, since $\xi(x, y)$ is differentiable with respect to x in the sense of square mean, therefore (see[7])

(3.12) $$\frac{\partial}{\partial x} E\xi(x, t) = E\left\{\frac{\partial}{\partial x}\xi(x, t)\right\}$$

and (see [6], p. 666-667):

$$(3.13)\quad V'_{\xi}(x) = \int_0^x E\left\{\frac{\partial}{\partial x}\xi(x, t)\right\}f(t)\mu(dt) = \int_0^\infty E\,\bar{\xi}(x, t)f(t)\mu(dt)$$

It is easy to see that $V'_{\xi}(x)$ is continuous. Moreover, since $x^* > a > 0$ and x^* is the unique stationary point of the function $V_{\xi}(x)$ on $(0,\infty)$, we have

$$(3.14)\quad \inf_{\varepsilon<|x-x^*|<\frac{1}{\varepsilon}} (x - x^*)\, V_{1\xi}(x) < 0, \quad \forall\varepsilon > 0$$

where

$$(3.15)\quad V_{1\xi}(x) = \begin{cases} V'_{\xi}(x), & \text{if } x > a \\ 1, & \text{if } x \leq a \end{cases}$$

Applying theorem 2.1 we have

$$(3.16)\quad E_\lambda\theta_\xi(x) = V'_{\xi}(x)$$

By (3.13), (3.3), (3.12) and (C) it is deduced that

$$(3.17)\quad |V'(x)| \leq \int_0^x \left|E\left\{\frac{\partial}{\partial x}\xi(x, t)\right\} f(t)\right|\mu(dt) \leq (Kx + D)\, \|f\|_{L^1(0,\infty)}$$

It is known that (see [4], p. 173).

$$(3.18)\quad E_\lambda\theta^2_\xi(x) = (g(\cdot),\, E\bar{\xi}(x,\cdot)\, [2\varphi^*(\cdot) - E\bar{\xi}(x,\cdot)])$$

where g is the unique solution of equation (3.1) and φ^* is the unique solution of the following equation:

$$(3.19)\quad \varphi^*(t) = E\xi(x, t) + [\tau^*\varphi^*](t)$$

where τ^* is the conjugate operator of the operator τ. We have

$$(3.20)\quad \|\varphi^*\|_{L^\infty(0,\infty)} = \left\|\sum_{i=0}^\infty \tau^{*i} E\bar{\xi}(x,\cdot)\right\|_{L^\infty(0,\infty)} \leq$$

$$\leq \sum_{i=0}^\infty \|\tau^{*i}\|\, (Kx + D) \leq A(Kx + D)$$

where

$$A \geq \sum_{i=0}^{n_0} \|\tau^{*i}\| + \frac{\delta_0^{\,n_0}}{1 - \delta_0}$$

and δ_0, n_0 are chosen by an argument similar to that used in the proof of Theorem 2.1.

By (3.20), (3.18) it follows that

$$(3.21)\quad E_\lambda \theta^2_\xi(x) \leq \|g\|_{L^1(0,\infty)} \|E\bar{\xi}(x,\cdot)\|_{L^\infty(0,\infty)} \left\{2\|\varphi^*\|_{L^\infty(0,\infty)} + \|E\bar{\xi}(x,\cdot)\|_{L^\infty(0,\infty)}\right\} \leq$$

$$\leq \|g\|_{L^1(0,\infty)} (Kx + D)\left\{2A\,(Kx + D) + (Kx + D)\right\} \leq$$

$$\leq d(1 + x^2), \quad \forall x \geq 0$$

where d is a sufficiently large positive constant.

It is easy to see that

$$(3.22)\quad E_\lambda \tilde{\theta}_\xi(x) = V_{1\xi}(x)$$

By (3.17), (3.22), (3.21), (3.14), applying theorem 5.7 (see [3], p. 243), we have (3.5) and (3.6). This completes the proof.

Put

$$(3.23)\quad \Delta(x,\delta) = f(x+\delta) - f(x), \quad x \geq 0$$

where δ is a positive constant and f is the solution of equation (1.5).

Theorem 3.2 Suppose that, when $\xi(x, y) = \bar{\gamma}(x, y)$, the assumptions of Theorem 3.1 (except assumption 3) are satisfied. In addition, suppose that there exists $\frac{\partial \eta(x)}{\partial x}$ (in the sense of square mean) being continuous and satisfying

$$(3.24)\quad E\left|\frac{\partial \eta(x)^2}{\partial x}\right| \leq B, \quad \forall x \geq 0, \quad B = \text{const}$$

and the function $\Delta(x,\delta)$, for each fixed δ, has the unique stationary point $x^* \in [0,\infty)$; moreover, $x^* > 0$ and $\Delta(x,\delta)$ reaches its maximum at x^*.

Then

$$(3.25)\quad \lim_{k\to\infty} E_\lambda |C_k - x^*|^2 = 0$$

$$(3.26)\quad \lambda\left\{\lim_{k\to\infty} C_k = x^*\right\} = 1$$

where

(3.27)
$$C_{k+1} = C_k + \gamma_k \tilde{\Psi}(C_k)$$

C_0 is an arbitrary number, $\{\gamma_k\}_{k=0}^{\alpha_0}$ satisfies (3.8),

(3.28)
$$\tilde{\Psi}(x) = \begin{cases} \Psi(x), & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}$$

(3.29)
$$\Psi(x) = \frac{\partial\eta(x+\Delta)}{\partial x} - \frac{\partial\eta(x)}{\partial x} + \theta_{\bar\gamma}(x+\delta) - \theta_{\bar\gamma}(x), \quad x > 0$$

P r o o f. We have (see [7]).

$$E_\lambda \left\{ \frac{\partial\eta(x)}{\partial x} = E\,\frac{\partial\eta(x)}{\partial x} \right\} = \frac{d}{dx}\, E\eta(x) = \frac{d}{dx}\, E_\lambda \eta(x)$$

It is easy to see that $f(x)$ is differentiable on $[0,\infty)$; therefore, from (3.29), (1.5), (3.4), (3.13) it is deduced that

(3.30)
$$E_\lambda \Psi(x) = \frac{d}{dx}\, \Delta(x,\delta)$$

By (3.24), (3.17), (3.30) we have

(3.31)
$$|E_\lambda \Psi(x)| = \left|\frac{\partial}{\partial x} E\eta(x+\delta) - \frac{\partial}{\partial x} E\eta(x) + \right.$$
$$\left. + E_\lambda\theta_{\bar\gamma}(x+\delta) - E_\lambda\theta_{\bar\gamma}(x)\right| \leq 2\sqrt{B} + |E_\lambda\theta_{\bar\gamma}(x+\delta)| + |E_\lambda\theta_{\bar\gamma}(x)| \leq$$
$$\leq A_1 x + B_1$$

where A_1, B_1 are positive constants.

By (3.21), (3.17), (3.24) it is deduced that

(3.32)
$$E_\lambda \Psi^2(x) = E_\lambda\Big\{\Big(\frac{\partial\eta(x)}{\partial x}\Big)^2 + \Big(\frac{\partial\eta(x+\delta)}{\partial x}\Big)^2 +$$
$$+ \theta_{\bar\gamma}^2(x+\delta) + \theta_{\bar\gamma}^2(x) - 2\theta_{\bar\gamma}(x+\delta)\,\frac{\partial\eta(x+\delta)}{\partial x} -$$
$$- 2\theta_{\bar\gamma}(x)\,\frac{\partial\eta(x+\delta)}{\partial x} -$$
$$- 2\theta_{\bar\gamma}(x+\delta)\,\frac{\partial\eta(x)}{\partial x} + 2\theta_{\bar\gamma}(x)\frac{\partial\eta(x)}{\partial x}\Big\} \leq$$

$$\leq 2B + d[1 + (x + \delta)^2] + d(1 + x^2) +$$

$$+ 2\sqrt{B}d \left\{[1 + (x + \delta)^2 + (1 + x^2)\right\} \leq d^*(1 + x^2)$$

where d* is a sufficiently large positive constant.

By (3.28) it follows that (3.31) holds for each $x \in (-\infty, \infty)$ and by the assumptions on $\overline{\gamma}(x, y)$ and $\eta(x)$ it is deduced that $V'_{\overline{\gamma}}(x)$ and $\frac{d}{dx} E\eta(x)$ are continuous. From this and (3.23) it implies that $\frac{\partial}{\partial x} \Delta(x,\delta)$ is continuous. By (3.28) and the assumptions on x^* it can be deduced that

(3.33) $$\inf_{\varepsilon<|x-x^*|<\frac{1}{\varepsilon}} (x - x^*)\Delta_1(x,\delta) < 0, \ \forall \varepsilon > 0$$

where

$$\Delta_1(x,\delta) = \begin{cases} \frac{d}{dx}\Delta(x,\delta), & \text{if } x > 0 \\ 1, & \text{if } x \leq 0 \end{cases}$$

and

(3.34) $$E_{\lambda}\widetilde{\Psi}(x) = \Delta_1(x,\delta)$$

From (3.28), (3.31), (3.24), (3.32), (3.33) we obtain (3.25), (3.26) (see[6], p. 243, Theorem 5.7). This completes the proof.

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O PEWNYCH WŁASNOŚCIACH PROBABILISTYCZNYCH ROZWIĄZANIA STOCHASTYCZNEGO RÓWNANIA CAŁKOWEGO

W pracy rozważa się zmienne losowe w przestrzeniach probabilistycznych generowane przez trajektorie łańcucha Markowa. Udowodnione zostało, że wartości oczekiwane $E\xi_N$, $E\xi_N(x)$ są asymptotycznie nieobciążonymi estymatorami iloczynu skalarnego $(f, E\xi)$ i rozwiązania $f(x)$ stochastycznego równania całkowego postaci: $f(x) = E\eta(x) + \int_0^x E\gamma(x, y)f(x - y)\,\mu(dy)$ w przestrzeniach $L^p(0,\infty)$ $(p \geq 1)$.